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Vertex-edge domination in graphs

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Abstract. In this paper we study graph parameters related to vertex-edge domination, where a vertex dominates the edges incident to it as well as the edges adjacent to these incident edges. First, we present new relationships relating the *ve*-domination to some other domination parameters, answering in the affirmative four open questions posed in the 2007 PhD thesis by Lewis. Then we provide an upper bound for the independent *ve*-domination number in terms of the *ve*-domination number for every nontrivial connected $K_{1,k}$-free graph, with $k \geq 3$, and we show that the independent *ve*-domination number is bounded above by the domination number for every nontrivial tree. Finally, we establish an upper bound on the *ve*-domination number for connected C_5-free graphs, improving a recent bound given for trees.

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1. Introduction

Let $G = (V, E)$ be a graph with order $n = |V|$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$, the *closed neighborhood* is $N[S] = N(S) \cup S$, and $G[S]$ is the *subgraph* induced by the vertices of S. The *private neighborhood* of a vertex u with respect to S is defined as $\text{pn}[u, S] = \{v \in V \mid N[v] \cap S = \{u\}\}$. The *degree* of a vertex v is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. We denote the *star* consisting of one central vertex and k leaves by $K_{1,k}$. If a graph G does not contain an induced subgraph that is isomorphic to some graph F, then we say that G is *F-free*. In particular, if $F = K_{1,3}$, we say that G is claw-free. Let $\text{diam}(G)$ denote the diameter of a graph G.

A vertex $u \in V$ is said to *ve-dominate* an edge $vw \in E$ if

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1. $u = v$ or $u = w$, that is, u is incident to vw, or
2. uv or uw is an edge in G, that is, u is incident to an edge that is adjacent to vw.

In other words, a vertex u *ve*-dominates all edges incident to any vertex in $N[u]$. A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply a *ve-dominating set*) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v *ve*-dominates e. The property for a subset of V to be *ve*-dominating is *superhereditary*, meaning that every superset of a *ve*-dominating set is also a *ve*-dominating set. A *ve*-dominating set S is, therefore, *minimal* if, for every vertex $v \in S$, $S - \{v\}$ is not a *ve*-dominating set in G. The minimum cardinality of a *ve*-dominating set of G is called the *vertex-edge domination number* $\gamma_{ve}(G)$ and the maximum cardinality of a minimal *ve*-dominating set of a graph G is called the *upper vertex-edge domination number* (or simply the *upper ve-domination number*) and is denoted by $\Gamma_{ve}(G)$. A set $S \subset V$ is *independent* if no two vertices in S are adjacent. A set $S \subseteq V$ is an *independent vertex-edge dominating set* (or simply an *independent ve-dominating set*) if S is both independent and *ve*-dominating. The *independent vertex-edge domination number* , $i_{ve}(G)$, of G is the minimum cardinality of an independent *ve*-dominating set and the *upper independent vertex-edge domination number* $\beta_{ve}(G)$ is the maximum cardinality of a minimal independent *ve*-dominating set of G.

A vertex $u \in S \subseteq V$ has a *private edge* $e = vw \in E$ (with respect to a set S), if:

1. u is incident to e or u is adjacent to either v or w, and
2. for all vertices $x \in S - \{u\}$, x is neither incident to e nor adjacent to v or w.

In other words, u *ve*-dominates the edge e and no other vertex in S *ve*-dominates e.

A set S is a *vertex-edge irredundant set* (or simply a *ve-irredundant set*) if every vertex $v \in S$ has a private edge. The property for a subset of V to be *ve*-irredundant is *hereditary*, meaning that every subset of a *ve*-irredundant set is also a *ve*-irredundant set. A *ve*-irredundant set S is, therefore, *maximal* if, for every vertex $v \notin S$, $S \cup \{v\}$ is not *ve*-irredundant. The *vertex-edge irredundance number* $ir_{ve}(G)$ is the minimum cardinality of a maximal *ve*-irredundant set in G, and the *upper vertex-edge irredundance number* $IR_{ve}(G)$ is the maximum cardinality of a *ve*-irredundant set in G.

