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Approximate Roberts orthogonality sets and (δ,ε)-(a,b)-isosceles-orthogonality preserving mappings

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Abstract. We introduce the concept of approximate Roberts orthogonality set and investigate the geometric properties of such sets. In addition, we introduce the notion of approximate a-isosceles-orthogonality and consider a class of mappings, which approximately preserve a-isosceles-orthogonality.

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1. Introduction

In an inner product space, two vectors are said to be orthogonal when their inner product vanishes. There are several concepts of orthogonality such as Roberts, Birkhoff–James, isosceles, Pythagorean, etc, in an arbitrary real normed space $(\mathcal{X}, \|\cdot\|)$, which can be regarded as generalizations of orthogonality in the inner product spaces, see [1,2,8]. Among them we recall the following ones (see [10]):

(i) *Roberts* $\perp_R$: $x \perp_R y$ if $\|x+ty\| = \|x-ty\|$ for all $t \in \mathbb{R}$;
(ii) *Birkhoff–James* $\perp_J$: $x \perp_J y$ if $\|x\| \leq \|x+ty\|$ for all $t \in \mathbb{R}$;
(iii) *Isosceles* $\perp_I$: $x \perp_I y$ if $\|x+y\| = \|x-y\|$;
(iv) *a-Isosceles* $\perp_{aI}$: $x \perp_{aI} y$ if $\|x+ay\| = \|x-ay\|$ for some fixed $a \neq 0$.

Let us consider the space $(\mathbb{R}^2, |||\cdot|||)$, where $|||(x,y)||| = \max\{|x|,|y|\}$ for all $(x,y) \in \mathbb{R}^2$ and let $a = \frac{1}{3}$. Suppose that $x = (1, \frac{1}{2}), y = (\frac{1}{2}, -1), z = (3,1)$ and $w = (0,3)$. One can observe that $x \perp_I y$, but not $x \perp_{aI} y$. Further, it is easy to check that $z \perp_{aI} w$, but not $z \perp_I w$. Thus neither $\perp_I \subseteq \perp_{aI}$ nor $\perp_{aI} \subseteq \perp_I$ holds in general.

Let $\varepsilon \in [0,1)$ and $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ be a real-valued inner product space. Two vectors x and y are said to be approximately orthogonal if $|\langle x|y\rangle| \leq \varepsilon\|x\|\,\|y\|$

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and we write $x \perp^{\varepsilon} y$. The condition $x \perp^{\varepsilon} y$ is equivalent to any one of the following:

(i) *left approximate Birkhoff–James orthogonality* ${}^{\varepsilon}\!\perp_J$:

$$\|x + ty\| \geq \sqrt{1-\varepsilon^2}\|x\| \quad \text{for all} \quad t \in \mathbb{R} \text{ (see [3])}.$$

(ii) *right approximate Birkhoff–James orthogonality* $\perp_J^{\varepsilon}$:

$$\|x + ty\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\,\|ty\| \quad \text{for all} \quad t \in \mathbb{R} \text{ (see [3])}.$$

(iii) *left approximate Roberts orthogonality* ${}^{\varepsilon}\!\perp_R$:

$$\Big|\|x+ty\| - \|x-ty\|\Big| \leq \frac{1-\sqrt{1-\varepsilon^2}}{\varepsilon}(\|x+ty\| + \|x-ty\|)$$
$$\text{for all} \quad t \in \mathbb{R} \text{ (see [13])}.$$

(iv) *right approximate Roberts orthogonality* $\perp_R^{\varepsilon}$:

$$\Big|\|x+ty\|^2 - \|x-ty\|^2\Big| \leq 4\varepsilon\|x\|\,\|ty\| \quad \text{for all} \quad t \in \mathbb{R} \text{ (see [13])}.$$

Motivated by approximate Birkhoff orthogonality sets as stated in [7], the definition of approximate Roberts orthogonality sets arises as follows.

Definition 1.1. Let $(\mathcal{X}, \|\cdot\|)$ be a real normed space and $x, y \in \mathcal{X}$. For any $\varepsilon \in [0, 1)$, the approximate Roberts orthogonality set of x with respect to y is defined as

$${}^{\varepsilon}F_{\|\cdot\|}(x, y) = \{s \in \mathbb{R} \colon y\, {}^{\varepsilon}\!\perp_R (x - sy)\}.$$

In the next section, we investigate the geometric properties of approximate Roberts orthogonality sets.

Inspired by approximate Birkhoff–James orthogonality and approximate Roberts orthogonality, we propose two definitions of approximate a-isosceles-orthogonality for some fixed $a \neq 0$.

