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Aequationes Mathematicae

Cauchy's functional equation and extensions: Goldie's equation and inequality, the Gołąb–Schinzel equation and Beurling's equation

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To Ranko Bojanić on his 90th birthday.

Abstract. The Cauchy functional equation is not only the most important single functional equation, it is also central to regular variation. Classical Karamata regular variation involves a functional equation and inequality due to Goldie; we study this, and its counterpart in Beurling regular variation, together with the related Gołąb–Schinzel equation.

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Gołąb–Schinzel functional equation.

1. Introduction

We are concerned with the fundamental functional equation, the *Cauchy Functional Equation* (CFE)

$$K(x+y) = K(x) + K(y), \quad k(xy) = k(x)k(y), \tag{CFE}$$

to give both the additive and multiplicative versions. For background, see the standard work by Kuczma [28]. This is known to be crucial to the theory of regular variation, in both its Karamata form (see Ch. 1 of [6], BGT below) and its Bojanić–Karamata/de Haan form (BGT Ch. 3, [13], or the recent [12]).[1] A close study of these involves a certain functional equation ([5], BGT), which we call here the *Goldie functional equation* [(GFE)—see below].[2] One of the themes of Kuczma's book is the interplay between functional equations

[1] For historical remarks, see the full version of this paper: arXiv:1405.3947.
[2] The equation occurs first in joint work by the first author and Goldie; the first author is happy to confirm that the argument is in fact due to Goldie, whence the name.

and inequalities; he focusses particularly on the Cauchy functional equation and Jensen's inequality. Even more closely linked to (CFE) is the functional inequality of *subadditivity*; see [28, Ch. 16] for classical background, [7,8,12] for more recent results, developed further here. The Goldie functional equation (GFE) has its counterpart in the *Goldie functional inequality* [(GFI)—see below].

The theme of the present paper is that one begins with the functional inequality, imposes a suitable side-condition (which serves to 'give the inequality the other way') and deduces the corresponding functional equation, which under suitable conditions one is able to solve. The functional equation and functional inequality we have in mind originate in the study of

$$F^*(u) := \limsup_{x\to\infty}(F(u+x)-F(x))/g(x)$$

(cf. BGT 3.1.1), where in the prototypical case below we will have

$$g(x) \equiv e^{\gamma x}.$$

They are those mentioned above:

$$F^*(u+v) \leqslant e^{\gamma v}F^*(u) + F^*(v) \quad (\forall u, v \in \mathbb{R}) \tag{GFI}$$

(BGT (3.2.5), cf. (3.0.11)), and

$$F^*(u+v) = e^{\gamma v}K(u) + F^*(v) \quad (\forall u \in \mathbb{A}, v \in \mathbb{R}) \tag{GFE}$$

[see BGT (3.2.7)] with $\mathbb{A}$ an additive subgroup (of $\mathbb{R}$). For the relationship here for $\gamma \neq 0$ between F^* and its *Goldie kernel* K, indeed in greater generality, see Theorem 3 in Sect. 2 below, and also Eq. (GBE-P) in Sect. 4; cf. [11, Prop. 1] for the *additive kernel* in the case $\gamma = 0$, which reduces (GFI) to subadditivity on $\mathbb{R}_+$, the context there. (GFI) captures an asymptotic relation in functional form, and so is key to establishing the Characterization Theorem of regular variation (BGT §1.4). Our focus here is on the extent to which the universal quantifiers occurring in the functional inequalities and functional equations under study can be weakened, in the presence of suitable side-conditions. The prototypical side-condition here is the *Heiberg–Seneta condition*

$$\limsup_{u\downarrow 0} F(u) \leqslant 0, \tag{HS(F)}$$

due to Heiberg [24] in 1971, and Seneta [34] in 1976 (BGT, Th. 3.2.4). This condition, best possible here (cf. [BGT, §1.3], [11]), is what is needed to reduce (GFI) to (GFE).

Two related matters occur here. One is the question of *quantifier weakening* above. This, together with (HS), hinges on the algebraic nature of the set on which one can assert equality. The second, *automatic continuity*, relates to the extent to which a solution of (GFE) is continuous (and hence easily of standard form—see BGT Ch. 3), or (in the most important case $\gamma = 0$) an *additive* function becomes *continuous*, and so *linear*. This is the instance of the subject of *automatic continuity* relevant here. Automatic continuity has a vast literature, particularly concerning homomorphisms of Banach algebras, for which see [20,21]. See also Helson [25] for Gelfand theory, Ng and Warner [29] and Hoffmann-Jørgensen [26]. The crux here is the *dichotomy* between additive functions with a hint of regularity, which are then linear, and those without, which are pathological; for background and references on dichotomies of this nature, Hamel pathologies and the like, see [9].

One of our themes here and in [11] is *quantifier weakening*: one weakens a universal quantifier $\forall$ by thinning the set over which it ranges. In what follows we will often have two quantifiers in play, and will replace "$\forall u \in \mathbb{A}$" by "$(u \in \mathbb{A})$" , etc., a convenient borrowing from mathematical logic.

One theme that this paper and [11] have in common is the great debt that the subject of regular variation, as it has developed since [5] and BGT, owes to the Goldie argument. It is a pleasure to emphasize this here. This argument originated in a study of *Frullani integrals*, important in many areas of analysis and probability ([5, I, II.6]; cf. BGT §1.6.4, [11, §1]).

When one specializes from functional *inequalities* to functional *equations* matters can be taken further. While our methods here are necessarily *analytic*, relying on ideas from the dynamics of *continuous flows* (see e.g. [4,30]), there one can use *algebraic* methods. We refer to resulting algebraicization to the companion paper [32], where these equations are transformed into *homomorphisms*, and to alternative approaches standard in the literature of functional equations, for instance [1] and the recent [19].

