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Limit properties in a family of quasi-arithmetic means

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Abstract. It is known that the Power Means tend to the maximum of their arguments when the exponents tend to $+\infty$. We give certain necessary and sufficient conditions for a 1-parameter family of quasi-arithmetic means generated by functions satisfying certain smoothness conditions to have an analogous property. Our results are deeply connected with operators introduced by Mikusiński and Páles in the late 1940s and late 1980s, respectively. The main result is a generalization of the author's earlier results.

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1. Introduction

There are several directions of exploration concerning means in mathematical analysis. Definitely the most frequent are inequalities among various families of means. It can be seen in the by-now-classical monography [1].

In the present paper we are going to discuss a limit property holding true in certain 1-parameter families of the enormously vast family of quasi-arithmetic means, introduced in the series of nearly simultaneous papers [2,4,6] as a generalization of Power Means. Namely, for any continuous, strictly monotone function $f\colon U \to \mathbb{R}$ (U – an interval) one may define, for every vector of *entries* $a = (a_1, \ldots, a_r) \in U^r$, $r \in \mathbb{N}$, with *weights* $w = (w_1, \ldots, w_r)$, where $w_i > 0$ for $i = 1, \ldots, r$ and $w_1 + \cdots + w_r = 1$, the *quasi–arithmetic mean*

$$\mathcal{A}^{[f]}(a, w) := f^{-1}\big(w_1 f(a_1) + w_2 f(a_2) + \cdots + w_r f(a_r)\big).$$

For $U = (0, +\infty)$ and $f := p_\alpha$, where

$$p_\alpha(x) = \begin{cases} x^\alpha & \alpha \neq 0, \\ \ln(x) & \alpha = 0, \end{cases}$$

one thus obtains the αth power mean.

We will now discuss some limit properties in this very rich family of means. It is well known that, for the p_α above and every all-positive-components vector a with corresponding weights w,

$$\lim_{\alpha \to -\infty} \mathcal{A}^{[p_\alpha]}(a,w) = \min(a), \qquad \lim_{\alpha \to +\infty} \mathcal{A}^{[p_\alpha]}(a,w) = \max(a).$$

More generally, for a given sequence of means $(M_n)_{n \in \mathbb{N}}$ one could study (whenever it exists) the pointwise limit $\lim_{n\to\infty} M_n$. In many families this limit is either the maximum or minimum. As mentioned before, this property is known best for the family of Power Means, but it is also pertinent to Gini means, Bonferroni means, mixed means etc. (These families, except for Power Means, are not quasi-arithmetic.)

It is much different for *general* quasi-arithemetic means. Some results concerning quasi-arithmetic means were proved by Kolesarova [3]. We proved in [8] certain results under an additional smoothness condition (the generating function is twice differentiable, having a nowhere vanishing first derivative). Another result is closely related to the previous result of Páles [7] (cf. Lemma 2.1), announced by him during a private conversation.

We will discuss when the family of quasi-arithmetic means generated by $(f_n)_{n\in\mathbb{N}}$, $f_n \colon U \to \mathbb{R}$ tends to the maximum pointwise. More precisely

$$\lim_{n\to\infty} \mathcal{A}^{[f_n]}(a,w) = \max(a)$$

for every admissible a and w. Hereafter such a family will be called *max-family*. Analogously we define min-family. These definitions are adapatable to many different means, but very often some natural adaptation is required (e.g., omitting weights, restricting the vector of arguments to a fixed length etc). For example, in this terminology, $(\mathcal{A}^{[p_n]})_{n\in\mathbb{N}}$ is a max-family, while $(\mathcal{A}^{[p_{-n}]})_{n\in\mathbb{N}}$ is a min-family.

We are going to present *three* necessary and sufficient conditions for the family of quasi-arithmetic means to be a max-family (each time a requiring different smoothness assumption).

2. Auxiliary results

In order to simplify many proofs in the present note we will restrict our consideration just to the two variable case. It will be denoted briefly by

$$\begin{aligned}\mathcal{A}_\xi^{[f]}(x,z) &:= \mathcal{A}^{[f]}\Big((x,z),(\xi,1-\xi)\Big)\\ &= f^{-1}(\xi f(x) + (1-\xi)f(z)), \qquad x,z\in U,\ \xi\in(0,1).\end{aligned}$$

We will prove the following equivalence-type lemma, involving -[general] weighted quasi-arithmetic means,-quasi-arithmetic means of two variables and -some operator introduced by Páles [7].

