Aequat. Math. 90 (2016), 25–34
0001-9054/16/010025-10
published online April 7, 2015
DOI 10.1007/s00010-015-0349-z

Aequationes Mathematicae

Van Vleck's functional equation for the sine

Henrik Stetkær

We dedicate this paper to professor Roman Ger on the occasion of his 70th birthday

Abstract. We solve Van Vleck's functional equation on semigroups with an involution in terms of multiplicative functions.

Mathematics Subject Classification. 39B32, 39B52.

Keywords. Functional equation, Sine function, Van Vleck, Involution.

1. Introduction

The American mathematician Van Vleck [14,15] studied around 1910 the continuous solutions $f : \mathbb{R} \to \mathbb{R}$, $f \neq 0$, of the functional equation

$$f(x - y + z_0) - f(x + y + z_0) = 2f(x)f(y), \quad x, y \in \mathbb{R}, \tag{1}$$

where $z_0 > 0$ is fixed, with a view to characterize the sine function on the real line. He showed first that all solutions are periodic with period $4z_0$, and then he selected for his study any continuous solution with minimal period $4z_0$. He proved that such a solution has to be the sine function

$$f_0(x) = \sin\left(\frac{\pi}{2z_0}x\right), \ x \in \mathbb{R}.$$

Actually the continuous non-zero solutions of (1) form a countable set. They are sine functions all of them, but their periods are integral fractions of the period of f_0 (this statement is derived in Example 9 below).

We shall in this paper solve a generalization of (1), in which $\mathbb{R}$ is replaced by a semigroup and the group inversion $x \mapsto -x$ of $(\mathbb{R}, +)$ by an involution of the semigroup. It turns out that this simple framework suffices for the study, because it allows us to establish a link between the generalization (2) and d'Alembert's functional equation (3). Knowledge of the solutions of (3) enables us to solve (2). Our way of proceeding is computational owing to the very general setup, while Van Vleck exploited properties of continuous, real-valued functions on the real line.

Throughout our paper the setup and the notation are as follows:

Set Up. *S is a semigroup, $\tau : S \to S$ is an involution of S, and $z_0 \in S$ denotes a fixed element in the center of S.*

We seek the solutions $f : S \to \mathbb{C}$ of the functional equation

$$f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in S. \tag{2}$$

We recall that a semigroup is a non-empty set equipped with an associative operation. An involution on a semigroup S is a map $\tau : S \to S$ such that $\tau(xy) = \tau(y)\tau(x)$ and $\tau(\tau(x)) = x$ for all $x, y \in S$. On a group the group inversion $x \mapsto x^{-1}$ is an involution. The solutions of (2) will be expressed in terms of multiplicative functions on S, i.e., maps $\chi : S \to \mathbb{C}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$. It is well known that the multiplicative functions $\chi \neq 0$ on a group G are its characters, i.e., the homomorphisms $\chi : G \to (\mathbb{C} \setminus \{0\}, \cdot)$.

The functional equation (2) is a natural generalization of (1), so we call it *Van Vleck's functional equation*. As mentioned above we shall relate it to a version of d'Alembert's functional equation, more precisely to

$$g(xy) + g(x\tau(y)) = 2g(x)g(y), \quad x, y \in S. \tag{3}$$

Concerning d'Alembert's functional equation we shall use results by Corovei [5, Teorema 1] and Kannappan [8, Theorem 2], or rather generalizations of their results (see [13, Proposition 8.14(a)] and [13, Theorem 9.21(a)] respectively for details). Corovei's paper contains a criterion for g to be abelian, while Kannappan's gives us the form of any abelian g.

Other recent advances in the study of d'Alembert's functional equation have been made by Davison [6], Stetkær [12] and Bahyrycz and Brzdęk [1]. [4] considered operator valued solutions.