2. Generalized Goldie equation

We begin by generalizing (GFE) by replacing the exponential function on the right by a more general function g, the *auxiliary function*. To avoid trivial solutions, without loss of generality for this section $g(0) = 1$. We further generalize by *weakening the quantifiers*, allowing them to range over a set $\mathbb{A}$ smaller than $\mathbb{R}$. It is appropriate to take $\mathbb{A}$ as a dense (additive) subgroup. (This is motivated by asymptotic analysis and additivity in the domain of the operation $\lim_{x\to\infty}$.) The functional equation in the result below, written there ($G_{\mathbb{A}}$), may be thought of as the second form of the Goldie functional equation above. As we see in Theorem 1 below, the two coincide in the principal case of interest—compare the insightful Footnote 3 of [13]. The notation H_γ below (originating in [13]) is from BGT §3.1.7 and 3.2.1, implying $H_0(t) \equiv t$. (See Sect. 4 for generalizations.) The identity

$$uv - u - v + 1 \equiv (1 - u)(1 - v)$$

(relevant to the *circle group* operation of [32]) gives that $(1-e^{-\gamma x})/\gamma$ is *subadditive* on $\mathbb{R}_+ := (0, \infty)$ for $\gamma \geqslant 0$, and *superadditive* on $\mathbb{R}_+$ for $\gamma \leqslant 0$. We will need Theorem 1 below, extending BGT Lemma 3.2.1. The Eq. ($G_{\mathbb{A}}$) below when $\mathbb{A} = \mathbb{R}$ is a special case of a generalized Pexider equation studied by Aczél [1]. In Theorem 1 (CEE) is the *Cauchy exponential equation*.

Theorem 1. ([13, (2.2)], BGT Lemma 3.2.1; cf. [2]). *For g with $g(0) = 1$, if $K \not\equiv 0$ satisfies*

$$K(u+v) = g(v)K(u) + K(v) \quad (u, v \in \mathbb{A}), \tag{$G_{\mathbb{A}}$}$$

with $\mathbb{A}$ a dense subgroup—then:

(i) *the following is an additive subgroup on which K is additive:*

$$\mathbb{A}_g := \{u \in \mathbb{A} : g(u) = 1\};$$

(ii) *if $\mathbb{A}_g \neq \mathbb{A}$ and $K \not\equiv 0$, there is a constant $\kappa \neq 0$ with*

$$K(t) \equiv \kappa(g(t) - 1) \quad (t \in \mathbb{A}), \tag{*}$$

and g satisfies

$$g(u+v) = g(v)g(u) \quad (u, v \in \mathbb{A}). \tag{CEE}$$

(iii) *So for $\mathbb{A}=\mathbb{R}$ and g locally bounded at 0 with $g \neq 1$ except at $0 : g(x) \equiv e^{-\gamma x}$ for some constant $\gamma \neq 0$, and so $K(t) \equiv cH_\gamma(t)$ for some constant c, where*

$$H_\gamma(t) := (1 - e^{-\gamma t})/\gamma.$$

Proof. Recall (see [28, §18.5]) that the *Cauchy nucleus* of K,

$$\mathcal{N}_K := \{x \in \mathbb{A} : K(x+a) = K(x) + K(a)(\forall a \in \mathbb{A})\},$$

is either empty or a subgroup (for a proof see [28, Lemma 18.5.1], or the related [3, Ch. 6, proof of Th. 1]). If $x \in \mathcal{N}_K$, choosing $a \in \mathbb{A}$ with $K(a) \neq 0$ yields $g(x) = 1$ from

$$K(a+x) = K(a) + K(x) = g(x)K(a) + K(x).$$

Conversely, $K(u+v) = K(u) + K(v)$ for $v \in \mathbb{A}_g$ and any $u \in \mathbb{A}$, so $v \in \mathcal{N}_K$: $\mathbb{A}_g = \mathcal{N}_K$. So $\mathbb{A}_g$ is a subgroup as $0 \in \mathbb{A}_g$; in particular K is additive on $\mathbb{A}_g$, so $K(0) = 0$.

As in [13, 2.2], BGT Lemma 3.2.1, and [2, Th. 1]: as $K(u+v) = K(v+u)$,

$$g(v)K(u) + K(v) = g(u)K(v) + K(u) : \quad K(u)[g(v) - 1] = K(v)[g(u) - 1].$$

For $\mathbb{A} \neq \mathbb{A}_g$, choose $v \in \mathbb{A} \backslash \mathbb{A}_g$ and take $\kappa := K(v)/(g(v) - 1)$; then

$$K(u) = \kappa[g(u) - 1] \quad (u \in \mathbb{A}).$$

For K not identically zero, there is $u \in \mathbb{A}$ with $K(u) \neq 0$; then $\kappa \neq 0$ and $u \notin \mathbb{A}_g$. Substitution (for $u, v \in \mathbb{A}$) yields first

$$\kappa[g(u+v) - 1] = \kappa g(v)[g(u) - 1] + \kappa[g(v) - 1],$$

and then, for $\kappa \neq 0$, (CEE) on $\mathbb{A}$.

If $\mathbb{A} = \mathbb{R}$ and $\mathbb{A}_g = \{0\}$, either $K \equiv 0$ (and $c = 0$), or $\kappa \neq 0$. Then (CEE) and local boundedness yield $g(x) \equiv e^{-\gamma x}$ for some γ (see [3, Ch. 3], or [28, §13.1]), and $\gamma \neq 0$ (otherwise $g \equiv 1$). Now take $c := -\kappa/\gamma$. □

Remarks. 1. Above, for g Baire/measurable, by the Steinhaus subgroup theorem (see e.g. [9, Th. S] for its general combinatorial form), $\mathbb{A}_g = \mathbb{R}$ iff $\mathbb{A}_g$ is non-negligible, in which case K is additive. The additive case is studied in [11] and here we have passed to $\mathbb{A}_g = \{0\}$ as a convenient context. But more is true. As an alternative to the last remark, for $\mathbb{A}_g$ negligible: by the Fubini/Kuratowski–Ulam Theorem [33, Ch. 14–15], the Eq. (CEE) above holds for *quasi all* $(u, v) \in \mathbb{R}^2$; consequently, by a theorem of Ger ([22], or [28, Th.18.71]), there is a homomorphism on $\mathbb{A}$ 'essentially extending' $\log g$ to $\mathbb{A}$. From here, again for g Baire/measurable, $g(x) = e^{-\gamma x}$ for some γ; similarly also, if the *kernel* (null space) $\mathbb{K}$ of K is negligible, since, for some κ, as above $K(x) \equiv \kappa(g(x) - 1)(x \notin \mathbb{K})$.