Lemma 2.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a family of continuous, strictly monotone functions defined on an interval U. Then the following conditions are equivalent*

(i) (f_n) *is a max-family,*
(ii) $\lim_{n\to\infty} \mathcal{A}_\xi^{[f_n]}(x,z) = \max(x,z)$ *for* $x,\, z \in U$ *and* $\xi \in (0,1)$,
(iii) $\lim_{n\to\infty} \frac{f_n(x)-f_n(y)}{f_n(z)-f_n(y)} = 0$ *for all* $x,\, y,\, z \in U$, $x<y<z$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. For the converse implication, notice that $\mathcal{A}^{[f_n]}$ is symmetric for each $n \in \mathbb{N}$ (one needs to simultaneously change entries and weights). Moreover, for every function f_n, vector $a \in U^r$ satisfying $a_1 \le a_2 \le \cdots \le a_r$ and corresponding weights w, we have (using the brief notation introduced in the beginning of the present section)

$$\max(a) \ge \mathcal{A}^{[f_n]}(a,w) \ge \mathcal{A}_{w_r}^{[f_n]}(a_r, a_1)\,.$$

Then, passing to the limit,

$$\lim_{n\to\infty} \mathcal{A}^{[f_n]}(a,w) = \max(a_1,a_r) = a_r = \max(a).$$

(ii) $\Leftrightarrow$ (iii) Let us assume that each f_n is increasing (replacing f_n by $-f_n$ if necessary; cf. Remark 2.3). Then

$$\frac{f_n(x)-f_n(y)}{f_n(z)-f_n(y)} < 0 \text{ for every } n \in \mathbb{N} \text{ and } x,\, y,\, z \in U \text{ satisfying } x<y<z.$$

Whence, for $x,\, y,\, z \in U$ satisfying $x<y<z$ and $\xi \in (0,1)$, one simply gets

$$\begin{aligned} y < \mathcal{A}_\xi^{[f_n]}(x,z) &\iff f_n(y) < \xi f_n(x) + (1-\xi) f_n(z) \\ &\iff \frac{f_n(x)-f_n(y)}{f_n(z)-f_n(y)} > \tfrac{\xi-1}{\xi}, \\ &\iff \frac{f_n(x)-f_n(y)}{f_n(z)-f_n(y)} \in \left(\tfrac{\xi-1}{\xi}, 0\right). \end{aligned}$$

Upon passing $y \to z$ and $\xi \to 1$ we obtain the ($\Leftarrow$) and the ($\Rightarrow$) part of proof, respectively. Standard consideration involving the definition of limit is omitted here. □

Now we are going to recall some specification of Mikusiński's result [5]. He and, independently, Łojasiewicz (cf. [5, footnote 2]) established a handy tool allowing us to compare quasi-arithmetic means. It is expressed in terms of the operator

$$A_f := \frac{f''}{f'}$$

defined for every twice differentiable function having a nowhere vanishing first derivative. They proved (in a much more general framework) the following

Lemma 2.2. *Let U be an interval, $f,\ g\colon U \to \mathbb{R}$ be twice differentiable functions with nowhere vanishing first derivatives. Then the following conditions are equivalent:*

- $A_f(x) \le A_g(x)$ *for every* $x \in U$,
- $\mathcal{A}^{[f]}(a,w) \le \mathcal{A}^{[g]}(a,w)$ *for every admissible* a, w,
- $\mathcal{A}_{\xi}^{[f]}(x,y) \le \mathcal{A}_{\xi}^{[g]}(x,y)$ *for every admissible* x, y, *and* ξ.

The operator A is so central that, to make the notation more compact, we will call a function [to be] $\mathcal{D}^2$ if it is twice differentiable with a nowhere vanishing first derivative – in fact it is the weakest possible assumption needed to define the operator A.