The concepts of vertex-edge domination and edge-vertex domination were introduced by Peters [5] in his 1986 PhD thesis. These two concepts were also studied in the 2007 PhD thesis by Lewis ([3, 2007]), where many new results were established, which led to the 2010 paper by Lewis et al. [4].

Throughout this paper, we only consider nontrivial connected graphs, called *ntc graphs.* We first show that every minimal *ve*-dominating set is a *ve*-irredundant set.

Proposition 1. *Let S be a ve-dominating set of an ntc graph G. Then S is a minimal ve-dominating set if and only if every vertex $v \in S$ has at least one private edge with respect to S.*

Proof. Suppose that S is a minimal *ve*-dominating set of G. Then for every vertex $v \in S$, $S - \{v\}$ does not *ve*-dominate G. Hence, there is an edge e which is not *ve*-dominated by $S - \{v\}$, implying that e is a private edge of v with respect to S.

Conversely, assume that S is a *ve*-dominating set of G such that every vertex of S has a private edge with respect to S. If S is not minimal, then there is a vertex $v \in S$ such that $S - \{v\}$ *ve*-dominates G. It follows that each edge of G is *ve*-dominated by $S - \{v\}$, contradicting the property of S. Therefore, S is a minimal *ve*-dominating set. □

Lewis et al. [4] proved the following property for *ve*-irredundant sets.

Proposition 2. (Lewis et al. [4]) *Every vertex in a ve-irredundant set S of an ntc graph G has a private neighbor in $V - S$.*

Corollary 3. *Every vertex in a minimal ve-dominating set S of an ntc graph G has a private neighbor in $V - S$.*

The six *ve* parameters are related by the following chain of inequalities, established by Lewis et al. [4].

Theorem 4. (Lewis et al. [4]) *For any ntc graph G of order n,*

$$ir_{ve}(G) \leq \gamma_{ve}(G) \leq i_{ve}(G) \leq \beta_{ve}(G) \leq \Gamma_{ve}(G) \leq IR_{ve}(G) \leq n/2.$$

The chain of inequalities in Theorem 4 is a variant of the well-known domination chain, first observed by Cockayne et al. [1]:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G),$$

where $ir(G)$ and $IR(G)$ denote the lower and upper irredundance numbers of a graph G, $\gamma(G)$ and $\Gamma(G)$ denote the lower and upper domination numbers of G, and $i(G)$ and $\beta_0(G)$ denote the independent domination number and vertex independence number of G, respectively.

In his PhD thesis, Lewis [3] raised the following four questions. For an ntc graph G of order n.

1. Is $IR_{ve}(G) + \gamma(G) \leq n$?
2. Is $\Gamma_{ve}(G) + i(G) \leq n$?
3. Is $IR(G) + \gamma_{ve}(G) \leq n$?
4. Is $\Gamma(G) + i_{ve}(G) \leq n$?

In this paper, we answer, in the affirmative, each of these four questions. We present an upper bound for the independent *ve*-domination number in terms of the *ve*-domination number for every ntc $K_{1,k}$-free graph, with $k \geq 3$, and we show that the independent *ve*-domination number is bounded above by the domination number for every nontrivial tree. Finally, we provide an upper bound on the *ve*-domination number for connected C_5-free graphs, improving a recent bound for trees by Krishnakumari et al. [2].

2. Answers to questions 1–4

In this section we prove two theorems, which answer the four open questions listed in Sect. 1. The first result relates the upper irredundance number to the independent *ve*-domination number for ntc graphs. Moreover, we provide a characterization of the graphs attaining this bound.

We recall that a set $S \subset V$ of vertices is called *irredundant* if for every vertex $v \in S$, $N[v] - N[S - \{v\}] \neq \emptyset$, that is, every vertex $v \in S$ has a *private neighbor* with respect to S. The maximum cardinality of an irredundant set is called the *upper irredundance number* of G and is denoted $IR(G)$. For any parameter $\mu(G)$ associated to a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set.

Our first result provides positive answers to Questions 3 and 4 from Sect. 1.