2. In Theorem 1, omitting (i), one may restrict to $\mathbb{A}_+ := \mathbb{A} \cap \mathbb{R}_+$; in (ii) g then satisfies (CEE) on $\mathbb{A}_+$, and so in (iii) with $\mathbb{A} = \mathbb{R}$, by an extension theorem of Aczél and Erdős [28, Th. 13.5.3], g is still exponential.

Theorem 2 below is a variant of Theorem 1 relevant to regular variation. Here there is no quantifier weakening to $\mathbb{A}$, and so we need $(G_{\mathbb{R}})$ in place of $(G_{\mathbb{A}})$. The result is an immediate corollary of the Lemma below and classical results concerning (CFE). It will be convenient in what follows to write 'positive/non-negative' for a function to mean 'positive/non-negative on $\mathbb{R}_+$' (whatever its domain), unless otherwise stated.

Theorem 2. *If both K and g in $(G_{\mathbb{R}})$ are positive/non-negative with $g \neq 1$ except at 0, then either $K \equiv 0$, or $g(x) \equiv e^{-\gamma x}$ for some $\gamma \neq 0$.*

Our positivity assumption above, motivated by regular variation, yields continuity at one point at least (via monotonicity).

Lemma. *If both K and g in $(G_{\mathbb{R}})$ are non-negative with $g \neq 1$ except at 0, then either $K \equiv 0$, or both are continuous.*

Proof of the Lemma. Suppose that $K \not\equiv 0$. Writing $w = u + v$,

$$K(w) - K(v) = g(v)K(w - v),$$

so K is (weakly) increasing and so continuous at some point $y > 0$ say. Now $K(0) = 0$ (as $g(0) = 1$), so $g(y) > 0$, as otherwise taking $v = y$ above yields $K(w) \equiv K(y) \equiv K(0) = 0$. But for any h

$$K(y + h) - K(y) = g(y)K(h),$$

and so, since $g(y) > 0$, K is continuous at 0, as $K(0) = 0$. Hence K is continuous at any point $t > 0$ (by a similar argument with t for y). Likewise so is g: fix $w > t$, so that $K(w - t) > 0$. Then

$$g(t) = [K(w) - K(t)]/K(w-t),$$

and the right-hand side is continuous in t for $K(w-t) > 0$. □

Proof of Theorem 2. As in Th. 1 (*), $K(x) = \kappa[g(x) - 1]$ for all x, for some constant κ; if $\kappa \neq 0$, then, as there, g satisfies (CEE). By the Lemma, g is continuous on $\mathbb{R}_+$. So again g is $e^{-\gamma x}$, as at the end of the proof of Theorem 1. □

In $(G_{\mathbb{R}})$ above for $x, \gamma \geqslant 0$ one has $g(x) = e^{-\gamma x} \leqslant 1$ on $\mathbb{R}_+$; generally, if $g(x) \leqslant 1$ on $\mathbb{R}_+$ and K positive satisfies $(G_{\mathbb{R}})$, then for $u, v \geqslant 0$

$$K(u+v) \leqslant K(u) + K(v),$$

and so K is subadditive on $\mathbb{R}_+$.

We now prove a converse—our main result. Here, in the context of subadditivity, the important role of the *Heiberg–Seneta condition*, discussed in Sect. 1, is performed by a weaker side-condition: *right-continuity at* 0, a consequence, established in [5]—see also BGT §3.2.1 and [10]. In Theorem 3 this yields coincidence on $\mathbb{R}_+$ of a continuous function, $G(u)$ below, with a (scalar multiple of a) function that was assumed right-continuous at 0. The bulk of the proof is devoted to establishing right-continuity everywhere. A further quantifier weakening occurs in (ii) below.

Theorem 3. (Generalized Goldie Theorem). *If for $\mathbb{A}$ a dense subgroup,*

(i) *$F^* : \mathbb{R}_+ \to \mathbb{R}$ is positive and subadditive with $F^*(0+) = 0$;*
(ii) *F^* satisfies the weakened Goldie equation*

$$F^*(u+v) = g(v)K(u) + F^*(v) \quad (u \in \mathbb{A})(v \in \mathbb{R}_+)$$

for some non-zero K satisfying $(G_{\mathbb{A}})$ with g continuous on $\mathbb{R}$ and $\mathbb{A}_g = \{0\}$;

(iii) *F^* extends K on $\mathbb{A}$:*

$$F^*(x) = K(x) \quad (x \in \mathbb{A}),$$

so that in particular F^* satisfies $(G_{\mathbb{A}})$, and indeed

$$F^*(u+v) = g(v)F^*(u) + F^*(v) \quad (u \in \mathbb{A})(v \in \mathbb{R}_+);$$

—then for some $c > 0, \gamma \geqslant 0$

$$g(x) \equiv e^{-\gamma x} \text{ and } F^*(x) \equiv cH_\gamma(x) = c(1 - e^{-\gamma x})/\gamma \quad (x \in \mathbb{R}_+).$$

Proof. Put

$$G(x) = \int_0^x g(t)dt : \quad G'(x) = g(x).$$

By continuity of g and Th. 1, $K(u+) = K(u)$ for all $u \in \mathbb{A}$, and so $K(0+) = 0$. Also note that F^* is right-continuous (and $F^*(u+) = K(u)$) on $\mathbb{A}$, and on $\mathbb{R}_+$ satisfies

$$\limsup_{v \downarrow 0} F^*(u+v) \leqslant F^*(u) + F^*(0+) = F^*(u).$$

Now proceed as in the Goldie proof of BGT §3.2.1. For any u, u_0 with $u_0 \in \mathbb{A}$ and $u_0 > 0$, define $i = i(\delta) \in \mathbb{Z}$ for $\delta > 0$ so that $(i-1)\delta \leqslant u < i\delta$, and likewise for u_0 define $i_0(\delta)$. As $\mathbb{A}_g = \{0\}$, put $c_0 := K(u_0)/[g(u_0) - 1]$. For $m \in \mathbb{N}$