Remark 2.3. *Lemma 2.2 has its own 'equal-type' version. Namely, in the setting of the previous lemma the following conditions are equivalent:*

- $A_f(x) = A_g(x)$ *for every* $x \in U$,
- $\mathcal{A}^{[f]}(a,w) = \mathcal{A}^{[g]}(a,w)$ *for every admissible* a, w,
- $\mathcal{A}_{\xi}^{[f]}(x,y) = \mathcal{A}_{\xi}^{[g]}(x,y)$ *for every admissible* x, y, *and* ξ,
- $f = \alpha g + \beta$ *for some* $\alpha, \beta \in \mathbb{R}$, $\alpha \ne 0$.

Let us define, for a family $(f_n)_{n\in\mathbb{N}}$, the following properties:

- $\mathcal{D}^2$ if f_n is $\mathcal{D}^2$ for all n,
- *increasing* if $\mathcal{A}^{[f_n]}(a,w) \ge \mathcal{A}^{[f_m]}(a,w)$ for every $n \ge m$ and admissible a and w [by Lemma 2.2 we obtain some equivalent definitions],
- *lower bounded* if all functions are differentiable, have nowhere vanishing derivatives and there exists a universal constant $C \in \mathbb{R}$ satisfying $f_n'(y)/f_n'(x) \ge e^{C(y-x)}$ for all $n \in \mathbb{N}$ and x, $y \in U$, $x < y$.

We also define their duals: min-, decreasing and upper bounded family. In fact each result presented in this paper has its dual wording, which are omitted, but may be similarly established and proved. Notice that the mapping $f(x) \mapsto f(-x)$ is a natural transformation between relevant definitions [and results]. Boundedness of a family is connected with the scale $\{\mathcal{E}_p\}_{p\in\mathbb{R}}$ of log - exp means (a subclass of quasi-arithmetic means with constant Mikusiński's operator; cf. [1, p. 269], [8]) defined for every $a \in \mathbb{R}^k$ ($k \in \mathbb{N}_+$) with corresponding weights w as

$$\mathcal{E}_p(a,w) := \begin{cases} \frac{1}{p}\ln\left(w_1 \cdot e^{p\cdot a_1} + w_2 \cdot e^{p\cdot a_2} + \cdots + w_k \cdot e^{p\cdot a_k}\right), & \text{if } p \ne 0,\\ w_1 a_1 + \cdots + w_k a_k, & \text{if } p = 0. \end{cases}$$

Then, in view of Lemma 2.2, the family $(f_n)_{n\in\mathbb{N}}$ of differentiable, strictly monotone functions defined on a common interval is lower bounded if and only if there exists a universal constant $C \in \mathbb{R}$ satisfying

$$\mathcal{A}^{[f_n]}(a,w) \ge \mathcal{E}_C(a,w) \text{ for all admissible } a,\ w \text{ and } n \in \mathbb{N}.$$

For a $\mathcal{D}^2$-family it will be also handy to write

$$X_\infty := \left\{ x \in U \colon \lim_{n \to \infty} A_{f_n}(x) = +\infty \right\}. \tag{2.1}$$

Notice that the set X_∞ depends on the family $(f_n)_{n \in \mathbb{N}}$, but in each usage of this notion the familly will be known. Let us recall the major result from [8]:

Proposition 2.4. *Let U be a closed, bounded interval, $(f_n)_{n \in \mathbb{N}}$ be an increasing $\mathcal{D}^2$-family defined on U.*

- *If $X_\infty = U$ then (f_n) is a max-family.*
- *If (f_n) is a max-family then X_∞ is a dense subset of U.*

The proof enclosed in [8] was not written precisely enough. Because of that this proposition will be reproved in Sect. 4.2 as one of the applications of Theorem 3.1. Moreover, there appears a natural question: how to close the gap between necessary and sufficient conditions (in [8] it was stated as an open problem).

The answer is fairly non-trivial. Precisely, the fact whether the family is a max-family cannot be completely characterized by the properties of X_∞ (compare Propositions 4.1 and 4.3). Some examples, counter-examples as well as the strengthening of Proposition 2.4 will be given in Sect. 4.1.

3. Main result

We prove the following equivalence-type result for the family of differentiable functions to be max.