Theorem 5. *Let G be an ntc graph of order n. Then $IR(G)+i_{ve}(G) \leq n$, with equality if and only if G is a star.*

Proof. Let D be an $IR(G)$-set. Then $\text{pn}[v, D] \neq \emptyset$ for all $v \in D$. Since G is an ntc graph, every vertex in D has a neighbor in $V - D$, and so, $V - D$ *ve*-dominates G. Let A be a maximal independent set in $G[V - D]$ and $B = V-(D \cup A)$. Observe that A is an independent *ve*-dominating set of $G[V - D]$.

Now if $B = \emptyset$, then since $A = V - D$, it follows that A is an independent *ve*-dominating set of G. Hence, $i_{ve}(G) \leq |A| = |V| - |D|$, and the result holds.

Thus we may assume that $B \neq \emptyset$. Let $A' = N(A) \cap D$, let B' be the set of isolated vertices in $G[D - A']$, and let $C = D - (A' \cup B')$. Note that every edge e incident to a vertex in B' is also incident to a vertex in $A' \cup B$, and so, e is *ve*-dominated by A.

Now if $C = \emptyset$, then A is an independent *ve*-dominating set of G. Hence, $i_{ve}(G) \leq |A| < |V| - |D|$, and $IR(G) + i_{ve}(G) < n$.

Thus we may assume that $C \neq \emptyset$. Since the subgraph induced by C has no isolated vertices, every vertex of C has a private neighbor in B with respect to D. Hence, $|C| \leq |B|$. Let C' be a maximal independent set in $G[C]$. Clearly, $|C'| < |C|$ and $C' \cup A$ is an independent *ve*-dominating set of G. Therefore, $i_{ve}(G) \leq |A \cup C'| < |A| + |C| \leq |A| + |B| = |V - D|$, and so $IR(G) + i_{ve}(G) < n$.

Suppose now that $IR(G) + i_{ve}(G) = n$, and let D be an $IR(G)$-set. Then $i_{ve}(G) = |V - D|$. Following the same notation as above, we deduce that $B = \emptyset$. Hence, $V - D = A$ is an $i_{ve}(G)$-set and $B' = C = \emptyset$. Note that since $B' = C = \emptyset$, it follows that $A' = N(A) \cap D = D$. Observe that if some vertex $u \in D$ has at least two neighbors in $V - D$, then $((V - D) - N(u)) \cup \{u\}$ is an independent ve-dominating set of G of cardinality smaller than $|V - D| = i_{ve}(G)$; contradiction. Hence, each vertex of D has exactly one neighbor in $V - D$. Let D_2 be the set of isolated vertices in $G[D]$ and $D_1 = D - D_2$. We first assume that $D_1 \neq \emptyset$. Since every vertex of D_1 has a neighbor in D_1, its private neighbors are in $V - D$. Thus, each vertex in D_1 has exactly one neighbor, its private neighbor, in $V - D$. Since $V - D$ is independent and G is connected, we deduce that $D_2 = \emptyset$. Now let x and y be two adjacent vertices of D_1, and let x' and y' be their private neighbors in $V - D$, respectively. Then $((V - D) - \{x', y'\}) \cup \{x\}$ is an independent ve-dominating set of cardinality smaller than $V - D$; a contradiction. Hence, $D_1 = \emptyset$. It follows that $D = D_2 \neq \emptyset$ and that D is an independent set. Now since G is connected, every vertex of D has exactly one neighbor in $V - D$, and $V - D = A$ is an $i_{ve}(G)$-set, we conclude that $|A| = 1$. Therefore, G is a star.

The converse, if G is a star of order n, then $IR(G) + i_{ve}(G) = n - 1 + 1 = n$, is obvious. □

Since for any graph G, $\gamma_{ve}(G) \leq i_{ve}(G)$ and $\Gamma(G) \leq IR(G)$, we obtain the following inequalities as a corollary to this theorem, answering Questions 3 and 4.

Corollary 6. *If G is an ntc graph of order n, then $IR(G) + \gamma_{ve}(G) \leq n$ and $\Gamma(G) + i_{ve}(G) \leq n$, with equality if and only if G is a star.*

We are now ready to settle Questions 1 and 2. In fact, we prove a stronger result, namely that $IR_{ve}(G) + i(G) \leq n$ for ntc graphs G of order n.