$$F^*(m\delta) - F^*((m-1)\delta) = g((m-1)\delta)K(\delta),$$

as $m\delta \in \mathbb{A}$, so that on summing

$$F^*(i(\delta)\delta) = K(\delta) \sum_{m=1}^{i} g((m-1)\delta), \tag{**}$$

as $F^*(0) = 0$. Note that as $\delta \to 0$,

$$\delta \sum_{m=1}^{i} g((m-1)\delta) \to \int_0^u g(x)dx \tag{RI}$$

(for 'Riemann Integral'). Without loss of generality $G(u_0) \neq 0$. (Indeed, otherwise $g = 0$ on $\mathbb{A} \cap \mathbb{R}_+$ and so on $\mathbb{R}_+$, so that $F^*(u+) = 0$ on $\mathbb{A} \cap \mathbb{R}_+$; this together with $F^*(u+v) = F^*(v)$ contradicts positivity of F^* on $\mathbb{R}_+$.) Taking limits as $\delta \to 0$ through positive $\delta \in \mathbb{A}$ with $K(\delta) \neq 0$ (see below for $K(\delta) = 0$), we then have, as $G(u_0) \neq 0$,

$$\frac{F^*(i(\delta)\delta)}{F^*(i_0(\delta)\delta)} = \frac{K(\delta)}{K(\delta)} \frac{\sum_{m=1}^{i} g((m-1)\delta)}{\sum_{m=1}^{i_0} g((m-1)\delta)} = \frac{\delta \sum_{m=1}^{i} g((m-1)\delta)}{\delta \sum_{m=1}^{i_0} g((m-1)\delta)} \to \frac{G(u)}{G(u_0)}.$$

Here by right-continuity at u_0

$$\lim F^*(i_0(\delta)\delta) = F^*(u_0) = K(u_0) = c_0[g(u_0) - 1] > 0.$$

So

$$F^*(i(\delta)\delta) \to G(u) \cdot F^*(u_0)/G(u_0).$$

Put $c_1 := c_0[g(u_0) - 1]/G(u_0)$. As before, as $u_0 \in \mathbb{A}$,

$$\begin{aligned} F^*(u) &\geqslant \limsup F^*(i(\delta)\delta) = G(u) \cdot F^*(u_0)/G(u_0) \\ &= G(u)K(u_0)/G(u_0) = G(u)c_0[g(u_0) - 1]/G(u_0) = c_1 G(u). \end{aligned}$$

Now specialize to $u \in \mathbb{A}$, on which, by the above, F^* is right-continuous. Letting $i(\delta)\delta \in \mathbb{A}$ decrease to u, the inequality above becomes an equation:

$$K(u) = F^*(u) = c_1 G(u).$$

This result remains valid with $c_1 = 0$ if $K(\delta) = 0$ for $\delta \in \mathbb{A} \cap I$ for some interval $I = (0, \varepsilon)$, as then $F^*(u) = 0$ by right-continuity on $\mathbb{A}$, because $F^*(i(\delta)\delta) = 0$ for $\delta \in \mathbb{A} \cap I$, by (**).

Now for *arbitrary* $u \in \mathbb{R}$, taking $v \uparrow u$ with $v \in \mathbb{A}$, we have (as $u - v > 0$) that

$$\begin{aligned} F^*(u) &= F^*(u - v + v) = K(v)g(u - v) + F^*(u - v) \quad \text{(by (ii), as } v \in \mathbb{A}) \\ &= c_1 G(v)g(u - v) + F^*(u - v) \to c_1 G(u), \end{aligned}$$

by continuity of G. Thus for *all* $u \in \mathbb{R}$,

$$F^*(u) = c_1 G(u).$$

Thus by (*) of Theorem 1, for some κ

$$c_1 G(u) = F^*(u) = K(u) = \kappa[g(u) - 1] \quad (u \in \mathbb{A}).$$

So, by density and continuity on $\mathbb{R}_+$ of g,

$$\kappa[g(u) - 1] = c_1 G(u) \quad (u \in \mathbb{R}_+).$$

So g is indeed differentiable; differentiation now yields

$$c_1 g(u) = \kappa g'(u) : \quad g'(u) = (c_1/\kappa)g(u) \quad (u \in \mathbb{R}_+),$$

as $\kappa \neq 0$ (otherwise $K(u) \equiv 0$, contrary to assumptions). So with $\gamma := -c_1/\kappa$

$$g(u) = e^{-\gamma u} \text{ and } G(u) = H_\gamma(u) : \quad F^*(u) = c_1 G(u) = c_1[1 - e^{-\gamma u}]/\gamma \quad (u \in \mathbb{R}).$$

As $(1 - e^{-\gamma x})/\gamma$ is subadditive on $\mathbb{R}_+$ iff $\gamma \geqslant 0$ (cf. before Th. 1), $c_1 > 0$. □

Remark. We use above the sequence $s_n = n\delta$, rather than the Beck sequence of Sect. 3 below which is not appropriate here, but see below in Theorem 7 for a Beck-sequence adaptation of the current argument.

Theorem 4. *If g, K are positive, F^* is subadditive on $\mathbb{R}_+$ with $F^*(0+) = 0$, and*

$$F^*(u + v) = g(v)K(u) + F^*(v) \quad (u \in \mathbb{A})(v \in \mathbb{R}_+)$$

—then F^* is increasing and continuous on $\mathbb{R}_+$, and so g is continuous on $\mathbb{R}_+$. In particular, the continuity on $\mathbb{R}_+$ assumed in Theorem 3 above is implied by the positivity of both g and K.

Proof. (i) Since

$$F^*(v + u) - F^*(v) = g(v)K(u) \quad (u \in \mathbb{A})(v \in \mathbb{R}_+),$$

then for $u > 0$ and $u \in \mathbb{A}$

$$F^*(v + u) > F^*(v) \quad (v \in \mathbb{R}).$$

So letting $u \downarrow 0$ through $\mathbb{A}$,

$$F^*(v) \leqslant \lim \sup_{u \downarrow 0 \text{ in } \mathbb{A}} F^*(v + u) \leqslant \limsup_{u \downarrow 0} F^*(v + u) \leqslant F^*(v) + F^*(0+) = F^*(v).$$