The present paper is not the first place to treat Van Vleck's functional equation since Van Vleck [14,15]. The solutions $f : \mathbb{R} \to \mathbb{C}$ of his classical functional equation (1) were derived in the textbooks Kannappan [9, Theorem 3.53] and Stetkær [13, Exercise 9.18]. The $L^\infty(\mathbb{R})$ solutions of (1) were listed by Gajda [7, Corollary 2], who got them in his studies of the functional equation

$$\int_G f(x+y-s)\,d\mu(s) + \int_G f(x-y+s)\,d\mu(s) = f(x)f(y), \quad x, y \in G,$$

where G is a locally compact, abelian group, and μ is a complex-valued, regular Borel measure on G of bounded variation. Nagy [10] studied Banach algebra valued solutions of (1).

The groups in [7] are abelian, but the literature also contains results for groups that need not be abelian. Stetkær [13, Exercise 9.18] found the complex-valued solutions of (2), when S is a group and τ the group inversion. Perkins and Sahoo [11] replaced the group inversion by the more general concept of an involution. They derived d'Alembert's functional equation (3) from (2) on groups with an involution [11, Theorem 1], and then used (3) to find the abelian, complex-valued solutions of (2) [11, Corollary 1].

The present paper extends the above results of Perkins and Sahoo [11] about Van Vleck's functional equation (2) and of [13, Exercise 9.18]. Novel features of our study of (2) are that we

1. for semigroups with an involution (instead of groups with an involution) derive an explicit formula for the solutions of (2) in terms of multiplicative maps (Theorem 4),
2. prove that the decomposition of any solution into multiplicative functions in Theorem 4 is essentially unique (Proposition 5), and
3. take continuity into account (Proposition 6) which facilitates the treatment of examples.

It follows from Theorem 4 that all solutions of (2) are abelian, so the restriction to abelian solutions in Perkins and Sahoo [11, Corollary 1] is not needed.

2. Preliminary results

In this section the setup and the corresponding notation are as described in the Introduction. In particular z_0 belongs to the center of S. Then so does $\tau(z_0)$, a fact that we use without explicit mentioning below.

Lemma 1 presents properties of any solution $f \neq 0$ of (2). Lemma 3 discusses the corresponding solution g of d'Alembert's functional equation (3).

Lemma 1. *Let $f \neq 0$ be a solution of (2). Then f is odd with respect to τ (meaning that $f \circ \tau = -f$), and the following formulas hold for all $x \in S$:*

$$f(z_0) \neq 0, \tag{4}$$

$$f(z_0^2) = 0, \tag{5}$$

$$f(x\tau(z_0)z_0) = f(z_0)f(x), \tag{6}$$

$$f(xz_0^2) = -f(z_0)f(x), \tag{7}$$

$$f(\tau(x)z_0) = f(xz_0). \tag{8}$$

Assume S is a group and τ the group inversion. Then $f(z_0) = 1$ for any solution $f \neq 0$ of (2). In particular $f(xz_0^2) = -f(x)$ for all $x \in S$, so that f is periodic with period z_0^4.

Proof. If we replace y by $\tau(y)$ in (2) the left hand side changes sign. This implies that $f(\tau(y)) = -f(y)$. In other words that $f \circ \tau = -f$.

We continue by deriving some identities involving f from (2). Let $x, y \in S$ be arbitrary. Taking $x = \tau(z_0)$ in (2) and using that f is odd with respect to τ we find (6).

Setting $y = z_0$ in (2) gives us $f(x\tau(z_0)z_0) - f(xz_0^2) = 2f(x)f(z_0)$. When we apply (6) to the left hand side of this we obtain (7).

We will next show (4). Replacing x by xz_0 in (2) and applying (7) yield (the right hand side first)

$$2f(xz_0)f(y) = f\left(x\tau(y)z_0^2\right) - f\left(xyz_0^2\right) = -f(z_0)f(x\tau(y)) + f(z_0)f(xy).$$

If $f(z_0) = 0$, then the right hand side vanishes. From the left hand side we see that $f(xz_0) = 0$ for all $x \in S$. Thus the left hand side of (2) vanishes. A glance at the right hand side of (2) reveals that $f = 0$. But this contradicts the hypothesis $f \neq 0$. We conclude that $f(z_0) \neq 0$.

