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Aequationes Mathematicae

Approximate convexity with the standard deviation's error

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The paper is dedicated to Professor János Aczél on the occasion of his 90th birthday

Abstract. Functions $f\colon D \to \mathbb{R}$ defined on an open convex subset of $\mathbb{R}^n$ satisfying the approximate type convexity condition with bound of the form $\varepsilon\sqrt{t(1-t)}\|x-y\|$ are considered. We discuss properties concerning such functions characteristic for convex functions.

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1. Introduction

Let D be a bounded open convex subset of $\mathbb{R}^n$. It is known that a function $f\colon D \to \mathbb{R}$ is convex if only if it satisfies the Jensen integral inequality

$$f(x_\mu) \le \int_D f d\mu \tag{1}$$

for all probabilistic measures μ on D, where $x_\mu = \int_D x d\mu = (\int_D x_1 d\mu, \ldots, \int_D x_n d\mu)$. The question arises: what about a function $f\colon D \to \mathbb{R}$ which satisfies (1) with some error depending on μ.

Let $B(D)$ be the σ-algebra of Borel subsets of $D \subset \mathbb{R}^n$, $\mathcal{M}(D)$ be the set of all Borel probabilistic measures on D and let $\varepsilon \ge 0$.

Assume that a Borel measurable function $f\colon D \to \mathbb{R}$ satisfies the inequality

$$f(x_\mu) \le \int_D f(x) d\mu + \varepsilon \left[\int_D \|x - x_\mu\|^2 d\mu\right]^{\frac{1}{2}}, \tag{2}$$

with some $\varepsilon > 0$, for all probabilistic measures on D such that there exist finite: x_μ, $\int_D f d\mu$, $\int_D \|x - x_\mu\|^2 d\mu$, where $x_\mu = \int_D x d\mu = (\int_D x_1 d\mu, \ldots, \int_D x_n d\mu)$.

In this paper we consider a class of functions for which this error is proportional to $[\int_D \|x-x_\mu\|^2 d\mu]^{\frac{1}{2}}$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Namely, taking in (1) for arbitrary fixed $x, y \in D$, $t \in [0,1]$, instead of μ the Dirac convex combination $\mu = t\delta_x + (1-t)\delta_y$ of Dirac measures, we get that f satisfies in particular the following inequality

$$f\left(tx+(1-t)y\right) \leq tf(x)+(1-t)f(y)+\varepsilon\sqrt{t(1-t)}\|x-y\|.$$

Definition 1.1. We say that a function $f\colon D \to \mathbb{R}$ defined on a convex subset $D \subset \mathbb{R}^n$ is *approximately ε-convex with respect to the standard deviation*, briefly *ε-sconvex*, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)+\varepsilon\sqrt{t(1-t)}\|x-y\| \tag{3}$$

for $x, y \in D$, $t \in [0,1]$.

If condition (3) holds for $t = \frac{1}{2}$ and all $x, y \in D$, i.e.

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \frac{1}{2}\varepsilon\|x-y\| \qquad \text{for } x, y \in D, \tag{4}$$

we say that f is *approximately ε-midconvex with respect to the standard deviation*, briefly *ε-smidconvex.*

The notion of ε-smidconvexity given in Definition 1.1 is a modification of the notion of approximate convexity. It was introduced by Hyers and Ulam [3] with constant error bound and next generalized and development by many authors, see for example: [2–6,9].

We give basic properties of ε-sconvex and ε-smidconvex functions. One of the main tools will be the following

Theorem TTZ [9, Thr. 2.2]. Let D be an open convex subset of $\mathbb{R}^n$ and let $f\colon D \to \mathbb{R}$ be an ε-smidconvex function locally bounded above at a point. Then f is locally uniformly continuous.

2. Results

Let D be an open convex subset of $\mathbb{R}^n$.

Proposition 2.1. *Let $\alpha, \beta \geq 0$. $f, g\colon D \to \mathbb{R}$. If f, g are respectively ε_1- and ε_2-sconvex (smidconvex) then $\alpha f + \beta g$ is $\alpha\varepsilon_1 + \beta\varepsilon_2$-sconvex (midconvex).*

Proof. Obvious. □

Proposition 2.2. *If $f\colon D \to \mathbb{R}$ is ε-sconvex then f is locally uniformly continuous.*

If f is ε-smidconvex and locally bounded at a point then f is locally uniformly continuous.

Proof. The first part follows from [9, Thr. 2.2].