So

$$F^*(v+) = F^*(v),$$

i.e. F^* is right-continuous everywhere on $\mathbb{R}_+$. Now for $u \in \mathbb{A}$ with $0 < u < w$,

$$F^*(w-u) < F^*((w-u)+u) = F^*(w).$$

So, for arbitrary $0 < v < w$, and $u \in \mathbb{A}$ with $u > 0$ such that $v < w-u < w$,

$$F^*(v) = F^*(v+) = \liminf\{F^*(w-u) : v < w-u < w, u \in \mathbb{A}\} < F^*(w),$$

as $\mathbb{A}$ is dense. So

$$F^*(v) < F^*(w),$$

i.e. F^* is increasing on $\mathbb{R}_+$.
(ii) Consider $u \in \mathbb{A}_+ := \mathbb{A} \cap \mathbb{R}_+, v \in \mathbb{R}_+$. By $F^*(0+) = 0$, right-continuity on $\mathbb{R}_+$ of F^*, and the weakened Goldie equation, taking limits through $a \in \mathbb{A}_+$

$$\begin{aligned} F^*(u+v) &= \lim_{a \downarrow u} F^*(a+v) = \lim_{a \downarrow u} g(v)F^*(a) + F^*(v) \\ &= g(v)F^*(u) + F^*(v) \quad (u \in \mathbb{A}_+)(v \in \mathbb{R}_+). \end{aligned}$$

So F^* satisfies the Goldie equation of Theorem 1 on $\mathbb{A}_+$.

To conclude: as F^* and g are positive, by the Lemma above, they are continuous on $\mathbb{R}_+$, and so Theorem 2 applies, by Remark 2 after Theorem 1. Alternatively, use [19, Theorem] to deduce the form of F^* and g. □

3. From the Goldie to the Beurling equation

In (GFE), take K and F^* the same—written K. We generalize the $e^{-\gamma\cdot}$ to g, which will serve as an *auxiliary function* (which will reduce to $e^{-\rho\cdot}$ in the case of interest). We now have the *Goldie equation* in the form

$$K(v+u) - K(v) = g(v)K(u) \quad (u, v \in \mathbb{R}_+).$$

For reasons that will emerge (see inter alia § 5), an important generalization arises if on the left the additive action of v on u is made dependent on g:

$$K(v + ug(v)) - K(v) = g(v)K(u) \quad (u, v \in \mathbb{R}_+), \tag{1}$$

so that while g appears twice, K still appears here three times. This form is closely related to a situation with all function symbols identical, φ say (which we will take non-negative):

$$\varphi(v + u\varphi(v)) = \varphi(u)\varphi(v) \quad (u, v \in \mathbb{R}_+). \tag{BFE}$$

Indeed, from here, writing g for φ and with $K(t) \equiv g(t) - 1$ (i.e. as in (*) with $\kappa = 1$), we recover (1).

This (BFE) is our *Beurling functional equation*, a special case of the *Gołąb–Schinzel equation* [(GS)—see [23]] in view of the non-negativity and of the domain being $\mathbb{R}_+$ rather than $\mathbb{R}$ (both considerations arising from the context of Beurling regular variation). Aspects of (GS) with some imposed restriction ('conditionality') on the domain have been studied—see the survey

[16], and the recent papers [18,19,27]. For instance, solutions of the 'conditional' Gołąb–Schinzel equation (i.e. with domain restricted to $\mathbb{R}_+$, but without the non-negativity restriction) were considered and characterized in [17], and shown to be extensible uniquely to solutions with domain $\mathbb{R}$. Note that for any extension to $\mathbb{R}_+ \cup \{0\}$, if $\varphi(0) = 0$, then (BFE) implies $\varphi \equiv 0$; we will therefore usually set $\varphi(0) = 1$, the alternative dictated by the equation $\varphi(0) = \varphi(0)^2$. Solutions $\varphi > 0$ are relevant to the Beurling theory of regular variation [10,31]; their study is much simplified by the following easy result, inspired by a close reading of [14, Prop. 2].

Theorem 5. *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (BFE), then $\varphi(x) \geqslant 1$ for all $x > 0$.*

Proof. Suppose that $\varphi(u) < 1$ for some $u > 0$; then $v := u/(1-\varphi(u)) > 0$ and so, since $v = u + v\varphi(u)$,

$$0 < \varphi(v) = \varphi(u + v\varphi(u)) = \varphi(u)\varphi(v).$$

So cancelling $\varphi(v)$, one has $\varphi(u) = 1$, a contradiction. □

The theorem above motivates the introduction of an important tool in the study of positive solutions φ: the *Beck sequence* $t_m = t_m(u)$, defined for any $u > 0$ recursively by

$$t_{m+1} = t_m + u\varphi(t_m) \text{ with } t_0 = 0,$$

so that

$$\varphi(t_{m+1}) = \varphi(u)\varphi(t_m).$$

By Th. 5, $\{t_m\}$ is divergent, as either $\varphi(u) = 1$ and $t_m = mu$, or

$$t_m = u\frac{\varphi(u)^m - 1}{\varphi(u) - 1} = (\varphi(u)^m - 1) \Big/ \frac{\varphi(u) - 1}{u}, \tag{2}$$

e.g. by Lemma 4 of [31] (cf. a lemma of Bloom: BGT Lemma 2.11.2). In either case, for $u, t > 0$ a unique integer $m = m_t(u)$ exists satisfying

$$t_m \leqslant t < t_{m+1}.$$

This tool will enable us to prove in Theorem 7 below that a positive solution of (BFE) takes the form $\varphi(t) = 1 + \rho t$ for some $\rho \geqslant 0$. Theorem 6 and its Corollary below lay the foundations.

Theorem 6. *If a function $\varphi \geqslant 0$ satisfies (BFE) on $\mathbb{R}_+$ with $\varphi(t) > 1$ for $t \in I = (0, \delta)$ for some $\delta > 0$, then φ is continuous and (strictly) increasing, and $\varphi > 1$.*

Proof. Take $K(t) = \varphi(t) - 1$; then $K > 0$ on I. Writing $x = u$ and $y = v\varphi(u)$,

$$\varphi(x + y) - \varphi(x) = K(y/\varphi(x))\varphi(x).$$

Fix $x \in I$; then $\varphi(x) > 1$, and so $y/\varphi(x) \in I$ for $y \in I$, so that $K(y/\varphi(x)) > 0$. As in Theorem 2, $\varphi(x+y) > \varphi(y)$ for $x, y \in I$, and φ is increasing on a subinterval of I. So φ is continuous at some point $u \in I$, $\varphi(u) > 0$ and

$$\varphi(u) = \lim_{v \downarrow 0} \varphi(u + v\varphi(u)) = \varphi(u) \lim_{v \downarrow 0} \varphi(v) : \quad \varphi(0+) = \lim_{v \downarrow 0} \varphi(v) = 1.$$

So for $x > 0$ with $\varphi(x) > 0$,

$$\lim_{v \downarrow 0} \varphi(x + v\varphi(x)) = \varphi(x) \lim_{v \downarrow 0} \varphi(v) = \varphi(x),$$

and so φ is right-continuous at any x with $\varphi(x) > 0$.