Theorem 7. *If G is an ntc graph of order n, then $IR_{ve}(G) + i(G) \leq n$.*

Proof. Let D be a ve-irredundant set of G of maximum cardinality $IR_{ve}(G)$. By Proposition 2, each vertex of D has at least one private neighbor in $V - D$. Hence, $V - D$ dominates D. Let A be a maximal independent set of $G[V - D]$, $B = (V - D) - A$ and $C = D - N(A)$. Note that A dominates B. Now if $C = \emptyset$, then A dominates G, and the result holds. Hence, assume that $C \neq \emptyset$. By Proposition 2, every vertex of C has a private neighbor in B, implying that $|C| \leq |B|$. Now if C' is a maximal independent set of $G[C]$, then $A \cup C'$ is a maximal independent set of G. Therefore, $i(G) \leq |A| + |C'| \leq |A| + |C| \leq |A| + |B| = |V| - |D|$. □

Since $\gamma(G) \leq i(G)$ and $\Gamma_{ve}(G) \leq IR_{ve}(G)$, the affirmative answers to Questions 1 and 2 follow as corollaries.

Corollary 8. *If G is an ntc graph of order n, then $IR_{ve}(G) + \gamma(G) \leq n$ and $\Gamma_{ve}(G) + i(G) \leq n$.*

3. Upper bounds on $i_{ve}(G)$ and $\gamma_{ve}(G)$

In this section, we give upper bounds on $\gamma_{ve}(G)$ and $i_{ve}(G)$. By Theorem 4, we have that $i_{ve}(G) \leq n/2$ for any ntc graph of order n. Using Corollary 3, we show that this bound is strict.

Proposition 9. *If G is an ntc graph of order $n \geq 3$, then $\gamma_{ve}(G) \leq i_{ve}(G) < n/2$.*

Proof. Let $i_{ve}(G) = k$, and let D be an $i_{ve}(G)$-set. If $n = 3$, then $i_{ve}(G) = 1$, and the result holds. Thus, let $n \geq 4$, and assume, to the contrary, that $i_{ve}(G) \geq n/2$. Theorem 4 implies that $i_{ve}(G) = k = n/2$. Thus, G has even order and $|D| = |V - D| = k \geq 2$. Since D is a minimal ve-dominating set, by Corollary 3, every vertex in D has a private neighbor in $V - D$. Since $|V - D| = |D| = k$, every vertex in D has exactly one private neighbor in $V - D$, implying that every vertex in $V - D$ has exactly one neighbor in D. Moreover, since G is connected and $n \geq 4$, the induced subgraph $G[V - D]$ is connected. Let S be a maximal independent set of $G[V - D]$. Since $G[V - D]$ is an ntc graph, $|S| < |V - D| = k$. But since every edge of G is incident to a vertex of $V - D$, it follows that S is an independent ve-dominating set of G with cardinality less than $k = i_{ve}(G)$; a contradiction. □

Our next result gives a bound on $i_{ve}(G)$ for $K_{1,k}$-free graphs G.

Theorem 10. *If G is an ntc graph with no induced subgraph isomorphic to $K_{1,k}$ for $k \geq 3$, then*

$$i_{ve}(G) \leq (k-2)\gamma_{ve}(G) - (k-3).$$

Proof. Let D be a $\gamma_{ve}(G)$-set, and let A be a maximal independent set of $G[D]$ and $B = D - A$. If $B = \emptyset$, then $D = A$ is both ve-dominating and independent, and so $i_{ve}(G) = \gamma_{ve}(G) = |D|$. Clearly, the result holds, since $i_{ve}(G) = \gamma_{ve}(G) \leq (k-2)\gamma_{ve}(G) - (k-3)$, for $k \geq 3$.