Theorem 3.1. *Let U be an interval, $(f_n)_{n \in \mathbb{N}}$ be a lower bounded family defined on U. Then the following conditions are equivalent*

(i) *$(f_n)_{n \in \mathbb{N}}$ is a max-family;*
(ii) *$\lim_{n \to \infty} \frac{f_n'(q)}{f_n'(p)} = +\infty$ for all $p, q \in U$, $p < q$.*

If, additionally, $(f_n)_{n \in \mathbb{N}}$ is a $\mathcal{D}^2$-family, there is an extra condition

(iii) *$\lim_{n \to \infty} \int_p^q A_{f_n}(x) dx = +\infty$ for all $p, q \in U$, $p < q$*

equivalent to (i) *and* (ii).

We will now prove Theorem 3.1. Throughout the proof we will assume that the constant C appearing in the definition of lower bounded family is negative. When the family is $\mathcal{D}^2$ then the equivalence (ii) $\iff$ (iii) is obvious. The equivalence (i) $\iff$ (ii) will be shown in the following two subsections.

3.1. Proof of (i) ⇒ (ii)

Let us assume that there exist x, $z \in U$, $x < z$ such that

$$\liminf_{n\to\infty} \frac{f_n'(z)}{f_n'(x)} < +\infty.$$

Then there exist $\bar{H} > 0$ and a subsequence $(n_1, n_2, \ldots)$ satisfying

$$\frac{f_{n_k}'(z)}{f_{n_k}'(x)} < \bar{H}, \quad k \in \mathbb{N}.$$

In particular, by the lower boundedness property, for every $k \in \mathbb{N}$ and $p, q \in [x, z]$,

$$\frac{f_{n_k}'(q)}{f_{n_k}'(p)} = \frac{f_{n_k}'(z)}{f_{n_k}'(x)} \cdot \frac{f_{n_k}'(x)}{f_{n_k}'(p)} \cdot \frac{f_{n_k}'(q)}{f_{n_k}'(z)} \le \frac{f_{n_k}'(z)}{f_{n_k}'(x)} e^{C(x-p)} e^{C(q-z)} \le \bar{H} e^{2C(x-z)}.$$

Hence, with $H := \bar{H} e^{2C(x-z)}$,

$$\frac{f_{n_k}'(q)}{f_{n_k}'(p)} < H, \qquad \text{for all } k \in \mathbb{N} \text{ and } p, q \in [x, z].$$

Fix $y \in (x, z)$ and (by Remark 2.3) assume $f_n'(y) = 1$, $f_n(y) = 0$ for all $n \in \mathbb{N}$ or, equivalently,

$$f_n(\tau) = \int_y^\tau \frac{f_n'(t)}{f_n'(y)} dt, \quad n \in \mathbb{N}, \tau \in U. \tag{3.1}$$

Then one has $f_{n_k}(\tau) \le H \cdot (\tau - y)$ for $\tau \in (y, z)$. In particular, by the continuity of f_{n_k},

$$f_{n_k}(z) \le H \cdot (z - y), \quad k \in \mathbb{N}. \tag{3.2}$$

Moreover, for every $k \in \mathbb{N}$ we have the following implications:

$$\begin{aligned} f_{n_k}'(y)/f_{n_k}'(t) &\le H, \quad t \in (x, y), \\ f_{n_k}'(t)/f_{n_k}'(y) &\ge \tfrac{1}{H}, \quad t \in (x, y), \\ f_{n_k}'(t) &\ge \tfrac{1}{H}, \quad t \in (x, y), \\ \int_x^y f_{n_k}'(t) dt &\ge \tfrac{y-x}{H}, \\ -f_{n_k}(x) &\ge \tfrac{y-x}{H}, \\ f_{n_k}(x) &\le \tfrac{x-y}{H}. \end{aligned} \tag{3.3}$$

Since $\frac{x-y}{H} < 0$, there exists $\xi \in (0, 1)$ such that

$$\xi \tfrac{x-y}{H} + (1 - \xi) H (z - y) < 0.$$

Whence, by (3.2) and (3.3),

$$\xi f_{n_k}(x) + (1 - \xi) f_{n_k}(z) < 0, \quad k \in \mathbb{N}.$$

But, by (3.1), f_{n_k} is increasing and $f_{n_k}(y) = 0$. Lastly, applying $f_{n_k}^{-1}$ to both sides,

$$\mathcal{A}_{\xi}^{[f_{n_k}]}(x,z) < y, \quad k \in \mathbb{N}.$$

Thus $(f_n)_{n\in\mathbb{N}}$ is not a max-family.