Observe, for the second one, that f is locally bounded. Indeed, let $S = \operatorname{conv}\{x_1, \ldots, x_{n+1}\}$ be an n-dimensional simplex contained in D. We show that

$$f(t_1x_1 + \cdots + t_kx_k) \leq t_1f(x_1) + \cdots + t_kf(x_k) + (k-1)\varepsilon \operatorname{diam} S \tag{5}$$

for $t_1, \ldots, t_k \geq 0$, $\sum_{i=1}^{k} t_i = 1$.

For $k = 1$ it is trivial, for $k = 2$ it follows from (3). Assuming that (5) holds for a certain $k \in \{2, \ldots, n\}$. Let $t_1, \ldots, t_{k+1} \geq 0$ such that $\sum_{i=1}^{k+1} t_i = 1$, and $t_{k+1} \neq 1$. Then, using (4) and the inequality $\sqrt{t_{k+1}(1 - t_{k+1})} \leq 1$, then applying (5) and the fact that the distance of two elements of S is not larger than $\operatorname{diam} S$ we have

$$\begin{aligned}
f(t_1x_1 + \cdots + t_{k+1}x_{k+1}) &= f\left((t_1 + \cdots + t_k)\sum_{i=1}^{k} \frac{t_i}{t_1 + \cdots + t_k}x_i + t_{k+1}x_{k+1}\right) \\
&\leq (t_1 + \cdots + t_k) f\left(\sum_{i=1}^{k} \frac{t_i}{t_1 + \cdots + t_k}x_i\right) + t_{k+1}f(x_{k+1}) \\
&\quad + \varepsilon \left\| \sum_{i=1}^{k} \frac{t_i}{t_1 + \cdots + t_k}x_i - x_{k+1} \right\| \\
&\leq \sum_{i=1}^{k+1} t_if(x_i) + (k-1)\varepsilon \operatorname{diam} S + \varepsilon \operatorname{diam} S \\
&= \sum_{i=1}^{k+1} t_if(x_i) + k\varepsilon \operatorname{diam} S.
\end{aligned}$$

Hence (5) holds for $k + 1$, because the case $t_{k+1} = 1$ is obvious. By (5) we obtain that

$$f(y) \leq \max\{f(x_1), \ldots, f(x_{n+1})\} + n\varepsilon \operatorname{diam} S \qquad \text{for } y \in S.$$

Hence f is bounded from above on S and consequently locally bounded from above in $\operatorname{int} S$. By [9, Thr. 2.2] f is locally uniformly continuous. □

Theorem 2.1. *Let P be an open interval in $\mathbb{R}$, $\varepsilon > 0$, and $f\colon P \to \mathbb{R}$ be a function. Then the following conditions are equivalent:*

(i) f *is ε-sconvex,*

(ii) $\frac{f(x_3)-f(x_1)}{x_3-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2} + \varepsilon\sqrt{\frac{x_2-x_1}{x_3-x_2}}$ *for $x_1 < x_2 < x_3$,*

(iii) $\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_1)}{x_3-x_1} + \varepsilon\sqrt{\frac{x_3-x_2}{x_2-x_1}}$ *for $x_1 < x_2 < x_3$,*

(iv) $\frac{f(x_2)-f(x_1)}{x_2-x_1} - \varepsilon\sqrt{\frac{x_3-x_2}{x_2-x_1}} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2} + \varepsilon\sqrt{\frac{x_2-x_1}{x_3-x_2}}$ *for $x_1 < x_2 < x_3$.*

Proof. In this case the definition of sconvexity of f is equivalent to:

$$f(x_2) \le \frac{x_3-x_2}{x_3-x_1}f(x_1) + \frac{x_2-x_1}{x_3-x_1}f(x_3) + \varepsilon\sqrt{(x_3-x_2)(x_2-x_1)}. \tag{6}$$

for $x_1 < x_2 < x_3$, $x_1, x_2, x_3 \in P$.

Indeed, since $x_2 = \frac{x_3-x_2}{x_3-x_1}x_1 + \frac{x_2-x_1}{x_3-x_1}x_3$, from (3) we get (6).

On the other hand, assuming (6), and putting in (6) $x_1 = x, x_2 = tx + (1-t)y, x_3 = y$, we have

$$x_3 - x_2 = t(y-x), x_2 - x_1 = (1-t)(y-x), x_3 - x_1 = y - x,$$

hence we obtain (3).

We show that (6) and (ii) are equivalent. Subtracting $f(x_3)$ from both sides of (6) we get

$$f(x_2) - f(x_3) \le \frac{x_3-x_2}{x_3-x_1}(f(x_1) - f(x_3)) + \varepsilon\sqrt{(x_3-x_2)(x_2-x_1)}.$$

Hence, by dividing by $x_3 - x_2$, (i) follows.