To show (5) we note that $f(\tau(x)x) = 0$ for all $x \in S$, since f is odd with respect to τ. Using first this and then (7) we find that

$$0 = f\left(\tau\left(z_0^2\right)z_0^2\right) = -f(z_0)f\left(\tau\left(z_0^2\right)\right) = f(z_0)f\left(z_0^2\right),$$

which implies that $f(z_0^2) = 0$, because $f(z_0) \neq 0$.

We get the formula $f(z_0)f(\tau(y)z_0) = f(z_0)f(yz_0)$, when we combine (2) with $x = z_0^2$ and (7). From that we obtain (8), since $f(z_0) \neq 0$.

Let finally S be a group and τ its group inversion. From (6) we infer

$$f(z_0)f(y) = f(\tau(z_0)yz_0) = f\left(z_0^{-1}yz_0\right) = f(y) \text{ for all } y \in S.$$

Since $f \neq 0$, we get $f(z_0) = 1$. □

In general $f(z_0) \neq 1$, even on abelian groups (Example 10).

Remark 2. Assume that S has a neutral element e. If $z_0 = e$, then the only solution of (2) is $f = 0$ (take $x = e$ in (2) and use that f is odd).

It is illuminating to test the conclusions of the abstract Lemma 1 on the concrete example $f(x) = \sin x$ which satisfies (1) in the case of $z_0 = \pi/2$. Indeed, $\sin x$ is odd, $\sin\frac{\pi}{2} = 1 \neq 0$, $\sin(2\frac{\pi}{2}) = 0$, $\sin(x + 2\frac{\pi}{2}) = -\sin\frac{\pi}{2}\sin x$ and $\sin(-x + \frac{\pi}{2}) = \sin(x + \frac{\pi}{2})$ for all $x \in \mathbb{R}$.

The formula $\cos x = \sin(x + \pi/2)$ expresses that translating sine on $\mathbb{R}$ by a suitable distance produces cosine. The proof of the next lemma reveals that an analogous result holds for the translate $x \mapsto f(xz_0)$ of any solution $f \neq 0$ of (2): A scalar multiple (called g) of the translate of f satisfies d'Alembert's functional equation (3) (also known as the cosine equation).

Lemma 3. *Let $f \neq 0$ be a solution of (2). We may define the function $g : S \to \mathbb{C}$ by*

$$g(x) := \frac{f(xz_0)}{f(z_0)} \quad \textit{for} \quad x \in S,$$

because $f(z_0) \neq 0$ (*Lemma* 1).

g may be written in the form $g = (\chi + \chi \circ \tau)/2$, *where* $\chi : S \to \mathbb{C}$, $\chi \neq 0$, *is a multiplicative function.*

Proof. We note that

$$f(z_0)^2[g(xy) + g(x\tau(y))] = f(z_0)[f(xyz_0) + f(x\tau(y)z_0)]$$
$$= f(z_0)f(x\tau(y)z_0) + f(z_0)f(xyz_0).$$

Applying (6) to the first term on the right and (7) to the second one we continue the computations as follows

$$= f(x\tau(y)z_0\tau(z_0)z_0) - f(xyz_0z_0^2)$$
$$= f((xz_0)\tau(yz_0)z_0) - f((xz_0)(yz_0)z_0) = 2f(xz_0)f(yz_0).$$

Dividing by $f(z_0)^2$ we arrive at (3). And $g \neq 0$, because

$$g(z_0^2) = f(z_0^2z_0)/f(z_0) = -f(z_0)f(z_0)/f(z_0) = -f(z_0) \neq 0.$$

We next prove that g is abelian. As a solution of (3) g is a pre-d'Alembert function by [13, Proposition 9.17(c)], so according to [13, Proposition 8.14(a)] (a generalization of Corovei [5, Teorema 1]) it suffices to prove that $g(z_0)^2 \neq d(z_0)$, where $d(x) := 2g(x)^2 - g(x^2)$ for $x \in S$. This is easily done. Indeed, $g(z_0) = f(z_0^2)/f(z_0) = 0/f(z_0) = 0$, while

$$d(z_0) = 2g(z_0)^2 - g(z_0^2) = 0 - (-f(z_0)) = f(z_0) \neq 0.$$

Now g is an abelian solution of d'Alembert's functional equation (3), so the existence of χ is immediate from [13, Theorem 9.21(a)] (a generalization of Kannappan [8, Theorem 2]). $\chi \neq 0$, because $g \neq 0$. □