3.2. Proof of (ii) ⇒ (i)

For the most part of this proof we will be dealing with a certain property of a quasi-arithmetic mean generated by a differentiable fuction f (having a nowhere vanishing first derivative) satisfying

$$\frac{f'(y)}{f'(x)} \geq e^{C\cdot(y-x)} \text{ for some } C \text{ and all } x,\, y \in U,\, y > x. \tag{3.4}$$

Note that the inequality above has already appeared in the definition of lower bounded family. Let us firstly establish the following

Lemma 3.2. *Let U be an interval, $f\colon U \to \mathbb{R}$ be a differentiable function with a nowhere vanishing derivative satisfying (3.4) for some $C < 0$. Let us take $\xi \in (0,1)$ and x, y, $z \in U$ satisfying $x < y < z$. Then there exists $\Phi = \Phi(\xi, C, \varepsilon, x, y)$ such that for every $\varepsilon \in (0, z-y)$*

$$\frac{f'(z-\varepsilon)}{f'(y)} \geq \Phi \Rightarrow \mathcal{A}_{\xi}^{[f]}(x,z) \geq y.$$

It could be observed that Lemma 3.2 implies the part (ii) ⇒ (i) of Theorem 3.1. Indeed, by the definition, $\mathcal{A}_{\xi}^{[f]}(x,z) = \mathcal{A}_{1-\xi}^{[f]}(z,x)$. So let us assume without loss of generality that $x < z$. Take an arbitrary $y \in (x,z)$ and $\varepsilon \in (0, z-y)$. There exists n_y such that

$$\frac{f_n'(z-\varepsilon)}{f_n'(y)} \geq \Phi(\xi, C, \varepsilon, x, y) \quad \text{for every } n > n_y.$$

Thus, by Lemma 3.2,

$$\mathcal{A}_{\xi}^{[f_n]}(x,z) \geq y \quad \text{for every } n > n_y.$$

Lastly, upon passing $y \to z$, one gets

$$\lim_{n\to\infty} \mathcal{A}_{\xi}^{[f_n]}(x,z) = z.$$

3.3. Proof of Lemma 3.2

In view of Remark 2.3, let us assume without loss of generality (like it was already done in (3.1)) that

$$f(\tau) = \int_y^\tau \frac{f'(t)}{f'(y)} dt \quad \text{for } \tau \in U.$$

We will establish a certain lower bound for $f(x)$ and, later, for $f(z)$. Since $f'(y) = 1$, inequality (3.4) implies (for $x = \tau$)

$$f'(\tau) \le e^{C(\tau - y)}, \quad \tau \in (x, y).$$

Whence

$$f(y) - f(x) = \int_x^y f'(\tau) d\tau \le \int_x^y e^{C(\tau - y)} d\tau = \tfrac{1}{C}(1 - e^{C(x-y)}).$$

Therefore, by $f(y) = 0$,

$$f(x) \ge \tfrac{1}{C}(e^{C(x-y)} - 1).$$

A bound for $f(z)$ looks fairly different. Fix $\varepsilon \in (0, z - y)$. One has

$$\begin{aligned} f(z) &= \int_y^z \frac{f'(t)}{f'(y)} dt \\ &\ge \int_{z-\varepsilon}^z \frac{f'(t)}{f'(y)} dt \\ &= \frac{f'(z-\varepsilon)}{f'(y)} \int_{z-\varepsilon}^z \frac{f'(t)}{f'(z-\varepsilon)} dt \\ &\ge \frac{f'(z-\varepsilon)}{f'(y)} \int_{z-\varepsilon}^z e^{C(t-z+\varepsilon)} dt \\ &= \frac{1}{C} \cdot \frac{f'(z-\varepsilon)}{f'(y)} \left(e^{C\varepsilon} - 1\right). \end{aligned}$$