Let $\varepsilon \in [0, 1)$ and $x, y \in \mathcal{X}$, let us put $x\, {}^{\varepsilon}\!\perp_{aI}\, y$ if

$$\Big|\|x+ay\| - \|x-ay\|\Big| \leq \varepsilon(\|x+ay\| + \|x-ay\|)$$

or equivalently,

$$\frac{1-\varepsilon}{1+\varepsilon}\|x-ay\| \leq \|x+ay\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|x-ay\|. \tag{1.1}$$

We also define $x \perp_{aI}^{\varepsilon} y$ if

$$\Big|\|x+ay\|^2 - \|x-ay\|^2\Big| \leq 4\varepsilon\|x\|\|ay\|. \tag{1.2}$$

For $\delta, \varepsilon \in [0, 1)$, a mapping $T \colon \mathcal{H} \to \mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are inner product spaces, is said to be approximately orthogonality preserving, or (δ, ε)-orthogonality preserving, if

$$x \perp^{\delta} y \Longrightarrow Tx \perp^{\varepsilon} Ty \quad (x, y \in \mathcal{H}).$$

It is known that approximate orthogonality preserving mappings may be nonlinear and discontinuous, but under the additional assumption of linearity, a mapping T is $(0,0)$-orthogonality preserving if and only if it is a scalar multiple of an isometry (see [4,12]). The same result is later obtained in [14] by using a different approach.

In the case when $\delta = 0$, Chmieliński and Turnšek [4,11] verified the properties of such a class of mappings. Kong and Cao [9] studied the stability of approximate orthogonality preserving mappings and some orthogonality equations as well.

Now, suppose that $\mathcal{X}$ and $\mathcal{Y}$ are real normed spaces and let $\delta, \varepsilon \in [0,1)$. We say that a linear mapping $T\colon \mathcal{X} \to \mathcal{Y}$ approximately preserves (a,b)-isosceles-orthogonality, or is (δ,ε)-(a,b)-isosceles-orthogonality preserving, if

$$x \,{}^{\delta}\!\perp_{aI}\, y \Longrightarrow Tx \,{}^{\varepsilon}\!\perp_{bI}\, Ty \quad (x, y \in \mathcal{X}).$$

Chmieliński and Wójcik [5] studied some properties of mappings that are $(0,\varepsilon)$-$(1,1)$-isosceles-orthogonality preserving. Recently the authors of the present paper [13] studied approximate Roberts orthogonality preserving mappings.

In the last section, we consider the class of (δ,ε)-(a,b)-isosceles-orthogonality preserving mappings.

2. Approximate Roberts orthogonality sets

We recall that in a real normed space $(\mathcal{X}, \|\cdot\|)$ and for $\varepsilon \in [0,1)$, we say x, y are approximately Roberts orthogonal, in short $x \,{}^{\varepsilon}\!\perp_R\, y$, if

$$\frac{1-\varepsilon}{1+\varepsilon}\|x - ty\| \le \|x + ty\| \le \frac{1+\varepsilon}{1-\varepsilon}\|x - ty\| \quad (t \in \mathbb{R}).$$

We need the following lemma.

Lemma 2.1. *Let $(\mathcal{X}, \|\cdot\|)$ be a real normed space and $x, y \in \mathcal{X}$. Then*

$$\begin{aligned} &{}^{\varepsilon}F_{\|\cdot\|}(x,y) \\ &\quad = \left\{ s \in \mathbb{R}\colon \frac{1-\varepsilon}{1+\varepsilon}\|x - (t+2s)y\| \le \|x + ty\| \le \frac{1+\varepsilon}{1-\varepsilon}\|x - (t+2s)y\|,\, t \in \mathbb{R} \right\} \end{aligned}$$

for any $\varepsilon \in [0,1)$.

Proof. Let $\varepsilon \in [0,1)$. By the definition of the approximate Roberts orthogonality set, a number $s \in \mathbb{R}$ belongs to ${}^{\varepsilon}F_{\|\cdot\|}(x;y)$ if and only if $y \,{}^{\varepsilon}\!\perp_R\, (x - sy)$, or equivalently, if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}\|y - t(x - sy)\| \le \|y + t(x - sy)\| \le \frac{1+\varepsilon}{1-\varepsilon}\|y - t(x - sy)\| \quad (t \in \mathbb{R}).$$

This holds if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}\left\|y-\frac{1}{t}(x-sy)\right\| \le \left\|y+\frac{1}{t}(x-sy)\right\| \\ \le \frac{1+\varepsilon}{1-\varepsilon}\left\|y-\frac{1}{t}(x-sy)\right\| \quad (t\in\mathbb{R}\smallsetminus\{0\}),$$

or equivalently, if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}\|x-(t+s)y\| \le \|x+(t-s)y\| \le \frac{1+\varepsilon}{1-\varepsilon}\|x-(t+s)y\| \quad (t\in\mathbb{R}).$$

The above inequality is valid if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}\|x-(t+2s)y\| \le \|x+ty\| \le \frac{1+\varepsilon}{1-\varepsilon}\|x-(t+2s)y\| \quad (t\in\mathbb{R}).$$

□

In the following we state some properties of approximate Roberts orthogonality sets.

