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Tabor groups with finiteness conditions

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Abstract. We analyse Tabor groups where every element has finite order and we characterise finite Tabor groups.

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1. Introduction

In this article $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers and $\mathbb{N}^*$ denotes $\mathbb{N}\backslash\{0\}$. Additionally, $\mathbb{Z}$ is the set of the integers.

Definition. A (semi)group S is called a **Tabor (semi)group** if and only if

(T) For all $x, y \in S$ there is an element $k \in \mathbb{N}^*$ such that
$$(x \cdot y)^{2^k} = x^{2^k} \cdot y^{2^k}.$$

For elements $x \in S$ and $n \in \mathbb{N}^*$ here $x^1 := x$ and $x^{n+1} := x^n \cdot x$ is defined recursively.

The notion of Tabor (semi)groups comes from [1] since Józef Tabor [6] pointed out the usefulness of the condition (T) for stability investigations of functional equations.

In this paper we will study groups that satisfy condition (T). These groups are called **Tabor groups**. In [2] the authors showed that groups where the order of every element is a power of 2, and groups whose elements have odd order, are Tabor groups.

In general we call a group a **torsion group** if and only if each of its elements has finite order. Moreover, an element of a group is called a **2-element** if and only if its order is a power of 2.

In the following section we will investigate the Tabor condition (T) for torsion groups. It turns out that 2-elements and elements of odd order commute in a torsion group that is a Tabor group. Furthermore, the set of all 2-elements forms a subgroup in such a torsion Tabor group.

According to [2] the stability of some functional equations may require a stronger condition than (T). The authors investigate (semi)groups S fulfilling

$$(\tilde{\mathrm{T}})\ \text{For all } x, y \in S \text{ there is an element } k \in \mathbb{N}^* \text{ such that}$$
$$(x \cdot y)^{2^k} = x^{2^k} \cdot y^{2^k} \text{ and } (x \cdot y \cdot y)^{2^k} = x^{2^k} \cdot y^{2^k} \cdot y^{2^k}.$$

Evidently (semi)groups satisfying $(\tilde{\mathrm{T}})$ are Tabor (semi)groups.
In Theorem 2.5 we will show that the converse is true for a torsion group G, if the set of all elements of odd order forms a subgroup of G.

In the last section we characterise finite Tabor groups and show that they satisfy $(\tilde{\mathrm{T}})$ and are soluble.

2. Torsion Tabor groups

The next definition also repeats some notions from the introduction.

Definition. Let G be a group.

(a) For all $g \in G$ we set $\langle g \rangle := \{g^i \mid i \in \mathbb{Z}\}$ to be the subgroup of G that is generated by g, and $o(g) := |\langle g \rangle| (\leq \infty)$ is called the **order** of g.
(b) A group is called a **torsion group** if and only if each of its elements has finite order.
(c) An element of a group is called a 2**-element** if and only if its order is finite and a power of 2.
(d) A group is called a 2**-group** if and only if each of its elements is a 2-element.

Lemma 2.1. *Let G be a torsion group satisfying one of the following conditions.*

(a) *Every element is a* 2*-element.*
(b) *Every element has odd order.*

Then G satisfies $(\tilde{\mathrm{T}})$ and G is a Tabor group.

Proof. This is Theorem 3 and Remark 3 of [2] in the special case of groups. □

Lemma 2.2. *If a torsion group G is a Tabor group, then the product of any two* 2*-elements is a* 2*-element.*

Proof. Suppose for a contradiction the theorem fails. Let $a, b \in G$ be 2-elements such that $a \cdot b$ is no 2-element and such that the order of a is minimal; then $a \neq 1$. By (T) there is an element $k \in \mathbb{N}^*$ such that $a^{2^k} \cdot b^{2^k} = (a \cdot b)^{2^k}$.

We see that $o(a^{2^k}) = \max\{\frac{o(a)}{2^k}, 1\}$, since a is a 2-element. This implies that a^{2^k} is a 2-element of order smaller than $o(a)$. Thus the minimal choice of a forces $a^{2^k} \cdot b^{2^k}$ to be a 2-element. Therefore we obtain a natural number n such that $1 = (a^{2^k} \cdot b^{2^k})^{2^n} = ((a \cdot b)^{2^k})^{2^n} = (a \cdot b)^{2^{k+n}}$. This is a contradiction, as $a \cdot b$ is no 2-element. □

Corollary 2.3. *Let G be a torsion group. If G is a Tabor group, then the set $\{g \in G \mid g$ is a 2-element$\}$ forms a subgroup of G.*

Proof. The result follows from Lemma 2.2 and the subgroup criteria, as inverse elements of 2-elements are 2-elements. □

Proposition 2.4. *If a torsion group G is a Tabor group then any two elements y and b, such that y has odd order and b is a 2-element, commute.*

Proof. Let G be a torsion group and suppose that G is a Tabor group. Let further $y, b \in G$ be elements such that y has odd order and $o(b) = 2^n$ for some natural number n.