We are going to prove that $\mathcal{A}_\xi^{[f]}(x, z) \ge y$ for a sufficiently large value of $f'(z - \varepsilon)/f(y)$. Indeed, we have a series of ($\Leftarrow$) implications:

$$\begin{aligned} &\mathcal{A}_\xi^{[f]}(x, z) \ge y \\ &\quad \Leftarrow \xi f(x) + (1-\xi) f(z) \ge f(y) \\ &\quad \Leftarrow \tfrac{\xi}{C} \cdot (e^{C(x-y)} - 1) + \tfrac{1-\xi}{C} \cdot \frac{f'(z-\varepsilon)}{f'(y)} \left(e^{C\varepsilon} - 1\right) \ge 0 \\ &\quad \Leftarrow \frac{f'(z-\varepsilon)}{f'(y)} \ge -\tfrac{\xi}{1-\xi} \cdot \frac{e^{C(x-y)} - 1}{e^{C\varepsilon} - 1} =: \Phi(\xi, C, \varepsilon, x, y). \end{aligned}$$

In the last $\Leftarrow$ implication it was important that $\xi \in (0, 1)$, $C < 0$, and $\varepsilon > 0$.

4. Applications

4.1. Relations between max-family and X_∞ set

We are heading now toward a possible strengthening of Proposition 2.4. We will present vary situations in a sequence of propositions (examples). Let us denote by λ, H^d and $\dim_{\mathrm{H}}$ the Lebesgue measure, d-dimensional Hausdorff measure, and Hausdorff dimension, respectively. Moreover, the definition (2.1) will be used.

Proposition 4.1. *Let U be an interval, V be an arbitrary subset of U. If there exists an open interval $W \subset U$ such that $\lambda(V \cap W) = 0$ then there exists an increasing $\mathcal{D}^2$-family $(f_n)_{n\in\mathbb{N}}$, $f_n\colon U \to \mathbb{R}$, $n \in \mathbb{N}$, which is not a max-family, although $X_\infty \supset V$.*

Proof. Without loss of generality, let us assume $U = W$. We will construct an increasing $\mathcal{D}^2$-family $(f_n)_{n\in\mathbb{N}}$, $f_n\colon U \to \mathbb{R}$, $n \in \mathbb{N}$ satisfying (i) $\|A_{f_n}\|_{L_1(U)} < 1$ for any $n \in \mathbb{N}$ and (ii) $X_\infty \supset V$. Then, by Theorem 3.1, this family will not be max.

By the regularity of the Lebesgue measure, there exist open sets $(G_k)_{k=0}^\infty$ and $(H_k)_{k=0}^\infty$ satisfying (i) $V \Subset G_k \Subset H_k \subseteq G_{k-1}$, $k \in \mathbb{N}_+$, (ii) $G_0 \subset U$ and (iii) $\lambda(H_k) < \frac{1}{2^k}$, $k \in \mathbb{N}$. It follows from Tietze's theorem that there exists a family $(s_k)_{k=1}^\infty$, $s_k\colon U \to [0, 1]$ of continuous functions

$$s_k(x) = \begin{cases} 1 & x \in G_k, \\ 0 & x \in U \backslash H_k. \end{cases}$$

Then $\|s_k\|_{L_1(U)} < \frac{1}{2^k}$ for any $k \in \mathbb{N}$. Let us define

$$A_{f_n} := s_1 + s_2 + \cdots + s_n.$$

One has $\|A_{f_n}\|_{L_1(U)} < 1$ for any $n \in \mathbb{N}$. Whence, by Theorem 3.1, $(f_n)_{n\in\mathbb{N}}$ is not a max-family. Nevertheless $A_{f_n}(x) = n$ for each $n \in \mathbb{N}$ and $x \in V$. In particular, $X_\infty \supset V$. □

Proposition 4.2. *Let U be an interval, $(f_n)_{n\in\mathbb{N}}$, $f_n\colon U \to \mathbb{R}$, $n \in \mathbb{N}$ be an increasing $\mathcal{D}^2$-family. If $\lambda(X_\infty \cap V) > 0$ for each open subset $V \subset U$, then (f_n) is a max-family.*