3. The main results

In this section the setup and the corresponding notation are as described in the Introduction. We solve Van Vleck's functional equation (2) by expressing its solutions in terms of multiplicative functions. Furthermore we show that the decomposition of a solution into multiplicative functions is essentially unique, and we discuss briefly continuous solutions.

Theorem 4. *Under our setup the non-zero solutions* $f : S \to \mathbb{C}$ *of* (2) *are the functions of the form*

$$f = \chi(\tau(z_0))\,\frac{\chi - \chi \circ \tau}{2}, \tag{9}$$

where $\chi : S \to \mathbb{C}$ *is a multiplicative function such that* $\chi(z_0) \neq 0$ *and* $\chi(\tau(z_0)) = -\chi(z_0)$. *Furthermore* $f(z_0) = \chi(z_0\tau(z_0))$.

Assume S *is a group and* τ *the group inversion. Then the conditions on* χ *reduce to* $\chi(z_0^2) = -1$. *In this case* $f(z_0) = 1$ *for any solution* $f \neq 0$ *of* (2).

Proof. Let $f \neq 0$ be a solution of (2). Taking $y = z_0$ in (2) we get by the definition of g in Lemma 3 that

$$f(x) = \frac{1}{2f(z_0)}[f(x\tau(z_0)z_0) - f(xz_0z_0)] = \frac{1}{2}[g(x\tau(z_0)) - g(xz_0)].$$

When we here substitute the formula $g = (\chi + \chi \circ \tau)/2$ from Lemma 3, we get after elementary reductions that

$$f = \frac{\chi(z_0) - \chi(\tau(z_0))}{2} \frac{\chi \circ \tau - \chi}{2}. \tag{10}$$

Note for use below that $\chi(z_0) \neq \chi(\tau(z_0))$, and that $\chi \neq \chi \circ \tau$, since $f \neq 0$.

Writing $f = c[\chi \circ \tau - \chi]$, where $c := [\chi(z_0) - \chi(\tau(z_0))]/4 \neq 0$, we find from (10) that

$$f(\tau(x)z_0) = c[\chi(\tau(z_0))\chi(x) - \chi(z_0)\chi(\tau(x))], \text{ and that}$$
$$f(xz_0) = c[\chi(\tau(z_0))\chi(\tau(x)) - \chi(z_0)\chi(x)].$$

We infer from the formula (8) that

$$\chi(\tau(z_0))\chi(x) - \chi(z_0)\chi(\tau(x)) = \chi(\tau(z_0))\chi(\tau(x)) - \chi(z_0)\chi(x),$$

which reduces to

$$[\chi(\tau(z_0)) + \chi(z_0)][\chi - \chi \circ \tau] = 0.$$

Since $\chi \neq \chi \circ \tau$, we conclude that $\chi(\tau(z_0)) = -\chi(z_0)$, so that (10) becomes (9) as desired.

The converse [that any function f of the form (9) is a solution of (2)] is done by an elementary computation [substitution of the formula (9) for f into (2) using the assumption $\chi(\tau(z_0)) = -\chi(z_0)$] is left out.

Furthermore

$$f(z_0) = \chi(\tau(z_0)) \frac{\chi(z_0) - \chi(\tau(z_0))}{2} = \chi(\tau(z_0)) \frac{\chi(z_0) + \chi(z_0)}{2} = \chi(z_0\tau(z_0)).$$

Assume S is a group and τ the group inversion. Then the condition on χ becomes $\chi(z_0^{-1}) = -\chi(z_0)$, which is equivalent to $\chi(z_0)^2 = -1$. Lemma 1 takes care of the last statement of the theorem. □

Example 9 demonstrates that the condition $\chi(z_0^2) = -1$ in Theorem 4 is a serious restriction on χ: Only a countable subset of the multiplicative functions on $\mathbb{R}$ passes this restriction.