Theorem 2.2. *Suppose that* $(\mathcal{X},\|\cdot\|)$ *is a real normed space and* $x,y\in\mathcal{X}$*. Then*

$${}^{\varepsilon}F_{\|\cdot\|}(rx+py;y) = r\,{}^{\varepsilon}F_{\|\cdot\|}(x,y)+p$$

for any $\varepsilon\in[0,1)$ *and any* $r,p\in\mathbb{R}$*.*

Proof. If $r=0$, then by Lemma 2.1 we deduce that $s\in{}^{\varepsilon}F_{\|\cdot\|}(py;y)$ if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}\|py-(t+2s)y\| \le \|py+ty\| \le \frac{1+\varepsilon}{1-\varepsilon}\|py-(t+2s)y\| \quad (t\in\mathbb{R}),$$

or equivalently, if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}|p-(t+2s)| \le |p+t| \le \frac{1+\varepsilon}{1-\varepsilon}|p-(t+2s)| \quad (t\in\mathbb{R}).$$

On the other hand, it follows from

$$\frac{1-\varepsilon}{1+\varepsilon}|p-(t+2p)| = \frac{1-\varepsilon}{1+\varepsilon}|p+t| \le |p+t| \\ \le \frac{1+\varepsilon}{1-\varepsilon}|p+t| = \frac{1+\varepsilon}{1-\varepsilon}|p-(t+2p)|$$

that $p\in{}^{\varepsilon}F_{\|\cdot\|}(py;y)$. Now if $s\in{}^{\varepsilon}F_{\|\cdot\|}(py;y)$, then for $t=-p$ we get

$$\frac{1-\varepsilon}{1+\varepsilon}|p-(-p+2s)| \le |p+(-p)| = 0.$$

Hence $s=p$ and so ${}^{\varepsilon}F_{\|\cdot\|}(py;y)=\{p\}$. Thus

$${}^{\varepsilon}F_{\|\cdot\|}(0x+py;y) = {}^{\varepsilon}F_{\|\cdot\|}(py;y) = \{p\} = 0\,{}^{\varepsilon}F_{\|\cdot\|}(x,y)+p.$$

So, we may assume that $r \neq 0$. By Lemma 2.1, we have $s \in {}^{\varepsilon}F_{\|\cdot\|}(rx;y)$ if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}\|rx-(t+2s)y\| \leq \|rx+ty\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|rx-(t+2s)y\| \quad (t \in \mathbb{R}),$$

or equivalently, if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}\left\|x-\left(\frac{t}{r}+2\frac{s}{r}\right)y\right\| \leq \left\|x+\frac{t}{r}y\right\| \leq \frac{1+\varepsilon}{1-\varepsilon}\left\|x-\left(\frac{t}{r}+2\frac{s}{r}\right)y\right\| \quad (t \in \mathbb{R}).$$

This occurs if and only if

$$\frac{1-\varepsilon}{1+\varepsilon}\left\|x-\left(t+2\frac{s}{r}\right)y\right\| \leq \|x+ty\| \leq \frac{1+\varepsilon}{1-\varepsilon}\left\|x-\left(t+2\frac{s}{r}\right)y\right\| \quad (t \in \mathbb{R}),$$

or equivalently, $s \in r\,{}^{\varepsilon}F_{\|\cdot\|}(x,y)$. Therefore ${}^{\varepsilon}F_{\|\cdot\|}(rx;y) = r\,{}^{\varepsilon}F_{\|\cdot\|}(x,y)$.

Thus $s \in {}^{\varepsilon}F_{\|\cdot\|}(rx+py;y)$ if and only if

$$\begin{aligned}\frac{1-\varepsilon}{1+\varepsilon}\|rx+py-(t+2s)y\| &\leq \|rx+py+ty\| \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\|rx+py-(t+2s)y\| \quad (t \in \mathbb{R}),\end{aligned}$$

or equivalently, if and only if

$$\begin{aligned}\frac{1-\varepsilon}{1+\varepsilon}\|rx-(t-p+2s)y\| &\leq \|rx+(p+t)y\| \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\|rx-(t-p+2s)y\| \quad (t \in \mathbb{R}).\end{aligned}$$

The above double inequality holds if and only if

$$\begin{aligned}\frac{1-\varepsilon}{1+\varepsilon}\|rx-(t+2(s-p))\,y\| &\leq \|rx+ty\| \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\|rx-(t+2(s-p))\,y\| \quad (t \in \mathbb{R}),\end{aligned}$$

or equivalently, if and only if $s \in {}^{\varepsilon}F_{\|\cdot\|}(rx;y)+p$, namely, $s \in r\,{}^{\varepsilon}F_{\|\cdot\|}(x,y)+p$. □

By Lemma 2.1 and Theorem 2.2 we obtain the following result.

Corollary 2.3. *Suppose that $(\mathcal{X}, \|\cdot\|)$ is a real normed space and $x, y \in \mathcal{X}$. Let $\varepsilon \in [0,1)$. Then*

$${}^{\varepsilon}F_{\|\cdot\|}(rx;py) = \frac{r}{p}\,{}^{\varepsilon}F_{\|\cdot\|}(x,y)$$

for any $r \in \mathbb{R}$ and any nonzero $p \in \mathbb{R}$.