Proof. Assume $A_{f_n}(x) > C$ for some $C < 0$, all $x \in U$, and all $n \in \mathbb{N}$. Fix $a, b \in U$, $a < b$. We have $\lambda(X_\infty \cap [a, b]) > 0$. For every $M > 0$, one has

$$\bigcup_{n\in\mathbb{N}} \{x \in U\colon A_{f_n}(x) > M\} \supset X_\infty.$$

In particular, by the regularity of the Lebesgue measure and monotonicity of $n \mapsto A_{f_n}$, there exists n_M such that

$$\lambda\big([a, b] \cap X_\infty \backslash \{x \in U\colon A_{f_{n_M}}(x) > M\}\big) < 1/M.$$

Equivalently,

$$\lambda\big([a,b]\cap X_\infty \cap \{x\in U\colon A_{f_{n_M}}(x)>M\}\big) > \lambda\big([a,b]\cap X_\infty\big)-1/M.$$

Whence,

$$\begin{aligned}\int_a^b A_{f_{n_M}}(x)dx &\geq C\cdot(b-a)+M\cdot\lambda\big(\{x\in U\colon A_{f_{n_M}}(x)>M\}\big)\\ &\geq C\cdot(b-a)+M\cdot\lambda\big([a,b]\cap X_\infty\cap\{x\in U\colon A_{f_{n_M}}(x)>M\}\big)\\ &\geq C\cdot(b-a)+M\cdot\big(\lambda\big([a,b]\cap X_\infty\big)-1/M\big)\\ &= C\cdot(b-a)-1+M\cdot\lambda\big([a,b]\cap X_\infty\big).\end{aligned}$$

Upon taking a limit $M\to+\infty$ one gets

$$\lim_{M\to\infty}\int_a^b A_{f_{n_M}}(x)dx=+\infty \text{ for every } a,\, b\in U,\, a<b.$$

Therefore, by the monotonicity property,

$$\lim_{n\to\infty}\int_a^b A_{f_n}(x)dx=+\infty \text{ for every } a,\, b\in U,\, a<b.$$

So, by Theorem 3.1, $(f_n)_{n\in\mathbb{N}}$ is a max-family. □

Proposition 4.3. *Let U be an interval. There exists an increasing max-family $(f_n)_{n\in\mathbb{N}}$, $f_n\colon U\to\mathbb{R}$ satisfying* $\dim_{\mathrm{H}}(X_\infty)=0$.

Proof. Let us enumerate all rational numbers contained in a set U:

$$\mathbb{Q}\cap U=(q_1,q_2,\ldots).$$

Let $B(x,r):=\{y\in\mathbb{R}\colon\ |x-y|<r\}$ and

$$Q_k=\bigcup_{i=1}^k B\left(q_i,\frac{1}{k^2\cdot 2^i}\right),\qquad \widehat{Q}_k=\bigcup_{i=1}^k B\left(q_i,\frac{2}{k^2\cdot 2^i}\right).$$

Then both Q_k and $\widehat{Q}_k$ are finite sums of open intervals, $\lambda(Q_k)\leq\frac{1}{k^2}$, and $\lambda(\widehat{Q}_k)\leq\frac{2}{k^2}$. For each $k\in\mathbb{N}$ there exists a continuous function $c_k\colon U\to[0,k^2]$,

$$c_k=\begin{cases}k^2 & Q_k,\\ 0 & U\backslash\widehat{Q}_k.\end{cases}$$

Consider a $\mathcal{D}^2$-family $(f_n)_{n\in\mathbb{N}}$ defined on U satisfying $A_{f_n}=c_1+\cdots+c_n$, $n\in\mathbb{N}$. Fix $x,\,y\in U$, $x<y$. By Theorem 3.1, to show that $(f_n)_{n\in\mathbb{N}}$ is a max-family, it is sufficient to prove that

$$\lim_{n\to\infty}\int_x^y A_{f_n}(u)du=+\infty. \tag{4.1}$$

Let us take $q_i \in \mathbb{Q} \cap (x, y)$ and k_0 such that $B(q_i, \frac{1}{k_0^2 \cdot 2^i}) \subset (x, y)$. Then, for $k > \max(i, k_0) =: k_1$, one gets

$$\int_x^y c_k(u)du \geq \int_{B(q_i,1/(k^2 \cdot 2^i))} c_k(u)du = \int_{B(q_i,1/(k^2 \cdot 2^i))} k^2 du > 2^{1-i}.$$