Next, assume that $B \neq \emptyset$. Note that A is an independent ve-dominating set of $G[D]$. Let $B' = N(B) - N[A]$ and B'' be a maximal independent set in $G[B']$. Since A is an independent dominating set of $G[D]$, we have that $B' \subseteq V - D$. Further, since D is a minimum ve-dominating set, we know by Corollary 3 that every vertex in B (and in A) has a private neighbor in $V - D$, with respect to the set D. Thus, B', and hence, B'', is not empty. Since G does not contain $K_{1,k}$ as induced subgraph and every vertex of B has at least one neighbor in A, each vertex of B has at most $(k-2)$ neighbors in B''. It follows that $|B''| \leq (k-2)|B|$. Now using the fact that $B'' \cup A$ is an independent ve-dominating set of G, we obtain:

$$\begin{aligned} i_{ve}(G) &\leq |A| + |B''| \\ &\leq |A| + (k-2)|B| \\ &\leq |A| + (k-2)|D| - (k-2)|A| \\ &\leq (k-2)|D| - (k-3)|A| \\ &\leq (k-2)\gamma_{ve}(G) - (k-3). \end{aligned}$$

As a consequence of Theorem 10, we obtain the following corollary. □

Corollary 11. *If G is an ntc claw-free graph, then $\gamma_{ve}(G) = i_{ve}(G)$.*

Obviously, $\gamma_{ve}(G) \leq \gamma(G)$ for every graph G. We show that the domination number is also an upper bound on the independent ve-domination number for any nontrivial tree.

Theorem 12. *For every nontrivial tree T, $\gamma_{ve}(T) \leq i_{ve}(T) \leq \gamma(T)$.*

Proof. We use induction on the order n of T to show that $i_{ve}(T) \leq \gamma(T)$. Clearly, the result holds if $n \in \{2, 3\}$ establishing the base cases. Assume that every nontrivial tree T' of order $n' < n$ satisfies $i_{ve}(T') \leq \gamma(T')$. Let T be a tree of order n. Since for stars (where $i_{ve}(T) = \gamma(T) = 1$) and double stars (trees consisting of two stars with an edge joining the centers of the two stars, for which $i_{ve}(T) = 1 < \gamma(T) = 2$) the result holds, we may assume that T has diameter at least four.

Root T at a vertex r of maximum eccentricity, that is, r is at distance $\text{diam}(T) \geq 4$ from another vertex of T. Let u be a support vertex at maximum distance from r and v be the parent of u in the rooted tree. Let D be a $\gamma(T)$-set that contains all support vertices of T; such a $\gamma(T)$-set always exists. We distinguish between two cases.

Case 1. v has degree at least three. Then v is a support vertex or v has another child besides u that is a support vertex. In any case, v is dominated by $D - \{u\}$. Let T' be the tree obtained from T by removing u and its adjacent leaves. Then $D - \{u\}$ dominates T', and so, $\gamma(T') \leq \gamma(T) - 1$. Also, if S is any $i_{ve}(T')$-set, then S plus any leaf neighbor of u is an independent ve-dominating set of T, implying that $i_{ve}(T) \leq i_{ve}(T') + 1$. Now by induction, we have $i_{ve}(T') \leq \gamma(T')$. Using the previous inequalities, we obtain the desired inequality.

Case 2. v has degree two. Let T' be the tree obtained from T by removing u and all of its neighbors (including v). Since $\text{diam}(T) \geq 4$, the subtree T' is nontrivial. Noting that $D - \{u\}$ dominates T', we have that $\gamma(T') \leq \gamma(T) - 1$. Also, since any $i_{ve}(T')$-set can be extended to an independent ve-dominating set of T by adding u to it, we obtain $i_{ve}(T) \leq i_{ve}(T') + 1$. By induction, we have $i_{ve}(T') \leq \gamma(T')$, and so $i_{ve}(T) \leq \gamma(T)$. □

We note that a characterization of trees with equal ve-domination and domination numbers is given in [4].

In a recent paper, Krishnakumari et al. [2] established an upper bound of $n/3$ on the ve-domination number of any tree of order n and characterized the family of trees attaining this upper bound. The authors define this family $\mathcal{F}$ of trees $T = T_k$, as follows. Let T_1 be the path P_3. For $k \geq 2$, T_{i+1} can be obtained recursively from T_i by adding a path P_3 and joining one of its leaves to a vertex of T_i that is adjacent to a path P_2 or P_3. We note that in [2], a vertex v is said to be *adjacent to a path* P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n.