Next we show that (6) and (iii) are equivalent. Subtracting $f(x_1)$ from both sides of (6) we obtain

$$f(x_2) - f(x_1) \le \frac{x_2-x_1}{x_3-x_1}(f(x_3) - f(x_1)) + \varepsilon\sqrt{(x_3-x_2)(x_2-x_1)}.$$

Dividing this by $x_2 - x_1$ we get (iii).

Finally we show the equivalence of (6) and (iv). Subtracting from both sides of (6) the expression $\frac{x_2-x_1}{x_3-x_1}f(x_2) + \frac{x_3-x_2}{x_3-x_1}f(x_1)$ we obtain

$$\frac{x_3-x_2}{x_3-x_1}(f(x_2) - f(x_1)) \le \frac{x_2-x_1}{x_3-x_1}(f(x_3) - f(x_2)) + \varepsilon\sqrt{(x_3-x_2)(x_2-x_1)}.$$

Multiplying this inequality by $\frac{x_3-x_1}{(x_3-x_2)(x_2-x_1)}$ we get

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_3)-f(x_2)}{x_3-x_2} + \varepsilon\frac{(x_3-x_2)+(x_2-x_1)}{\sqrt{(x_3-x_2)}\sqrt{(x_2-x_1)}}.$$

Hence we get (iv). □

Theorem 2.2. *If $f\colon D \to \mathbb{R}$ is ε-smidconvex then*

$$f\left(\frac{k}{2^n}x + \left(1 - \frac{k}{2^n}\right)y\right) \le \frac{k}{2^n}f(x) + \left(1 - \frac{k}{2^n}\right)f(y) + 2\varepsilon\frac{\sqrt{k(2^n-k)}}{2^n}\|x-y\| \tag{7}$$

for $x, y \in D$, $n \in \mathbb{N}$, $k \in \{0, \ldots, 2^n\}$.

Proof. For $n = 1$ condition (7) is true (for $k = 1$ by (4), for $k = 0$ or $k = 2$ it is obvious). Assume, for the proof by induction, that (7) holds for some $n \in \mathbb{N}$ and all $x, y \in D$, $k \in \{0, \ldots, 2^n\}$. We check that this condition holds for $n+1$. Let $x, y \in D$, $k \in \{0, \ldots, 2^{n+1}\}$. Without loss of generality we may assume that $k \in \{1, \ldots, 2^{n+1}\}$. Changing x and y, if necessary, we may assume that $k \in \{0, \ldots, 2^n\}$. Making use of (4) and the inductive hypothesis we obtain

$$\begin{aligned}
&f\left(\frac{k}{2^{n+1}}x + \left(1 - \frac{k}{2^{n+1}}\right)y\right)\\
&\quad= f\left(\frac{\frac{k}{2^n}x + \left(1 - \frac{k}{2^n}\right)y + y}{2}\right)\\
&\quad\le \frac{f\left(\frac{k}{2^n}x + \left(1 - \frac{k}{2^n}\right)y\right) + f(y)}{2} + \frac{\varepsilon}{2}\frac{k}{2^n}\|x-y\|\\
&\quad\le \frac{1}{2}\left(\frac{k}{2^n}f(x) + \left(1 - \frac{k}{2^n}\right)f(y) + 2\varepsilon\frac{\sqrt{k(2^n-k)}}{2^n}\|x-y\| + f(y)\right)\\
&\qquad+ \frac{\varepsilon}{2}\frac{k}{2^n}\|x-y\|\\
&\quad= \frac{k}{2^{n+1}}f(x) + \left(1 - \frac{k}{2^{n+1}}\right)f(y)\\
&\qquad+ \left(\frac{\varepsilon\sqrt{k(2^n-k)}}{2^n} + \frac{\varepsilon}{2^{n+1}}\right)\|x-y\|.
\end{aligned}$$

Hence we need to show that

$$\frac{\sqrt{k\,(2^n-k)}}{2^n} + \frac{1}{2^{n+1}} \le 2\frac{\sqrt{k(2^{n+1}-k)}}{2^{n+1}} \qquad \text{for } k \in \{1, \ldots, 2^n\},$$

i.e.

$$2\sqrt{k(2^n-k)} + 1 \le 2\sqrt{k(2^{n+1}-k)} \qquad \text{for } k \in \{1, \ldots, 2^n\}.$$