Theorem 2.4. *Suppose that $(\mathcal{X}, \|\cdot\|)$ is a real normed space such that $\|\cdot\|$ is induced by an inner product $\langle\cdot|\cdot\rangle$. Let $x, y \in \mathcal{X}\smallsetminus\{0\}$. Then*

$$^{\varepsilon}F_{\|\cdot\|}(x,y) = \left(\frac{\langle x|y\rangle}{\|y\|^2} - r, \frac{\langle x|y\rangle}{\|y\|^2} + r\right)$$

for any $\varepsilon \in [0,1)$, where $r = \frac{2\varepsilon}{(1-\varepsilon^2)\|y\|^2}\sqrt{\|x\|^2\,\|y\|^2 - \langle x|y\rangle^2}$.

Proof. By the definition of the approximate Roberts orthogonality set, a number $s \in \mathbb{R}$ belongs to $^{\varepsilon}F_{\|\cdot\|}(x,y)$ if and only if $y\,^{\varepsilon}\!\perp_R (x - sy)$.

Furthermore, by [13, Theorem 2.7], $y\,^{\varepsilon}\!\perp_R (x - sy)$ if and only if

$$|\langle y|x - sy\rangle| \leq \frac{2\varepsilon}{1+\varepsilon^2}\|y\|\|x - sy\|,$$

or equivalently, if and only if

$$\langle x|y\rangle^2 + s^2\|y\|^4 - 2s\langle x|y\rangle\|y\|^2 \leq \frac{4\varepsilon^2}{(1+\varepsilon^2)^2}\|y\|^2(\|x\|^2 + s^2\|y\|^2 - 2s\langle x|y\rangle).$$

This holds if and only if

$$s^2\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\right)^2 - 2s\frac{\langle x|y\rangle}{\|y\|^2}\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\right)^2 \leq \frac{4\varepsilon^2\|x\|^2}{(1+\varepsilon^2)^2\|y\|^2} - \frac{\langle x|y\rangle^2}{\|y\|^4},$$

or equivalently, if and only if

$$s^2 - 2s\frac{\langle x|y\rangle}{\|y\|^2} \leq \left(\frac{1+\varepsilon^2}{1-\varepsilon^2}\right)^2\left(\frac{4\varepsilon^2\|x\|^2}{(1+\varepsilon^2)^2\|y\|^2} - \frac{\langle x|y\rangle^2}{\|y\|^4}\right).$$

This, in turn, is valid if and only if

$$\left(s - \frac{\langle x|y\rangle}{\|y\|^2}\right)^2 \leq \left(\frac{1+\varepsilon^2}{1-\varepsilon^2}\right)^2\left(\frac{4\varepsilon^2\|x\|^2}{(1+\varepsilon^2)^2\|y\|^2} - \frac{\langle x|y\rangle^2}{\|y\|^4} + \left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\right)^2\frac{\langle x|y\rangle^2}{\|y\|^4}\right),$$

or equivalently, if and only if

$$\left(s - \frac{\langle x|y\rangle}{\|y\|^2}\right)^2 \leq \frac{4\varepsilon^2}{(1-\varepsilon^2)^2}\left(\frac{\|x\|^2}{\|y\|^2} - \frac{\langle x|y\rangle^2}{\|y\|^4}\right).$$

The later inequality holds if and only if

$$\left|s - \frac{\langle x|y\rangle}{\|y\|^2}\right| \leq \frac{2\varepsilon}{(1-\varepsilon^2)\|y\|^2}\sqrt{\|x\|^2\,\|y\|^2 - \langle x|y\rangle^2}.$$

□

Lemma 2.5. [6, Proposition 3.2] *Let $\mathcal{X}$ be a real normed space and let $f, g, h\colon \mathcal{X} \longrightarrow \mathbb{R}$ be mappings such that*

$$2h(z) \leq f(z+w) + f(z-w) - 2f(w) \leq 2g(z) \quad (z, w \in \mathcal{X}).$$

Then, there exists a quadratic mapping $Q\colon \mathcal{X} \longrightarrow \mathbb{R}$*, i.e., a mapping satisfying*

$$Q(z+w)+Q(z-w)=2Q(z)+2Q(w) \quad (z,w\in\mathcal{X})$$

such that

$$h(z)\le Q(z)\le g(z) \quad (z\in\mathcal{X}).$$

Recall that a normed space $(\mathcal{X},\|\cdot\|)$ satisfies the δ-parallelogram law for some $\delta\in[0,1)$, if the inequalities

$$2(1-\delta)\|z\|^2\le\|z+w\|^2+\|z-w\|^2-2\|w\|^2\le 2(1+\delta)\|z\|^2 \tag{2.1}$$

hold for all $z,w\in\mathcal{X}$ (see [2]).

In the next result we use some ideas of [6].