Let $J \supseteq I$ be a maximal interval $(0, \eta)$ on which φ is increasing, and suppose that η is finite. Consider any t with $0 < t < \eta < t + \delta$; then $v := (\eta - t)/\varphi(t) < \eta - t < \delta$, as $\varphi(t) > 1$. As $\varphi(v) > 1$

$$\varphi(\eta) \geqslant \varphi(\eta)/\varphi(v) = \varphi(t + v\varphi(t))/\varphi(v) = \varphi(t) > 1.$$

So φ is bounded above by $\varphi(\eta)$ on $(0, \eta)$. We check φ is left-continuous at η. Let $z_n \downarrow 0$ with $\eta - z_n > 0$; then, as above, $\varphi(z_n) \to 1$. As $\varphi > 1$ on $(0, \eta)$, $u_n := \varphi(\eta - z_n) - 1$ is non-negative and bounded, so by right-continuity at η

$$\varphi(\eta - z_n) = \varphi((\eta - z_n) + z_n\varphi(\eta - z_n))/\varphi(z_n) = \varphi(\eta + z_n u_n)/\varphi(z_n) \to \varphi(\eta).$$

As $\varphi(\eta) > 1$, by right-continuity at $\eta, \varphi > 1$ to the right of η so increasing there, a contradiction. So $J = \mathbb{R}_+$, and φ is right-continuous and increasing.

That φ is left-continuous at any $x > 0$ follows as above but with x replacing η, noting now that as φ is increasing, $u_n := \varphi(x - z_n) - 1 \leqslant \varphi(x) - 1$ is non-negative and bounded. □

Corollary. *If $\varphi > 0$, then φ is continuous, and either $\varphi > 1$, or the value 1 is repeated densely and so $\varphi \equiv 1$.*

Proof. By Theorem 5 $\varphi \geqslant 1$, so φ is (weakly) increasing and so continuous (by the argument for Theorem 6). If $\varphi > 1$ is false, then by Theorem 6 there is no interval $(0, \delta)$ with $\delta > 0$ on which $\varphi > 1$. So there are arbitrarily small $u > 0$ with $\varphi(u) = 1$. Fix $t > 0$. For any u with $\varphi(u) = 1$, choose $n = n_t(u)$ with $t_n := nu \leqslant t < (n+1)u$, as above. Then $\varphi(t_n) = 1$ and $0 \leqslant t - t_n < u$. So the value 1 is taken densely, and so by continuity $\varphi(t) \equiv 1$. □

We now adapt Goldie's argument above to give an easy proof of the following. Theorem 7 below can be derived from [14, Cor 3] or [15, Th1]. There algebraic considerations are key; an analytical proof was provided in [31], but by a different and more complicated route.[3] We include the proof below for completeness, as it is analogous to the Goldie Theorem above and so thematic here. We use a little less than Theorem 6 provides.

[3] For a fuller account of this argument and simplifications of work of Brzdęk and of Brzdęk and Mureńko see the longer version of this paper: arXiv:1405.3947.

Theorem 7. *If $\varphi(t) > 1$ holds for all t in some interval $(0, \delta)$ with $\delta > 0$, and satisfies (BFE) on $\mathbb{R}_+$, then φ is differentiable, and takes the form*

$$\varphi(t) = 1 + \rho t.$$

Proof. Fix $x_0 > 0$ with $\varphi(x_0) \neq 1$. Put

$$K(t) := \varphi(t) - 1.$$

By Theorem 6 K is continuous, so $K(t) \neq 0$ for t sufficiently close to x_0; we may assume also that $\varphi(u) \neq 1$ for all small enough $u > 0$, and so $K(u) \neq 0$ for sufficiently small u.

Let x be arbitrary; in the analysis below x and x_0 play similar roles, so it will be convenient to also write x_1 for x.

For $j = 0, 1$ and any $u > 0$, referring to the Beck sequence $t_m = t_m(u)$ as above, select $i_j = i_j(u) := m_{x_j}(u)$ so that

$$t_{i_j} \leqslant x_j < t_{i_j+1} \quad (j = 0, 1);$$

then

$$\varphi(t_{m+1}) - \varphi(t_m) = \varphi(u)\varphi(t_m) - \varphi(t_m) = K(u)\varphi(t_m).$$

Summing,

$$K(t_m) = \varphi(t_m) - 1 = \varphi(t_m) - \varphi(t_0) = K(u) \sum_{n=0}^{m-1} \varphi(t_n).$$

As noted, for all small enough u, $K(t_{i_0})$ is non-zero (this uses compactness of $[0, x_0]$). Cancelling $K(u)$ below (as also $K(u)$ is non-zero), introducing u in its place (to get the telescoping sums), and recalling $t_{n+1} - t_n = u\varphi(t_n)$,

$$\frac{K(t_{i_1})}{K(t_{i_0})} = \frac{K(u)\sum_{n=0}^{i_1-1}\varphi(t_n)}{K(u)\sum_{n=0}^{i_0-1}\varphi(t_n)} = \frac{\sum_{n=0}^{i_1-1} u\varphi(t_n)}{\sum_{n=0}^{i_0-1} u\varphi(t_n)} = \frac{\sum_{n=0}^{i_1-1}(t_{n+1} - t_n)}{\sum_{n=0}^{i_0-1}(t_{n+1} - t_n)} = \frac{t_{i_1}}{t_{i_0}}.$$