If $n = 0$, then $b = 1$ and we observe $y \cdot b = b \cdot y$.
Assume now $n \geq 1$ and that any elements x and a such that x has odd order and $o(a) = 2^m$ with $m < n$ commute.

As G is a Tabor group there is some $k \in \mathbb{N}^*$ such that $(by)^{2^k} = b^{2^k} \cdot y^{2^k}$. From $\gcd(o(y), 2^k) = 1 = \gcd(o(y), 2^n)$ we obtain that $y^{-1} \in \langle y \rangle = \langle y^{2^k} \rangle = \langle y^{2^n} \rangle$.

If we have $k \geq n$, then we see that $b^{2^k} = (b^{2^n})^{2^{k-n}} = 1^{2^{k-n}} = 1$ and hence $(by)^{2^k} = b^{2^k} \cdot y^{2^k} = y^{2^k}$. Thus we deduce $y^{-1} \in \langle y^{2^k} \rangle = \langle (by)^{2^k} \rangle \leq \langle by \rangle$ in this case.

In the other case, if $k < n$, we obtain $o(b^{2^k}) = 2^{n-k}$ and $n - k < n$. As y^{2^k} has odd order, the elements b^{2^k} and y^{2^k} commute by our assumption. Therefore we conclude for $l := n - k$ that

$$(by)^{2^n} = \left((by)^{2^k}\right)^{2^{n-k}} = \left(b^{2^k} \cdot y^{2^k}\right)^{2^l} = \left(b^{2^k}\right)^{2^l} \cdot \left(y^{2^k}\right)^{2^l} = b^{2^n} \cdot y^{2^n} = y^{2^n}.$$

In particular we see $y^{-1} \in \langle y \rangle = \langle y^{2^n} \rangle = \langle (by)^{2^n} \rangle \leq \langle by \rangle$.

In both cases we have shown that $y^{-1} \in \langle by \rangle$. From the fact that cyclic groups are abelian we deduce

$$\begin{aligned} b \cdot y = \left(y \cdot y^{-1}\right) \cdot (b \cdot y) = y \cdot \left(y^{-1} \cdot (b \cdot y)\right) &= y \cdot \left((b \cdot y) \cdot y^{-1}\right) \\ &= y \cdot \left(b \cdot \left(y \cdot y^{-1}\right)\right) = y \cdot b. \end{aligned}$$

Finally induction yields the assertion of the proposition. □

Theorem 2.5. *Let G be a torsion group and let $K := \{x \in G \mid o(x)$ is odd$\}$ be a subgroup of G. Then the following conditions are equivalent.*

(a) *G is a Tabor group.*
(b) *$G \cong K \times T$, where $T := \{g \in G \mid g$ is a 2-element$\}$ is a subgroup of G.*
(c) *G fulfils condition $(\tilde{\mathrm{T}})$.*

Proof. Let first G be a Tabor group. Then $T := \{g \in G \mid g$ is a 2-element$\}$ is a subgroup of G by Corollary 2.3. Furthermore, Proposition 2.4 yields that $G \cong K \times T$.

Let now $T := \{g \in G \mid g \text{ is a 2-element}\}$ be a subgroup of G and $G \cong K \times T$. Let further g, h be elements of G. Then we obtain elements $a, b \in T$ and $x, y \in K$ such that $g = a \cdot x$ and $h = b \cdot y$. Additionally, each of a and b commutes with both x and y, as we have $G \cong K \times T$.

Let $m = o(x) \cdot o(y) \cdot o(xy) \cdot o(xy^2)$. Then m is an odd number. Analogously to the proof of Theorem 3 in [2] we deduce from Fermat's Theorem (c.f. Theorem 6–1 of [4]) that m divides $2^{\varphi(m)} - 1$, where φ denotes Euler's function. We set $k := \varphi(m)$. Then all of $o(x)$, $o(y)$, $o(xy)$ and $o(xy^2)$ divide $2^k - 1$. Thus we have for all $z \in \{x, y, xy, xy^2\}$ that: $z^{2^k} = z^{2^k-1} \cdot z = 1 \cdot z = z$.

Since T is a 2-group, we see that $o(a) \cdot o(b) \cdot o(ab) \cdot o(ab^2) \cdot 2 =: 2^n$ is a power of 2. Hence we have $a^{2^n} = b^{2^n} = (ab)^{2^n} = (ab^2)^{2^n} = 1$.