Theorem 2.6. *Suppose that* $\varepsilon\in[0,1)$ *and* $(\mathcal{X},\|\cdot\|)$ *is a real normed space such that* $\|\cdot\|$ *satisfies the* δ*-parallelogram law for some* $\delta\in[0,1)$*. Then there exists a norm* $|||\cdot|||$ *in* $\mathcal{X}$ *coming from an inner product* $\langle\cdot|\cdot\rangle$ *such that*

$$\left(\frac{\langle x|y\rangle}{|||y|||^2}-r_1,\frac{\langle x|y\rangle}{|||y|||^2}+r_1\right)\subseteq {}^{\varepsilon}F_{\|\cdot\|}(x,y)\subseteq\left(\frac{\langle x|y\rangle}{|||y|||^2}-r_2,\frac{\langle x|y\rangle}{|||y|||^2}+r_2\right),$$

for all $x,y\in\mathcal{X}\smallsetminus\{0\}$*, where* $r_k=\frac{2\varepsilon+(-1)^k(1+\varepsilon^2)}{\sqrt{1-\delta^2(1-\varepsilon^2)|||y|||^2}}\sqrt{|||x|||^2\,|||y|||^2-\langle x|y\rangle^2}$.

Proof. By the δ-parallelogram law (2.1) we get

$$2(1-\delta)\|z\|^2\le\|z+w\|^2+\|z-w\|^2-2\|w\|^2\le 2(1+\delta)\|z\|^2 \quad (z,w\in\mathcal{X}).$$

Let $h(z)=(1-\delta)\|z\|^2$, $f(z)=\|z\|^2$ and $g(z)=(1+\delta)\|z\|^2$ for $z\in\mathcal{X}$. Then

$$2h(z)\le f(z+w)+f(z-w)-2f(w)\le g(z) \quad (z,w\in\mathcal{X}).$$

By Lemma 2.5, there exists a real quadratic mapping Q satisfying

$$(1-\delta)\|z\|^2\le Q(z)\le(1+\delta)\|z\|^2 \quad (z\in\mathcal{X}). \tag{2.2}$$

It follows from (2.2) that $Q(z)>0$ for $z\in X\smallsetminus\{0\}$ and $Q(0)=0$. Let us define

$$\langle z|w\rangle:=\frac{1}{4}[Q(z+w)-Q(z-w)]$$

for all $z,w\in\mathcal{X}$. It follows from (2.2) that $\langle\cdot|\cdot\rangle$ is locally bounded with respect to each variable. Since Q is quadratic, $\langle\cdot|\cdot\rangle$ is symmetric and biadditive. Thus $\langle\cdot|\cdot\rangle$ is linear in each variable. Hence $\langle\cdot|\cdot\rangle$ is an inner product in $\mathcal{X}$ generating the norm $|||z|||:=\sqrt{Q(z)}$, $z\in\mathcal{X}$. Now, we can write (2.2) as

$$\sqrt{1-\delta}\|z\|\le|||z|||\le\sqrt{1+\delta}\|z\| \quad (z\in\mathcal{X})$$

i.e., the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

Easy computations show that ${}^{\varepsilon}F_{\|\cdot\|}(x,y)\subseteq{}^{\zeta}F_{|||\cdot|||}(x,y)$ where

$$\zeta=\frac{\sqrt{1+\delta}-\sqrt{1-\delta}+\varepsilon(\sqrt{1+\delta}+\sqrt{1-\delta})}{\sqrt{1+\delta}+\sqrt{1-\delta}+\varepsilon(\sqrt{1+\delta}-\sqrt{1-\delta})}.$$

So, by Theorem 2.4, we get

$$^{\varepsilon}F_{\|\cdot\|}(x,y) \subseteq \left(\frac{\langle x|y\rangle}{|||y|||^2} - r_2, \frac{\langle x|y\rangle}{|||y|||^2} - r_2\right),$$

where

$$r_2 = \frac{2\zeta}{(1-\zeta^2)|||y|||^2}\sqrt{|||x|||^2\,|||y|||^2 - \langle x|y\rangle^2}.$$

A straightforward computation shows that $\frac{2\delta}{1-\delta^2} = \frac{2\varepsilon+(1+\varepsilon^2)}{\sqrt{1-\delta^2}(1-\varepsilon^2)}$. Hence

$$r_2 = \frac{2\varepsilon + (1+\varepsilon^2)}{\sqrt{1-\delta^2}(1-\varepsilon^2)|||y|||^2}\sqrt{|||x|||^2\,|||y|||^2 - \langle x|y\rangle^2}.$$

Also, we can write (2.2) as

$$\frac{1}{\sqrt{1+\delta}}|||x||| \le \|x\| \le \frac{1}{\sqrt{1-\delta}}|||x||| \quad (x \in \mathcal{X}).$$