This inequality is a consequence of the following one

$$2\sqrt{k(2^n-k)} + k \le 2\sqrt{k(2^{n+1}-k)} \qquad \text{for } k \in \{1, \ldots, 2^n\},$$

which is equivalent to:

$$\begin{aligned}
&4k(2^n-k) + k^2 + 4k\sqrt{k(2^n-k)} \le 4k(2^{n+1}-k), \qquad \text{for } k \in \{1, \ldots, 2^n\},\\
&\quad 4\sqrt{k(2^n-k)} \le 4 \cdot 2^n - k, \qquad \text{for } k \in \{1, \ldots, 2^n\}\\
&\quad 16k(2^n-k) \le 16 \cdot 4^n + k^2 - 8k2^n, \qquad \text{for } k \in \{1, \ldots, 2^n\}\\
&\quad 17k^2 - 24 \cdot 2^n k + 16 \cdot 4^n \ge 0, \qquad \text{for } k \in \{1, \ldots, 2^n\}.
\end{aligned}$$

The last inequality holds, because its discriminant $\Delta := (24^2 - 4\cdot17\cdot16)4^n = -512 \cdot 4^n < 0$. □

Corollary 2.1. *If $f\colon D \to \mathbb{R}$ is ε-smidconvex and continuous at a point then f is 2ε-sconvex.*

Proof. By Proposition 2.2 f is continuous. By Theorem 2.2 inequality (3) with ε replaced by 2ε holds for every dyadic number $t \in [0, 1]$. By the continuity of f, it holds also for every $t \in [0, 1]$. □

Example 2.1. The Takagi function $T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} dist(2^k x, \mathbb{Z})$, $x \in [0,1]$, satisfies (4) with $\varepsilon = \frac{1}{2}$ [1]. It follows by Corollary 2.1 that T is 1-sconvex. This example shows that, even if an sconvex function is continuous, it can be non-differentiable at every point.

Example 2.2. Assume that $f\colon D \to \mathbb{R}$ is ε-sconvex and let $D \subset G \subset \overline{G} \subset D$, where the set G is open convex with the compact closure. Let $\omega \in C_c^\infty(\mathbb{R}^n)$ be such that $\omega \geq 0$, $\omega(x) = 0$ if $\|x\| \geq 1$ and $\int_{\mathbb{R}^n} \omega(x)dx = 1$. Let $\omega_\delta(x) = \frac{1}{\delta^n}\omega(\frac{1}{\delta}x)$, for $\delta > 0$.

For $\delta < dist(G, \partial D)$ the regularization $f_\delta(x) = \int_{B(0,\delta)} f(x-z)\omega_\delta(z)dz$ is ε-sconvex in G. Indeed, for $x, y \in G$, $t \in [0,1]$

$$\begin{aligned} f_\delta(tx + (1-t)y) &= \int_{B(0,\delta)} f_\delta(tx + (1-t)y - z)\omega_\delta(z)dz \\ &\leq \int_{B(0,\delta)} \Big[tf(x-z)\omega_\delta(y-z) + (1-t)\omega_\delta(y-z) \\ &\quad + \varepsilon\sqrt{t(1-t)}\|x-y\| \Big] dz \\ &= tf_\delta(x) + (1-t)f_\delta(y) + \varepsilon\sqrt{t(1-t)}\|x-y\|. \end{aligned}$$

Let for $x_0 \in \mathbb{R}^n$, $R > 0$

$m(B(x_0, R))$ denote the volume of the ball $B(x_0, R)$ in the Euclidean space $\mathbb{R}^n$, and $\sigma(S(x_0, R))$ - the area of the sphere $S(x_0, R)$, the boundary of $B(x_0, R)$.

Proposition 2.3. *If $f\colon D \to \mathbb{R}$ is ε-sconvex then for each $B(x_0, R) \subset D$ the following Hadamard-type inequality holds*

$$\frac{1}{m(B(x_0,R))} \int_{B(x_0,R)} f dm \leq \frac{1}{\sigma(S(x_0,R))} \int_{S(x_0,R)} f d\sigma + \varepsilon n R c_n,$$

where $c_n = \int_0^1 r^{n-1}\sqrt{1-r^2}dr$.