Consider the multiplicative function χ in Theorem 4. The formula (10) shows that the solution f remains the same, if χ and $\chi \circ \tau$ are interchanged. The following proposition proves that this is the only ambiguity in the choice of χ, given the solution $f \neq 0$.

Proposition 5. *Let $\chi_1, \chi_2 : S \to \mathbb{C}$ be multiplicative maps, and let $c_1, c_2 \in \mathbb{C}$. If*

$$c_1 \frac{\chi_1 - \chi_1 \circ \tau}{2} = c_2 \frac{\chi_2 - \chi_2 \circ \tau}{2} \neq 0, \tag{11}$$

then $\chi_1 = \chi_2$ *or* $\chi_1 = \chi_2 \circ \tau$.

Proof. Multiplying out we get $c_1\chi_1 - c_1\chi_1 \circ \tau - c_2\chi_2 + c_2\chi_2 \circ \tau = 0$. If χ_1, χ_2, $\chi_1 \circ \tau$, $\chi_2 \circ \tau$ are four different multiplicative maps, then the individual terms vanish by [13, Theorem 3.18(a)]. In particular $c_1\chi_1 = c_1\chi_1 \circ \tau = 0$. But that contradicts the assumption (11). So at least two of the four multiplicative maps agree. The possibilities $\chi_1 = \chi_1 \circ \tau$ and $\chi_2 = \chi_2 \circ \tau$ contradict (11), and left are the ones of the proposition. □

Finally we discuss continuous solutions.

Proposition 6. *Assume in addition to the setup that S is a topological semigroup and that $\tau : S \to S$ is continuous. Then the solution $f \neq 0$ of (2) described in (9) of Theorem 4 is continuous, if and only if χ is continuous.*

Proof. Clearly χ continuous implies that f is continuous. Conversely, if $f \neq 0$ is continuous, then $\chi - \chi \circ \tau$ is continuous. Now [13, Theorem 3.18(d)] tells us that χ is continuous. □

Remark 7. Van Vleck [14] observed that if f is a solution of (1), then $-f$ is a solution of the version of (1) in which z_0 is replaced by $-z_0$. That is the reason why he could assume without loss of generality that z_0 was positive.

The same observation can be made in the general setup. Let f be a solution of (2). Since f is odd with respect to τ (Lemma 1) and abelian (by the formula of Theorem 4) we find for all $x, y \in S$ that

$$\begin{aligned}(-f)(x\tau(y)\tau(z_0)) &- (-f)(xy\tau(z_0)) = (f \circ \tau)(x\tau(y)\tau(z_0)) - (f \circ \tau)(xy\tau(z_0))\\ &= f(z_0 y\tau(x)) - f(z_0\tau(y)\tau(x)) = -(f(\tau(x)\tau(y)z_0) - f(\tau(x)yz_0))\\ &= -2f(\tau(x))f(y) = 2f(x)f(y) = 2(-f(x))(-f(y)),\end{aligned}$$

which shows that $-f$ satisfies the version of (2) in which z_0 is replaced by $\tau(z_0)$.

4. Examples

Example 8. The solution formula (9) is an extension to our general setup of Euler's formula for sine. To illustrate this point we consider Van Vleck's original functional equation (1) with $z_0 = \pi/2$, i.e.,

$$f(x - y + \pi/2) - f(x + y + \pi/2) = 2f(x)f(y), \ x, y \in \mathbb{R}. \tag{12}$$

Let $\chi(x) := \exp(ix)$, $x \in \mathbb{R}$. The condition on χ in Theorem 4 is satisfied, because $\chi(z_0) = \exp(i\pi/2) = i$ and $\chi(\tau(z_0)) = \chi(-z_0) = \exp(-i\pi/2) = -i$, so the solution formula (9) provides a solution of (12). That solution is

$$f(x) = \frac{1}{i}\,\frac{e^{ix} - e^{-ix}}{2} \quad \text{for} \quad x \in \mathbb{R}.$$

The right hand side is indeed Euler's classic formula for sine. We have in this roundabout way found that $f(x) = \sin x$ is a solution of (12), but this fact is of course easy to verify directly.