Hence for $\theta = \frac{1-\sqrt{1-\delta^2}-\varepsilon\delta}{\varepsilon(1-\sqrt{1-\delta^2})-\delta}$, we get $^{\theta}F_{|||\cdot|||}(x,y) \subseteq {}^{\mu}F_{\|\cdot\|}(x,y)$, where

$$\mu = \frac{\frac{1}{\sqrt{1-\delta}} - \frac{1}{\sqrt{1+\delta}} + \theta\left(\frac{1}{\sqrt{1-\delta}} + \frac{1}{\sqrt{1+\delta}}\right)}{\frac{1}{\sqrt{1-\delta}} + \frac{1}{\sqrt{1+\delta}} + \theta\left(\frac{1}{\sqrt{1-\delta}} - \frac{1}{\sqrt{1+\delta}}\right)}.$$

By Theorem 2.4 we deduce that

$$\left(\frac{\langle x|y\rangle}{|||y|||^2} - r_1, \frac{\langle x|y\rangle}{|||y|||^2} + r_1\right) \subseteq {}^{\mu}F_{\|\cdot\|}(x,y),$$

where

$$r_1 = \frac{2\theta}{(1-\theta^2)|||y|||^2}\sqrt{|||x|||^2\,|||y|||^2 - \langle x|y\rangle^2}.$$

Simple computations show that $\mu = \varepsilon$ and $\frac{2\theta}{1-\theta^2} = \frac{2\varepsilon-(1+\varepsilon^2)}{\sqrt{1-\delta^2}(1-\varepsilon^2)}$. Thus

$$r_1 = \frac{2\varepsilon - (1+\varepsilon^2)}{\sqrt{1-\delta^2}(1-\varepsilon^2)|||y|||^2}\sqrt{|||x|||^2\,|||y|||^2 - \langle x|y\rangle^2}.$$

□

Corollary 2.7. *Suppose that* $(\mathcal{X}, \|\cdot\|)$ *is a real normed space such that* $\|\cdot\|$ *is induced by an inner product* $\langle\cdot|\cdot\rangle$. *If* $\varepsilon \in [0,1)$, $r, p \in \mathbb{R}$ *and* $q \in \mathbb{R}\setminus\{0\}$, *then*

$$^{\varepsilon}F_{\|\cdot\|}(rx+py;qz) \subseteq \frac{r}{q}{}^{\varepsilon}F_{\|\cdot\|}(x;z) + \frac{p}{q}{}^{\varepsilon}F_{\|\cdot\|}(y;z) \quad (x,y,z \in \mathcal{X}).$$

Proof. We may assume that $z \neq 0$. By Theorem 2.4 and simple computations, we deduce that

$$\begin{aligned}
{}^{\varepsilon}F_{\|\cdot\|}(x+y;z) &= \Big\{ s \in \mathbb{R} \colon \Big| s - \frac{\langle x+y|z\rangle}{\|z\|^2} \Big| \\
&\leq \frac{2\varepsilon}{(1-\varepsilon^2)\|z\|^2} \sqrt{\|x+y\|^2\,\|z\|^2 - \langle x+y|z\rangle^2} \Big\} \\
&\subseteq \Big\{ s \in \mathbb{R} \colon \Big| s - \frac{\langle x|z\rangle}{\|z\|^2} \Big| \leq \frac{2\varepsilon}{(1-\varepsilon^2)\|z\|^2} \sqrt{\|x\|^2\,\|z\|^2 - \langle x|z\rangle^2} \Big\} \\
&\quad + \Big\{ s \in \mathbb{R} \colon \Big| s - \frac{\langle y|z\rangle}{\|z\|^2} \Big| \leq \frac{2\varepsilon}{(1-\varepsilon^2)\|z\|^2} \sqrt{\|y\|^2\,\|z\|^2 - \langle y|z\rangle^2} \Big\} \\
&= {}^{\varepsilon}F_{\|\cdot\|}(x;z) + {}^{\varepsilon}F_{\|\cdot\|}(y;z) \quad (x,y,z \in \mathcal{X}).
\end{aligned}$$

So, the subadditivity property holds. Corollary 2.3 yields that

$${}^{\varepsilon}F_{\|\cdot\|}(rx+py;qz) \subseteq {}^{\varepsilon}F_{\|\cdot\|}(rx;qz) + {}^{\varepsilon}F_{\|\cdot\|}(py;qz) = \frac{r}{q}\,{}^{\varepsilon}F_{\|\cdot\|}(x;z) + \frac{p}{q}\,{}^{\varepsilon}F_{\|\cdot\|}(y;z).$$

□

3. Approximate (a, b)-orthogonality preserving mappings

We start this section with some properties of approximate a-isosceles-orthogonality. Let $\mathcal{H}$ be an inner product space and $x, y \in \mathcal{H}$. It is easy to check that

$$x\, {}^{\varepsilon}\!\perp_{aI}\, y \Longleftrightarrow |\langle x, y\rangle| \leq \frac{\varepsilon}{|a|(1+\varepsilon^2)}[\|x\|^2 + \|ay\|^2] \tag{3.1}$$

and

$$x \perp_{aI}^{\varepsilon} y \Longleftrightarrow |\langle x, y\rangle| \leq \varepsilon\|x\|\|y\|. \tag{3.2}$$

Furthermore, in an arbitrary normed space simple computations show that $\perp_{aI}^{\varepsilon} \subseteq {}^{\varepsilon}\!\perp_{aI}$, but the converse is not true even in inner product spaces.