Passing to the limit as $u \to 0$, by continuity

$$K(x_1)/K(x_0) = x_1/x_0.$$

Setting $\rho_0 := K(x_0)/x_0$,

$$\varphi(x) - 1 = K(x) = \rho_0 x : \varphi(x) = 1 + \rho_0 x.$$

□

4. Extensions of the Goldie and Beurling equations

Below we consider two generalizations of the Beurling equation inspired by Goldie's equation, relevant to Beurling regular variation [10]. Our notation here is adjusted to coincide with that used in the companion paper [32]. The first uses three functions:

$$K(v+u\eta(v))-K(v)=\kappa(u)\eta(v) \quad (u,v\in\mathbb{R}_+). \tag{GBE}$$

Here the choice $\eta = K = \varphi$ with $\kappa = \varphi - 1$ recovers the Beurling equation. One can also form a Pexider-like generalization ([28, 13.3], or [3, 4.3]) for the right-hand side above, replacing the occurrence of η there with an additional function ψ:

$$K(v+u\eta(v))-K(v)=\kappa(u)\psi(v) \quad (u,v\in\mathbb{R}_+). \tag{GBE-P}$$

This equation when $\eta = 1$ is studied by algebraic means in [19] generalizing [1]. Here $\eta = 1$ yields Goldie's equation $G_{\mathbb{R}}$ with auxiliary $g = \psi$ when $\kappa = K$, and also yields the weakend Goldie equation of Theorem 3 with auxiliary $g = \psi$ for $K = F^*, \kappa$ the Goldie kernel of F, and $\mathbb{A} = \mathbb{R}$ (cf. (GFE) in §1); furthermore, $\psi = \eta = K, \kappa = \psi - 1$ yields the Beurling equation (in ψ). Below we assume without loss of generality that $K(0) = 0$ (otherwise replace $K(x)$ in either equation by $K(x) - K(0)$).

Theorem 8. *Consider the functional equation (GBE-P) with $\kappa(u) > 0$ for $u > 0$ near $0, \eta, \psi$ continuous on $\mathbb{R}_+\cup\{0\}$ with ψ non-negative and η positive thereon. With*

$$H(x) := \int_0^x \psi(t)\frac{dt}{\eta(t)}, \quad \textit{for} \quad x \geqslant 0,$$

any solution K is differentiable, and subject to $K(0) = 0$ takes the form $K(x) = cH(x)$, for some constant $c \geqslant 0$; furthermore, if $\psi(0) \neq 0$, then κ is differentiable with $\kappa(0) = 0, \kappa'(0) = c$, and $\kappa(x) = bH(x\eta(0))$ for $b := c/\psi(0)$.

Proof. Since for $v, w > 0$, assuming $\kappa(0+)$ exists,

$$K(v+w)-K(v)=\kappa(w/\eta(v))\psi(v),$$
$$K(v+)-K(v)=\kappa(0+)\psi(v), \quad K(u)-K(u-)=\kappa(0+)\psi(u) \quad (u,v\in\mathbb{R}_+),$$

by continuity of ψ and η, and for any $v > 0$ and all small enough $w > 0$,

$$K(v+w) \geqslant K(v).$$

So K is locally increasing (i.e. 'non-decreasing') on $\mathbb{R}_+$, and so is increasing and so continuous on a dense set $D \subseteq \mathbb{R}_+$. So $\kappa(0+)$ exists and for $u, v \in D$,

$$\kappa(0+)\psi(v)=K(v+)-K(v)=0, \quad \kappa(0+)\psi(u)=K(u-)-K(u)=0.$$

So, either $\kappa(0+) = 0$, or $\psi \equiv 0$ on D, and then $\psi \equiv 0$ on $[0, \infty)$, by continuity. In the former case, $K(v+) = K(v) = K(v-)$ for all $v \in \mathbb{R}_+$, so that K is continuous on $\mathbb{R}_+$. In the latter case, for $w > v > 0$ substituting $u = w-v/\eta(v)$ into (GBE-P) gives $K(w) - K(v) = 0$, so K is constant on $\mathbb{R}_+$. So in either case K is continuous on $\mathbb{R}_+$.

Now consider for $u > 0$ the Beck sequence

$$t_{n+1}(u)=t_n(u)+u\eta(t_n(u)), \quad t_0=0,$$

which is increasing as $\eta > 0$. For any $t, u > 0$ we claim there is $m = m_t(u)$ with

$$t_m(u) \leqslant t < t_{m+1}(u).$$

For otherwise, with t, u fixed as above, the increasing sequence $\{t_n(u)\}_n$ is bounded by t and, putting $\tau := \sup t_n(u) \leqslant t$,

$$\eta(t_n(u)) = \frac{1}{u}[t_{n+1}(u) - t_n(u)] \to 0,$$

contradicting lower boundedness of η near τ (as η is continuous and positive). Next observe that, since η is bounded on $[0, t]$, by M_t say,

$$t_{m+1}(u) - t_m(u) = u\eta(t_m(u)) \leqslant uM_t \to 0 \quad \text{as } u \downarrow 0.$$

Now fix x_0, $x_1 > 0$. Select $i_0 = i_0(u)$ and $i_1 = i_1(u)$ so that for $j \in \{0, 1\}$

$$t_{i_j} \leqslant x_j < t_{i_j+1}.$$

Then

$$K(t_{m+1}) - K(t_m) = \kappa(u)\psi(t_m).$$

Summing, and setting $f(t) := \psi(t)/\eta(t) \geqslant 0$ (as η is positive),

$$K(t_m) = K(t_m) - K(t_0) = \kappa(u)\sum_{n=0}^{m-1}\psi(t_n) = \frac{\kappa(u)}{u}\sum_{n=0}^{m-1}u\eta(t_n)f(t_n).$$

For all small enough u we have $\kappa(u)$ non-zero, so

$$\begin{aligned}\frac{K(t_{i_1})}{K(t_{i_0})} &= \frac{\kappa(u)\sum_{n=0}^{i_1-1}\psi(t_n)}{\kappa(u)\sum_{n=0}^{i_0-1}\psi(t_n)} = \frac{\sum_{n=0}^{i_1-1}u\eta(t_n)f(t_n)}{\sum_{n=0}^{i_0-1}u\eta(t_n)f(t_n)} \\ &= \frac{\sum_{n=0}^{i_1-1}(t_{n+1}-t_n)f(t_n)}{\sum_{n=0}^{i_0-1}(t_{n+1}-t_n)f(t_n)} \to \frac{\int_0^{x_i} f(t)dt}{\int_0^{x_0} f(t)dt} = \frac{H(x_1)}{H(x_0)},\end{aligned}$$

where passage to the limit in the rightmost terms is as $u \downarrow 0$. Here, as in Theorem 3, we assume without loss of generality that $H(x_0) > 0$ (otherwise $\psi \equiv 0$ on $[0, \infty)$, implying that K is constant and yielding, as above, the trivial case $K \equiv 0$). Passing to the limit as $u \downarrow 0$ in the leftmost term, by continuity of K,

$$K(x_1)/K(x_0) = H(x_1)/H(x_0).$$

Setting $c := K(x_0)/H(x_0)$,

$$K(x) = cH(x),$$

is valid for $x \geqslant 0$, as $K(0) = 0$. This is differentiable.