Altogether, we see for all $c \in \{a, b, ab, ab^2\}$ that $c^{2^{kn}} = (c^{2^n})^{2^{(k-1)n}} = 1$ and for all $z \in \{x, y, xy, xy^2\}$ we have:

$$z^{2^{kn}} = \left(z^{2^k}\right)^{2^{k(n-1)}} = z^{2^{k(n-1)}} = \left(z^{2^k}\right)^{2^{k(n-2)}} = \cdots = z^{2^{k\cdot 1}} = z.$$

We finally conclude that

$$\begin{aligned}(g \cdot h)^{2^{kn}} &= (ax \cdot by)^{2^{kn}} &&= (ab \cdot xy)^{2^{kn}}\\ &= (ab)^{2^{kn}} \cdot (xy)^{2^{kn}} &&= 1 \cdot xy\\ &= 1 \cdot x \cdot 1 \cdot y &&= a^{2^{kn}} \cdot x^{2^{kn}} \cdot b^{2^{kn}} \cdot y^{2^{kn}}\\ &= (ax)^{2^{kn}} \cdot (by)^{2^{kn}} &&= g^{2^{kn}} \cdot h^{2^{kn}}\end{aligned}$$

and

$$\begin{aligned}\left(gh^2\right)^{2^{kn}} &= (ax \cdot by \cdot by)^{2^{kn}} &&= \left(ab^2 \cdot xy^2\right)^{2^{kn}}\\ &= \left(ab^2\right)^{2^{kn}} \cdot \left(xy^2\right)^{2^{kn}} &&= 1 \cdot xy^2\\ &= 1 \cdot x \cdot 1 \cdot y \cdot 1 \cdot y &&= a^{2^{kn}} \cdot x^{2^{kn}} \cdot b^{2^{kn}} \cdot y^{2^{kn}} \cdot b^{2^{kn}} \cdot y^{2^{kn}}\\ &= (ax)^{2^{kn}} \cdot (by)^{2^{kn}} \cdot (by)^{2^{kn}} &&= g^{2^{kn}} \cdot h^{2^{kn}} \cdot h^{2^{kn}}.\end{aligned}$$

Thus ($\tilde{\mathrm{T}}$) is satisfied.

Finally, if G fulfills ($\tilde{\mathrm{T}}$), then obviously G is a Tabor group. □

3. The finite case

Lemma 3.1 (Cauchy's Theorem). *Let G be a finite group and let p be a prime dividing $|G|$. Then G contains an element of order p.*

Proof. This is 3.2.1 of [5]. □

Theorem 3.2. *Let G be a finite group.*
Then G is a Tabor group if and only if $G \cong K \times T$ where K is a subgroup of G, such that $|K|$ is odd, and T is a 2-subgroup of G.

Proof. Let G be a Tabor group. Then $T := \{g \in G \mid g \text{ is a 2-element}\}$ is a 2-subgroup of G by Corollary 2.3. Moreover, if $g \in G$ and $b \in T$ such that $o(b) = 2^n$, then $(g^{-1} \cdot b \cdot g)^{2^n} = g^{-1} \cdot b^{2^n} \cdot g = g^{-1} \cdot g = 1$. Thus, $b^g := g^{-1} \cdot b \cdot g$ is a 2-element and hence an element of T.

Therefore, T is a normal subgroup of G and from Lemma 3.1 we deduce that $|T|$ is a power of 2.

Suppose for a contradiction that $|G/T|$ is even. Then Lemma 3.1 provides an element $Tg \in G/T$ such that $o(Tg) = 2$. This implies $Tg \cdot Tg = T$ and so $g^2 \in T$. We obtain from the definition of T that also $g \in T$ and deduce that $o(Tg) = 1$, which is a contradiction. It follows that $|T|$ and $|G/T|$ are relatively prime.

These are exactly the conditions for the Theorem of Schur-Zassenhaus 6.2.1 of [5]. According to this theorem, the group G has a subgroup K such that $K \cap T = \{1\}$ and $G = K \cdot T$. We conclude that $|K| = |G/T|$ is odd and Proposition 2.4 yields $G \cong T \times K$.

Let now G have a subgroup K, with $|K|$ odd, and a 2-subgroup T such that $G \cong K \times T$. Then we conclude $K = \{x \in G \mid o(x) \text{ is odd}\}$ and T is the set of all 2-elements. Thus we may apply Theorem 2.5 to obtain that G is a Tabor group. □

Corollary 3.3. *Finite Tabor groups satisfy condition* $(\tilde{T})$.

Proof. This follows directly from Theorem 2.5 and Theorem 3.2. □

Corollary 3.4. *Finite Tabor groups are soluble.*

Proof. Let G be a finite Tabor group.
Then Theorem 3.2 yields that G contains a subgroup K, with $|K|$ odd, and a 2-subgroup T such that $G \cong K \times T$. From 5.1.3 and 5.1.6(iii) of [5] we obtain that T is soluble. Additionally K is soluble by the Odd-Order-Theorem [3]. Finally 6.1.6 of [5] implies the assertion. □

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