Example 9. In this example we return to Van Vleck's original functional equation (1), where $S = (\mathbb{R}, +)$ and $\tau(x) = -x$ for any $x \in \mathbb{R}$. We shall produce the continuous solutions $f : \mathbb{R} \to \mathbb{C}$ of (1) by the help of Theorem 4 and Proposition 6. If $z_0 = 0$, the only solution is $f = 0$ (by Remark 2), so from now on we let $z_0 \in \mathbb{R} \setminus \{0\}$.

According to Theorem 4 and Proposition 6 the continuous solutions $f \neq 0$ are the functions of the form

$$f(x) = \chi(-z_0)\frac{\chi(x) - \chi(-x)}{2}, \tag{13}$$

where $\chi : \mathbb{R} \to \mathbb{C}$ is any continuous character such that $\chi(2z_0) = -1$. It is known that the continuous characters of $\mathbb{R}$ are the exponential functions $x \mapsto \exp(\lambda x)$, where $\lambda \in \mathbb{C}$ (see for instance [13, Example 3.7(a)]). The condition $\chi(2z_0) = -1$ restricts the set of continuous characters to

$$\chi_n(x) = \exp\left(i\,\frac{(2n+1)\pi}{2z_0}\,x\right),\ x \in \mathbb{R},\ \text{where } n \text{ ranges over } \mathbb{Z}.$$

The corresponding solutions are by (13) the sine functions

$$f_n(x) = (-1)^n \sin\left(\frac{(2n+1)\pi}{2z_0}\,x\right),\ x \in \mathbb{R},\ n \in \mathbb{N} \cup \{0\}.$$

We have here restricted the index set from $\mathbb{Z}$ to $\mathbb{N} \cup \{0\}$, because $f_{-n} = f_{n-1}$ for all $n \in \mathbb{Z}$.

It might be noticed that f_0 has period $4z_0$, while the period of f_n is an integral fraction of $4z_0$, viz, $4z_0/(2n+1)$.

Example 9 is discussed in the references [7,9] and [13] plus of course the original papers [14,15] by Van Vleck.

Example 10. We have in Lemma 1 seen that $f(z_0) = 1$ for all non-zero solutions of (2), when S is a group and τ the group inversion. The present example reveals that in general $f(z_0) \neq 1$.

We consider the additive group $(\mathbb{R}^2, +)$ equipped with the involution $\tau(x, y) = (x, -y)$. Take $z_0 = (1, \pi/2)$ and

$$\chi(x, y) := e^{x+iy},\ x, y \in \mathbb{R}^2.$$

An easy computation shows that $\chi(\tau(z_0)) = -\chi(z_0)$, so the hypothesis of Theorem 4 is satisfied. According to Theorem 4 the corresponding solution f of (2) has

$$f(z_0) = \chi(z_0 + \tau(z_0)) = \chi(2, 0) = e^2,$$

so we have here an example in which $f(z_0) \neq 1$.

Example 11. By this example we indicate that our theory applies not just to groups, and that non-zero solutions of (2) may exist in more generality. Let S be $\mathbb{R}^2$ equipped with the composition rule given by

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1x_2 + y_1y_2, x_1y_2 + y_1x_2).$$

S is an abelian semigroup with neutral element $(1, 0)$, but it is not a group. It cannot even be embedded into a group, because $(0, 0)$ can have no inverse. The map $\tau(x, y) := (x, -y)$ is an involution of S. See [2] or [3].

The multiplicative functions of S are the functions of the form $\chi(x, y) = M_1(x + y)M_2(x - y)$, where $M_i(xy) = M_i(x)M_i(y)$ for all $x, y \in \mathbb{R}$ and $i = 1, 2$ ([3, Corollary 2.5] or [13, Exercise A.11]). The continuous functions $M : \mathbb{R} \to \mathbb{C}$ for which $M(xy) = M(x)M(y)$ for all $x, y \in \mathbb{R}$ are described in [13, Example 3.9(c)].