Example 3.1. Let $\mathcal{H}$ be an inner product space and $\varepsilon \in [0,1)$. Since $\lim_{t\to 0} \frac{|t|}{1+t^2} = 0$, there exists $t_0 \in \mathbb{R}$ such that $\frac{|t_0|}{1+{t_0}^2} \leq \frac{\varepsilon}{|a|(1+\varepsilon^2)}$. Now for $x \in \mathcal{H}\setminus\{0\}$ we have

$$\begin{aligned}
\varepsilon\|x\|\,\|t_0x\| < \|x\|^2\,|t_0| = |\langle x|t_0x\rangle| &\leq \frac{\varepsilon(1+{t_0}^2)}{|a|(1+\varepsilon^2)}\|x\|^2 \\
&= \frac{\varepsilon}{|a|(1+\varepsilon^2)}(\|x\|^2 + \|t_0x\|^2).
\end{aligned}$$

Thus, by (3.1), $x\,{}^{\varepsilon}\!\perp_{aI}\, t_0x$ whereas, by (3.2), $x \perp_{aI}^{\varepsilon} t_0x$ does not hold.

Employing some strategies of [5], we establish now the main result of this section.

Theorem 3.2. *Let* $0 < b \leq a$ *and* $\varepsilon, \delta \in [0, \frac{b}{a})$. *Let* $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ *be a nonzero linear* (δ, ε)*-*(a, b)*-isosceles-orthogonality preserving mapping. Then* $\delta \leq \frac{a-b+\varepsilon(a+b)}{a+b-\varepsilon(a-b)}$ *and* T *is injective, continuous and satisfies*

$$\frac{(1+\delta)(b-\varepsilon a)}{(1-\delta)(a+\varepsilon b)}\gamma\|x\| \leq \|Tx\| \leq \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}\gamma\|x\|$$

for all $x \in \mathcal{X}$ *and for all* $\gamma \in [[T], \|T\|]$.

Proof. Let $x\, {}^{\delta}\!\perp_{aI}\, y \implies Tx\, {}^{\varepsilon}\!\perp_{bI}\, Ty$ for all $x, y \in \mathcal{X}$. Thus

$$\begin{gathered}\frac{1-\delta}{1+\delta}\|x-ay\| \leq \|x+ay\| \leq \frac{1+\delta}{1-\delta}\|x-ay\| \\ \implies \\ \frac{1-\varepsilon}{1+\varepsilon}\|Tx-bTy\| \leq \|Tx+bTy\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|Tx-bTy\|, \end{gathered} \tag{3.3}$$

for all $x, y \in \mathcal{X}$.

By replacing x and y by $\frac{x+y}{2}$ and $\frac{x-y}{2a}$, respectively, (3.3) can be written in the following form:

$$\begin{aligned}&\frac{1-\delta}{1+\delta}\|y\| \leq \|x\| \leq \frac{1+\delta}{1-\delta}\|y\| \\ &\qquad\qquad \implies \\ &\frac{1-\varepsilon}{1+\varepsilon}\left\|\frac{a-b}{2}Tx + \frac{a+b}{2}Ty\right\| \leq \left\|\frac{a+b}{2}Tx + \frac{a-b}{2}Ty\right\| \\ &\quad \leq \frac{1+\varepsilon}{1-\varepsilon}\left\|\frac{a-b}{2}Tx + \frac{a+b}{2}Ty\right\|, \end{aligned} \tag{3.4}$$

for all $x, y \in \mathcal{X}$.

Fix $y \in \mathcal{X}$ with $\|y\| = 1$. For every $x \in \mathcal{X}$ with $\|x\| = 1$, since

$$\frac{1-\delta}{1+\delta}\|y\| \leq \left\|\frac{1+\delta}{1-\delta}\,x\right\| \leq \frac{1+\delta}{1-\delta}\|y\|,$$

from (3.4) we get

$$\begin{aligned}\frac{1-\varepsilon}{1+\varepsilon}\left\|\frac{a-b}{2}T\left(\frac{1+\delta}{1-\delta}\,x\right) + \frac{a+b}{2}Ty\right\| &\leq \left\|\frac{a+b}{2}T\left(\frac{1+\delta}{1-\delta}\,x\right) + \frac{a-b}{2}Ty\right\| \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\left\|\frac{a-b}{2}T\left(\frac{1+\delta}{1-\delta}\,x\right) + \frac{a+b}{2}Ty\right\|. \end{aligned} \tag{3.5}$$