Proof. Using polar coordinates and (3) we have

$$\begin{aligned} \int_{B(x_0,R)} f dm &= \int_0^R \int_{S(x_0,r)} f(x) d\sigma(x) dr \\ &= \int_0^R \int_{S(x_0,r)} \Big[f(\frac{R+r}{2R}\left(x_0 + \frac{R(x-x_0)}{r}\right) \\ &\quad + \frac{R-r}{2R}\left(x_0 - \frac{R(x-x_0)}{r}\right)\Big] d\sigma(x) dr \end{aligned}$$

$$\leq \int_0^R \int_{S(x_0,r)} \left[\frac{R+r}{2R} f\left(x_0 + \frac{R(x-x_0)}{r}\right) \right.$$
$$+ \frac{R-r}{2R} f\left(x_0 - \frac{R(x-x_0)}{r}\right)$$
$$\left. + \varepsilon \frac{\sqrt{R^2-r^2}}{2R} \left\| \frac{2R(x-x_0)}{r} \right\| \right] d\sigma(x)dr.$$

For $x \in S(x_0, r)$ we have $\|x - x_0\| = r$. Hence

$$\int_{B(x_0,R)} f dm \leq \frac{R+r}{2R} \int_0^R \int_{S(x_0,r)} f\left(x_0 + \frac{R(x-x_0)}{r}\right) d\sigma(x)dr$$
$$+ \frac{R-r}{2R} \int_0^R \int_{S(x_0,r)} f\left(x_0 - \frac{R(x-x_0)}{r}\right) d\sigma(x)dr$$
$$+ \varepsilon \int_0^R \int_{S(x_0,r)} \sqrt{R^2-r^2} d\sigma(x)dr.$$

By the symmetry of $S(x_0, r)$ with respect to x_0, in the last equality the first two integrals are equal. For the third integral, using that $\sigma(S(x_0, r)) = r^{n-1}$ $\sigma(S(0, 1))$, we obtain

$$\int_0^R \int_{S(x_0,r)} \sqrt{R^2-r^2} d\sigma(x)dr = \int_0^R \sigma(S(x_0,r))\sqrt{R^2-r^2}dr$$
$$= \sigma(S(0,1)) \int_0^R r^{n-1}\sqrt{R^2-r^2}dr$$
$$= \sigma(S(0,1))R^{n+1} \int_0^1 r^{n-1}\sqrt{1-r^2}dr.$$

Hence

$$\int_{B(x_0,R)} f dm \leq \int_0^R \int_{S(x_0,r)} f\left(x_0 + \frac{R(x-x_0)}{r}\right) d\sigma(x)dr + \varepsilon\sigma(S(0,1))R^{n-1}R^2 c_n$$
$$= \frac{1}{R^{n-1}} \int_0^R \int_{S(x_0,R)} f(x)d\sigma(x)r^{n-1}dr + \varepsilon R^2 \sigma(S(x_0,R))c_n$$
$$= \frac{R}{n} \int_{S(x_0,R)} f d\sigma + \varepsilon R^2 \sigma(S(x_0,R))c_n$$
$$= \frac{m(B(x_0,R))}{\sigma(S(x_0,R))} \int_{S(x_0,R)} f d\sigma + \varepsilon R^2 \sigma(S(x_0,R))c_n.$$

Hence

$$\frac{1}{m(B(x_0,R)} \int_{B(x_0,R)} f dm \leq \frac{1}{\sigma(S(x_0,R)} \int_{S(x_0,R)} f d\sigma + \varepsilon R^2 \frac{\sigma(S(x_0,R))}{m(B(x_0,R))} c_n.$$

This ends the proof, because $\frac{\sigma(S(x_0,R)}{m(B(x_0,R))} = \frac{n}{R}$. □

Remark 2.1. The constants c_n from Proposition 2.3 can be found using well known properties of the beta and gamma functions, see for example [7, ch. 8]. By substituting $t = r^2$ in the integral defining c_n we get

$$c_n = \frac{1}{2}\int_0^1 t^{\frac{n}{2}-1}(1-t)^{\frac{1}{2}}\,dt = \frac{1}{2}B\left(\frac{n}{2},\frac{3}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)}, \quad n \in \mathbb{N}.$$

Using Legendre's formula

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\frac{\Gamma(2n)}{\Gamma(n)} = \frac{\sqrt{\pi}}{2^{2n-1}}\frac{(2n-1)!}{(n-1)!}$$

for $n \in \mathbb{N}$ and basic properties of the gamma function we obtain

$$c_{2n+1} = \frac{1}{4}\frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(n+2)} = \frac{\pi}{2^{2n+1}}\frac{(2n-1)!}{(n+1)!(n-1)!}, \quad n \in \mathbb{N}.$$

(Obviously, by definition, $c_1 = \frac{\pi}{4}$.) Similarly,

$$c_{2n} = \frac{1}{2}\frac{\Gamma(n)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)} = \frac{1}{4}\frac{\Gamma(n)\sqrt{\pi}}{(n+\frac{1}{2})\Gamma\left(n+\frac{1}{2}\right)} = \frac{2^{2n-3}}{\left(n+\frac{1}{2}\right)}\frac{((n-1)!)^2}{(2n-1)!}, \quad n \in \mathbb{N}.$$

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