Let us be content with noting that the condition of Theorem 4 on χ can be satisfied for some $z_0 \in S$, so that (2) has non-zero solutions on S. If we choose

$$z_0 = \left(\frac{e+1}{2}, \frac{e-1}{2}\right), \ M_1(t) = |t|^{1+i\pi/2} \text{ and } M_2(t) = |t|^{1-i\pi/2},$$

then

$$\begin{aligned}\chi(z_0) &= \chi\left(\frac{e+1}{2}, \frac{e-1}{2}\right) = M_1\left(\frac{e+1}{2} + \frac{e-1}{2}\right) M_2\left(\frac{e+1}{2} - \frac{e-1}{2}\right) \\ &= M_1(e)M_2(1) = e^{1+i\pi/2} = ei.\end{aligned}$$

Similarly we find that

$$\chi(\tau(z_0)) = \chi\left(\frac{e+1}{2}, -\frac{e-1}{2}\right) = e^{1-i\pi/2} = -ei,$$

which shows that the condition of Theorem 4 is satisfied for these choices.

References

[1] Bahyrycz, A., Brzdęk, J.: A note on d'Alembert's functional equation on a restricted domain. Aequationes Math. **88**(1–2), 169–173 (2014)
[2] Berrone, L.R., Dieulefait, L.V.: A functional equation related to the product in a quadratic number field. Aequationes Math. **81**(1–2), 167–175 (2011)
[3] Bouikhalene, B., Elqorachi, E., Charifi, A.: On the approximate solution of D'Alembert type equation originating from number theory. Extracta Math. **28**(2), 157–167 (2013)
[4] Chojnacki, W.: On uniformly bounded spherical functions in Hilbert space. Aequationes Math. **81**(1–2), 135–154 (2011)
[5] Corovei, I.: The functional equation $f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y)$ for nilpotent groups. (Romanian) Bul. Ştiinţ. Instit. Politehn. Cluj-Napoca Ser. Mat.-Fiz.-Mec. Apl. **20**, 25–28 (1978)
[6] Davison, T.M.K.: D'Alembert's functional equation on topological monoids. Publ. Math. Debrecen **75**(1/2), 41–66 (2009)
[7] Gajda, Z.: A generalization of d'Alembert's functional equation. Funkcial. Ekvac. **33**(1), 69–77 (1990)

[8] Kannappan, P.: The functional equation $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ for groups. Proc. Am. Math. Soc. **19**, 69–74 (1968)
[9] Kannappan, P.: Functional Equations and Inequalities with Applications. Springer Monographs in Mathematics. Springer, New York (2009)
[10] Nagy, B.S.: A sine functional equation in Banach algebras. Publ. Math. Debrecen **24**(1–2), 77–90 (1977)
[11] Perkins, A.M., Sahoo, P.K.: On two functional equations with involution on groups related to sine and cosine functions. Aequationes Math. (2014). doi:10.1007/s00010-014-0309-z
[12] Stetkær, H.: d'Alembert's functional equation on groups. Recent developments in functional equations and inequalities, pp. 173–191, Banach Center Publ., vol. 99. Polish Acad. Sci. Inst. Math., Warsaw (2013)
[13] Stetkær, H.: Functional Equations on Groups. World Scientific Publishing Co, Singapore (2013)
[14] Vleck, E.B.Van : A functional equation for the sine. Ann. Math. Second Ser. **11**(4), 161–165 (1910)
[15] Vleck, E.B. Van: A functional equation for the sine. Additional note. Ann. Math. Second Ser. **13**(1/4), 154 (1911–1912)

Henrik Stetkær
Department of Mathematics
Aarhus University
Ny Munkegade 118, 8000 Aarhus C
Denmark
e-mail: stetkaer@imf.au.dk

Received: January 29, 2015
Revised: February 23, 2015