Theorem 13. (Krishnakumari et al. [2]) *If T is a tree of order $n \geq 3$, then $\gamma_{ve}(T) \leq n/3$, with equality if and only if $T \in \mathcal{F}$.*

The *P_2-corona* of a graph G is the graph of order $3|V(G)|$ obtained from G by attaching a distinct path P_2 to each vertex $v \in V$ by adding an edge between v and a leaf of its corresponding path P_2. We note that any tree $T = T_k$, for $k \geq 2$, in the family $\mathcal{F}$ given by Krishnakumari et al. [2], has the property that every vertex in $T_k \in \mathcal{F}$ is exactly one of: (i) a leaf, (ii) a support vertex of degree two, or (iii) a vertex having exactly one support vertex in its neighborhood, that is, adjacent to exactly one path P_2. Since $T_1 = P_3$ can be also considered as a P_2-corona of P_1, the trees of $\mathcal{F}$ are precisely, the P_2-coronas of trees.

Corollary 14. *A tree T of order n satisfies $\gamma_{ve}(T) = n/3$ if and only if T is a P_2-corona of some tree H.*

Our next observation improves the previous upper bound. Let $L(T)$ and $S(T)$ denote the set of leaves and support vertices of a tree T, respectively. For every tree T of order $n \geq 3$ and $\mathrm{diam}(T) \geq 3$ (in other words, T is not a star), let T^* be the tree obtained from T by removing all except one leaf adjacent to every support vertex of T. Clearly, T^* has order $n^* = n - |L(T)| + |S(T)| > 3$, and $\gamma_{ve}(T) = \gamma_{ve}(T^*)$.

Observation 15. *If T is a tree of order $n \geq 3$ and $\mathrm{diam}(T) \geq 3$, then $\gamma_{ve}(T) \leq (n - |L(T)| + |S(T)|)/3$, with equality if and only if T^* is the P_2-corona of some nontrivial tree H.*

Proof. Let T be a nontrivial tree of order n and $\mathrm{diam}(T) \geq 3$. Clearly, $n \geq 4$. Now consider the tree T^*. Obviously, $\gamma_{ve}(T) = \gamma_{ve}(T^*)$. Since T^* has order $n^* > 3$, by Theorem 13, $\gamma_{ve}(T^*) \leq n^*/3$ with equality if and only if T^* is the P_2-corona of some nontrivial tree. □

Our final result shows that $n/3$ is also an upper bound on the ve-domination number of connected C_5-free graphs of order $n \geq 3$, which improves the bound given in [2]. According to Proposition 1, every vertex v in a minimal ve-dominating set of a graph G has at least one private edge. Such private edges may be incident with either v or a vertex adjacent to v. To aid our discussion, we say that a vertex v in a $\gamma_{ve}(G)$-set is a *Type*-1 *vertex* if all its private edges are incident with v, and is a *Type*-2 *vertex*, otherwise. □

Theorem 16. *If G is a connected C_5-free graph of order $n \geq 3$, then $\gamma_{ve}(G) \leq n/3$.*

Proof. Among all $\gamma_{ve}(G)$-sets, let D be one with as few isolated vertices in $G[D]$ as possible. We shall show that $|V - D| \geq 2|D|$, which proves the theorem.

By Proposition 1, every vertex of D has at least one private edge. We note that it is possible for a private edge of u to be adjacent to a private edge of v for two vertices u and v in D. Let D_i be the subset of D containing the Type-i vertices for $i \in \{1, 2\}$. Then $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. Suppose a vertex, say u, of D_1 has a neighbor in D. Then $D - \{u\}$ is a ve-dominating set with cardinality less than $\gamma_{ve}(G)$; a contradiction. Thus, we conclude that D_1 is an independent set and no vertex of D_1 has a neighbor in D_2.