Therefore, for $n > k_1$,

$$\int_x^y A_{f_n}(u)du \geq \sum_{k=k_1+1}^{n} \int_x^y c_k(u)du > (n - k_1)2^{1-i}.$$

Whence (4.1) holds. So (f_n) is a max-family.

We will now prove that the Hausdorff dimension of the set X_∞ equals 0. Indeed,

$$c_1(x) + \cdots + c_{n-1}(x) < n^3 \quad \text{for every } n \in \mathbb{N} \text{ and } x \in U.$$

Thus,

$$X_\infty \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \operatorname{supp} c_k \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \widehat{Q}_k.$$

Therefore, by the definition of $\widehat{Q}_k$, one obtains

$$X_\infty \subseteq \bigcup_{k=n}^{\infty} \widehat{Q}_k = \bigcup_{i=1}^{n} B\left(q_i, \frac{2}{n^2 \cdot 2^i}\right) \cup \bigcup_{i=n+1}^{\infty} B\left(q_i, \frac{2}{i^2 \cdot 2^i}\right), \quad n \in \mathbb{N}.$$

Whence, for every $d > 0$ and $n \in \mathbb{N}$, one gets

$$\begin{aligned} H^d(X_\infty) &\leq \sum_{i=1}^{n} \left(\frac{4}{n^2 \cdot 2^i}\right)^d + \sum_{i=n+1}^{\infty} \left(\frac{4}{i^2 \cdot 2^i}\right)^d \\ &\leq \frac{4^d}{n^{2d}} \sum_{i=1}^{\infty} \frac{1}{2^{id}} = \frac{4^d}{n^{2d}(1 - 2^{-d})}. \end{aligned}$$

As $n \to \infty$, we get $H^d(X_\infty) = 0$ for every $d > 0$. So $\dim_{\mathrm{H}}(X_\infty) = 0$. □

Remark 4.4. *It is known that X_∞ is a G_δ-set for every max-family (cf. [8, pp.204–205]. Therefore, X_∞ could not be the set of rational numbers.*

4.2. New proof of Proposition 2.4

The first part is simply implied by Proposition 4.2. To prove the second part, we shall show that if X_∞ were not dense, then there would exist a closed, non-trivial (different from one point) interval I such that

$$M_I := \sup_{x \in I} \lim_{n \to \infty} A_{f_n}(x) < +\infty.$$

If this is so, one would get

$$\sup_{n\in\mathbb{N}} \int_I A_{f_n}(x)dx < |I| \cdot M_I < +\infty$$

and, by Theorem 3.1, the family (f_n) would not be a max-family.

Assume to the contrary that for every closed, non-trivial interval I one has $M_I = +\infty$.

Then for every closed, non-trivial interval $I_0 \subset U$ one can find a sequence $I_0 \supset I_1 \supset \dots$ of closed, non-trivial intervals satisfying

$$\lim_{n\to\infty} A_{f_n}(x) > j, \quad \text{for every } j \in \mathbb{N} \cup \{0\} \text{ and } x \in I_j.$$

Indeed, for every $j \in \mathbb{N} \cup \{0\}$, in view of $M_{I_j} = +\infty$, there exist $x_j \in I_j$ and n_j such that $A_{f_{n_j}}(x_j) > j+1$. In particular, one can take some closed neighbourhood $I_{j+1} \ni x_j$, $I_{j+1} \subset I_j$, satisfying

$$A_{f_{n_j}}(x) > j+1 \quad \text{for every } x \in I_{j+1}.$$

Whence, $X_\infty \supset \bigcap_{j=0}^{\infty} I_j \neq \emptyset$, so that $X_\infty \cap I_0 \neq \emptyset$. As I_0 was arbitrary, X_∞ is dense.

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