It follows from (3.5) that

$$\begin{aligned}\frac{(a+b)(1+\delta)}{2(1-\delta)}\|Tx\| &\le \left\|\frac{a+b}{2}T\left(\frac{1+\delta}{1-\delta}\,x\right)+\frac{a-b}{2}Ty\right\|+\frac{a-b}{2}\|Ty\|\\ &\le \frac{1+\varepsilon}{1-\varepsilon}\left\|\frac{a-b}{2}T\left(\frac{1+\delta}{1-\delta}\,x\right)+\frac{a+b}{2}Ty\right\|+\frac{a-b}{2}\|Ty\|\\ &\le \frac{(1+\varepsilon)(a-b)(1+\delta)}{2(1-\varepsilon)(1-\delta)}\|Tx\|+\frac{(1+\varepsilon)(a+b)}{2(1-\varepsilon)}\|Ty\|\\ &\quad +\frac{a-b}{2}\|Ty\|.\end{aligned}$$

Thus

$$\|Tx\| \le \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}\|Ty\| \quad (\|x\|=\|y\|=1), \tag{3.6}$$

which implies $1 \le \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}$ (or equivalently, $\delta \le \frac{a-b+\varepsilon(a+b)}{a+b-\varepsilon(a-b)}$), the injectivity and continuity of T. By (3.6) and passing to the supremum over $\|x\|=1$, we get

$$\|Tx\| \le \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}\|Ty\| \quad (\|y\|=1).$$

By the above inequality and passing to the infimum over $\|y\|=1$, we obtain

$$\|T\| \le \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}[T]. \tag{3.7}$$

Now, let $\gamma \in [\,[T], \|T\|\,]$ and $x \in \mathcal{X}$. Therefore we have from (3.7) that

$$\begin{aligned}\frac{(1+\delta)(b-\varepsilon a)}{(1-\delta)(a+\varepsilon b)}\gamma\|x\| &\le \frac{(1+\delta)(b-\varepsilon a)}{(1-\delta)(a+\varepsilon b)}\|T\|\,\|x\|\\ &\le \frac{(1+\delta)(b-\varepsilon a)}{(1-\delta)(a+\varepsilon b)}\times\frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}[T]\,\|x\|\\ &\le \|T\|\,\|x\| \le \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}[T]\,\|x\|\\ &\le \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}\gamma\|x\|.\end{aligned}$$

□

As a consequence of Theorem 3.2, we have the following result.

Corollary 3.3. *Let $0 < b \le a$ and $\varepsilon, \delta \in [0, \frac{b}{a})$. Let $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a linear (δ,ε)-(a,b)-isosceles-orthogonality preserving mapping with $0 \le \frac{a-b+\varepsilon(a+b)}{a+b-\varepsilon(a-b)} < \delta$. Then $T=0$.*

Corollary 3.4. *Let* $0 < b \leq a$ *and* $\varepsilon, \delta \in [0, \frac{b}{a})$. *Let* $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ *be a nonzero linear* (δ, ε)-(a, b)-*isosceles-orthogonality preserving mapping. If a linear mapping* $S : \mathcal{X} \to \mathcal{Y}$ *satisfies* $\|S - T\| \leq \theta\|T\|$, *then* $\|S\| \leq \eta[S]$, *where* $\eta = \frac{(1-\delta)^2(a+\varepsilon b)^2+\theta(1-\delta^2)(a+\varepsilon b)(b-\varepsilon a)}{(1+\delta)^2(b-\varepsilon a)^2-\theta(1-\delta^2)(a+\varepsilon b)(b-\varepsilon a)}$.

Proof. Let $\|S - T\| \leq \theta\|T\|$. For any $z \in \mathcal{X}$ we have

$$\Big| \|Sz\| - \|Tz\| \Big| \leq \|Sz - Tz\| \leq \|S - T\| \, \|z\| \leq \theta\|T\| \, \|z\|,$$

whence

$$-\theta\|T\| \, \|z\| \leq \|Sz\| - \|Tz\| \leq \theta\|T\| \, \|z\|. \tag{3.8}$$

Since T is a nonzero (δ, ε)-(a, b)-isosceles-orthogonality preserving mapping, by Theorem 3.2, for $\gamma = \|T\|$, we have

$$\frac{(1+\delta)(b-\varepsilon a)}{(1-\delta)(a+\varepsilon b)}\|T\| \, \|z\| \leq \|Tz\| \leq \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}\|T\| \, \|z\|. \tag{3.9}$$

Therefore by (3.8) and (3.9) we get

$$\left(\frac{(1+\delta)(b-\varepsilon a)}{(1-\delta)(a+\varepsilon b)} - \theta\right) \|T\| \, \|z\| \leq \|Sz\| \leq \left(\frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)} + \theta\right) \|T\| \, \|z\|. \tag{3.10}$$

For any $x, y \in \mathcal{X}$, by (3.10), we therefore have

$$\|Sx\| \, \|y\| \leq \left(\frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)} + \theta\right) \times \left(\frac{(1+\delta)(b-\varepsilon a)}{(1-\delta)(a+\varepsilon b)} - \theta\right)^{-1} \|Sy\| \, \|x\|.$$

Now the assertion follows passing to the supremum over $\|x\| = 1$ and passing to the infimum over $\|y\| = 1$ from the above inequality. □

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