For a vertex $x \in D$, let $\text{pn}_2(x, D)$ be the set of vertices of $V - D$ that are incident to a private edge of x but not incident to a private edge of another vertex in $D - \{x\}$. Note that $\text{pn}[x, D] \cap (V - D) \subseteq \text{pn}_2(x, D)$, and so, Corollary 3 implies that $|\text{pn}_2(x, D)| \geq 1$ for every $x \in D$. Our goal is to build a set S_x for each $x \in D$, such that $|S_x| \geq 2$ and $S_x \cap S_y = \emptyset$ for all $y \in D - \{x\}$. To build such a set, begin with $S_x = \text{pn}_2(x, D)$. If $|S_x| \geq 2$ for all $x \in D$, then we are finished. Suppose that there exists an $x \in D$ such that $|S_x| = 1$, and let $S_x = \{x'\}$. We show that either we have a contradiction, or that we can add a vertex v to S_x, that is, we can replace S_x with $S_x \cup \{v\}$ such that $S_x \cap S_y = \emptyset$ for all $y \in D - \{x\}$.

We consider two cases.

Case 1. $x \in D_1$. Since x has no neighbor in D, either x is a leaf of G or x has a common neighbor, say w, with another vertex $v \in D$. Assume first that x is a leaf in G. Since G is connected, x' has another neighbor z, and necessarily, $z \in V - D$. Moreover, since x is a Type-1 vertex, the edge $x'z$ is ve-dominated by a neighbor of z in D. But then $D' = (D - \{x\}) \cup \{z\}$ is a $\gamma_{ve}(G)$-set with fewer isolated vertices in $G[D']$ than in $G[D]$, contradicting our choice of D.

Hence, we may assume that x is not a leaf, that is, x and v share a common neighbor $w \neq x'$. Since $x \in D_1$, it follows that $w \in V - D$. Again, $(D - \{x\}) \cup \{w\}$ is a $\gamma_{ve}(G)$-set with fewer isolated vertices in the subgraph it induces than D has, contradicting our choice of D.

Case 2. $x \in D_2$. Then x has a private edge uv such that u and v are in $V - D$, that is, at least one of u and v is a private neighbor of x with respect to D. If $u \neq x'$ and $v \neq x'$, then $|S_x| \geq 2$, a contradiction. Hence, we may assume that all the private edges of x are incident to x'. Consider the private edge $x'v$ of x. Since $v \notin S_x$ implies that $v \notin \text{pn}_2(x, D)$, it follows that v is incident to a private edge of at least one vertex in $D - \{x\}$. Let $Y = \{y_1, ..., y_t\}$ be the subset of $D - \{x\}$ such that each $y_i \in Y$ has a private edge incident to v.

By our previous comments, $Y \neq \emptyset$. Let vw_i be a private edge of y_i for each $y_i \in Y$. Note that $v \notin N(y_i)$ for any $y_i \in Y$, because $x'v$ is a private edge of x. Thus, $w_i \in \text{pn}[y_i, D]$ and $y_i \in D_2$.

We first show that for all $y_i \in Y$, either $|S_{y_i}| \geq 2$ or we can add a vertex to S_{y_i} creating a new set that has no vertex in common with any S_a where $a \in D - \{y_i\}$. Suppose, to the contrary, that $|S_{y_i}| = 1$, that is, $S_{y_i} = \{w_i\}$, for some $y_i \in Y$. Assume first that there exists a vertex $z \in V - D$ for which $N(z) \cap D = \{x, y_i\}$. Note that $z \notin S_a$ for any $a \in D$. Moreover, every edge incident to z is ve-dominated by both x and y_i, so none of these edges are private edges for any vertex in D. Hence, we can add z to S_{y_i} forming a new S_{y_i}, and so, $|S_{y_i}| \geq 2$ and $S_{y_i} \cap S_a = \emptyset$ for all $a \in D - \{y_i\}$. Moreover, since $N(z) \cap D = \{x, y_i\}$, z is not a candidate to be added to another S_{y_j} for $j \neq i$. Thus, if $|S_{y_i}| = 1$, then we may assume that no such vertex z exists.

Note that x is not adjacent to v or to w_i, and y_i is not adjacent to v or x'. If x' is adjacent to w_i, then $(D - \{x, y_i\}) \cup \{x'\}$ is a ve-dominating set of G with cardinality less than $\gamma_{ve}(G)$, a contradiction. Hence, we may assume that x' and w_i are not adjacent. But then if x and y_i are adjacent, the subgraph induced by $\{x, x', v, w_i, y_i\}$ is a cycle C_5, a contradiction. Hence, x is not adjacent to y_i. Now, $(D - \{x, y_i\}) \cup \{v\}$ is a ve-dominating set of G with cardinality less than $\gamma_{ve}(G)$, a contradiction.