As K is continuous at 0, Eq. (GBE-P) holds also for $v = 0$. So if $\psi(0) \neq 0$,

$$\kappa(u) = \frac{c}{\psi(0)}\int_0^{u\eta(0)} f(t)dt = c/\psi(0)H(u\eta(0)) = bH(u\eta(0)).$$

So $c \geqslant 0$ as ψ and η are non-negative and $\kappa(u) > 0$ for $u > 0$ near 0. The right-hand side is differentiable in u; so, for some $\theta = \theta(u)$ with $0 < \theta(u) < 1$,

$$\kappa(u) = c \cdot u\eta(0)/\psi(0) \cdot f(u\theta\eta(0)) : \quad \kappa(u)/u = cf(u\theta_0\eta(0))/f(0).$$

As ψ and η are continuous at 0, taking limits as $u \downarrow 0$:

$$\lim_{u\downarrow 0} \kappa(u)/u = c,$$

i.e. $\kappa'(0) = c$ as $\kappa(0) = 0$ (right-sidedly). □

Theorem 8′ below is somewhat more than a partial converse; injectivity below is assumed for the convenience of solving (†) below for f directly. The full analysis of this equation in [18] reveals that we omit here only the obvious 'trivial' cases $f \equiv 1$ and $f \equiv 0$. (Here f is a relative flow velocity—see [4, §4.30].)

In Theorem 8 above one can simplify further to assume without loss of generality $\eta(0) = \psi(0) = 1$ (replacing: ψ by $\psi/\psi(0)$, η by $\eta/\eta(0)$, u by $\eta(0)u$ and $\kappa(\cdot)$ by $\kappa(./\eta(0))\psi(0)$); then $b = c$. This we do below.

Theorem 8′. *With the assumptions of Theorem 8 and with $\psi(0) = \eta(0) = 1$, if $\kappa = K \equiv cH$, and $f = \psi/\eta$ is injective, then $\eta(t) \equiv 1 + \rho t$ for some $\rho \geqslant 0$, and so for some γ with suitable interpretation for $\gamma = 0$:*

$$\psi(t) \equiv e^{\gamma t}, \quad H(x) \equiv \int_0^x e^{\gamma t} dt = (e^{\gamma x} - 1)/\gamma, \quad \text{if } \rho = 0;$$

$$\psi(t) \equiv (1 + \rho t)^{\gamma}, \quad H(x) \equiv \int_0^x (1 + \rho t)^{\gamma - 1} dt = ((1 + \rho x)^{\gamma} - 1)/\rho\gamma, \quad \text{if } \rho \in (0, \infty).$$

Proof. Note that $f(t) \geqslant 0$. Substituting for K and κ into (GBE-P),

$$\int_v^{v+u\eta(v)} f(t)dt = \psi(v) \int_0^u f(t)dt.$$

Differentiating w.r.t. $u, \eta(v)f(v + u\eta(v)) = \psi(v)f(u)$, i.e., as η is positive,

$$f(v + u\eta(v)) = f(u)f(v). \tag{†}$$

Define a *circle operation* by

$$x \circ y := v + u\eta(v)$$

(for the extensive algebraic background, including *Popa groups*, see [32]). As

$$f(u \circ v) = f(u)f(v) = f(v \circ u)$$

and f is injective,

$$v + u\eta(v) = u + v\eta(u) : \quad (1 - \eta(v))/v = (1 - \eta(u))/u \quad (u, v > 0).$$

So $\eta(t) = 1 + \rho t$ for some $\rho \geqslant 0$, and so $\circ$ is a group operation on the Popa group $\mathbb{G}_\rho = \mathbb{R}\backslash\{1/\rho\}$, which is isomorphic either to $(\mathbb{R}\backslash\{0\}, \times)$ for $\rho > 0$ under the 'shift-scale' map η, or to $(\mathbb{R},+)$ for $\rho=0$ (identically)—cf. [32]. Substituting ψ/η for f,

$$\frac{\psi(v+u\eta(v))}{\eta(v+u\eta(v))}=\frac{\psi(u)\psi(v)}{\eta(u)\eta(v)},$$

so

$$\psi(v+u\eta(v))=\psi(u)\psi(v),$$

since η satisfies (BFE). So

$$\psi(v+u\eta(v))=\psi(u\circ v)=\psi(u)\psi(v).$$

To conclude: passage through the relevant isomorphisms mentioned above converts this to a Cauchy functional equation determining ψ as asserted. Alternatively, as $\psi:\mathbb{G}_\rho\to\mathbb{G}_\infty$ is a continuous homomorphism, its form may be read off from [18, Prop. 2.1], or more simply [32, Prop. A]. □

With the usual L'Hospital convention, $\gamma=0$ is permissible in the first case above (yielding a linear function), and also in the second case (yielding there $\log(1+\rho x)/\rho$).

Taking $\psi=\eta$ so that $f\equiv 1$, which is already continuous, we obtain as a corollary Theorem 9 below, which needs only local boundedness above and away from 0, rather than continuity in η (to justify the use of the Beck sequence). The proof is similar to but simpler than that of Theorem 8.

Theorem 9. *For the functional equation* (GBE) *above with* $\eta>0$ *locally bounded above and away from* 0 *on* $\mathbb{R}_+$, *and* $\kappa(u)>0$ *for* $u>0$ *near* 0*:*

(i) *any solution with* $K(0)=0$ *is linear:* $K(x)=cx$*;*
(ii) $\kappa(u)=cu$.

In particular, for $K=\eta$ *and* $\kappa=\eta-1$, *the solution of the Beurling equation is* $\eta(u)=1+cu$.

Proof. Omitted—details in the full version of the paper: arXiv:1405.3947. □

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