Thus, for each $y_i \in Y$, we have a set S_{y_i} such that $|S_{y_i}| \geq 2$ and $S_{y_i} \cap S_a = \emptyset$ for all $a \in D - \{y_i\}$. Moreover, we note that $v \notin S_a$ for any $a \in D - \{x\}$. In other words, if $|S_x| = |S_y| = 1$ for vertices x and y in D, then no private edge of x is adjacent to a private edge of y. Now we can add v to S_x so that $|S_x| \geq 2$ and $S_x \cap S_a = \emptyset$ for all $a \in D - \{x\}$.

In both cases, we have demonstrated that for each vertex in D, we can count at least two distinct vertices in $V - D$, that is, $|V - D| \geq 2|D|$. Thus, $\gamma_{ve}(G) \leq n/3$. □

The following corollaries are immediate from Theorem 16.

Corollary 17. *If G is a bipartite graph of order $n \geq 3$, then $\gamma_{ve}(G) \leq n/3$.*

Corollary 18. (Krishnakumari et al. [2])*If T is a tree of order $n \geq 3$, then $\gamma_{ve}(T) \leq n/3$.*

Observe that the C_5-free condition is necessary for Theorem 16. For a simple example, consider the graph C_5 for which $\gamma_{ve}(C_5) = 2 > 5/3$. However, this is the only example of a graph that we have found for which the bound of Theorem 16 does not hold. We leave it as an open problem to either prove that the theorem holds for all ntc graphs of order $n \geq 6$, or give a family of graphs that are counterexamples. It is also worth mentioning that the bound given in Theorem 16 is not valid for the independent vertex-edge domination number $i_{ve}(G)$. Indeed, let G be the graph obtained from k $(k \geq 4)$ paths

P_5 by adding all edges between the centers of the paths. Clearly, G is C_5-free having $n = 5k$ vertices and $i_{ve}(G) = 2k - 1 > n/3$.

4. Open questions

We conclude this paper with a list of open questions.

1. Characterize all nontrivial trees T with $i_{ve}(T) = \gamma(T)$.
2. In Theorem 5, we prove that if G is an ntc graph of order n, then $IR(G) + i_{ve}(G) \leq n$. This raises the following question.
   Is $IR(G) + \beta_{ve}(G) \leq n$?
3. Let $S \subset V$. A vertex $v \in S$ is called an *enclave* in S if $N[v] \subseteq S$. Let $\Psi(G)$ equal the maximum cardinality of a set S containing no enclave. It is known that for any graph G, $IR(G) \leq \Psi(G)$.
   Is $\Psi(G) + i_{ve}(G) \leq n$?
4. In Theorem 7, we prove that if G is an ntc graph of order n, then $IR_{ve}(G) + i(G) \leq n$. This raises the following question:
   Is $\Gamma_{ve}(G) + \beta_0(G) \leq n$?
5. In Theorem 7, when is the equality $IR_{ve}(G) + i(G) = n$ achieved?

References

[1] Cockayne, E.J., Hedetniemi, S.T., Miller, D.J.: Properties of hereditary hypergraphs and middle graphs. Can. Math. Bull. **21**, 461–468 (1978)

[2] Krishnakumari, B., Venkatakrishnan, Y.B., Krzywkowski, M.: Bounds on the vertex-edge domination number of a tree. C. R. Acad. Sci. Paris Ser. I **352**, 363–366 (2014)

[3] Lewis, J.R.: Vertex-Edge and Edge-Vertex Domination in Graphs. Ph.D. Thesis, Clemson University, Clemson (2007)

[4] Lewis, J.R., Hedetniemi, S.T., Haynes, T.W., Fricke, G.H.: Vertex-edge domination. Util. Math. **81**, 193–213 (2010)

[5] Peters, J.W.: Theoretical and Algorithmic Results on Domination and Connectivity. Ph.D. Thesis, Clemson University, Clemson (1986)

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