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An orthogonality in normed linear spaces based on angular distance inequality

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Abstract. In this paper, we present a new orthogonality in a normed linear space which is based on an angular distance inequality. Some properties of this orthogonality are discussed. We also find a new approach to the Singer orthogonality in terms of an angular distance inequality. Some related geometric properties of normed linear spaces are discussed. Finally a characterization of inner product spaces is obtained.

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1. Introduction

The notion of orthogonality goes a long way back in time and various extensions have been introduced over the last decades. In particular, proposing the notion of orthogonality in normed linear spaces has been the object of extensive efforts of many mathematicians. The most natural notion of orthogonality arises in the case where the norm $\|.\|$ derives from an inner product. In this case $x \perp y$ if and only if $\langle x, y\rangle = 0$. The notion of orthogonality in an inner product space has the following interesting properties:

- $\lambda x \perp \mu x$ if and only if $\|\lambda\mu x\| = 0$ for all $\lambda, \mu \in \mathbb{R}$ (Non-degeneracy).
- Let $\{x_i\}_{i=1}^{\infty}$, $\{y_i\}_{i=1}^{\infty}$ be two sequences such that $x = \lim_{i\to\infty} x_i$ and $y = \lim_{i\to\infty} y_i$. If $x_i \perp y_i$ for each $i \in \mathbb{N}$, then $x \perp y$ (Continuity).
- $x \perp y$ implies $\mu x \perp \mu y$ for all $x, y \in X$ and $\mu \in \mathbb{R}$ (Simplification).
- For every $x, y \in X$ linearly independent, there exists a real number a such that $y \perp x + ay$ $(x + ay \perp y)$ (*Right* (*left*) *existence*).
- $x \perp y$ implies $\mu x \perp \lambda y$ for all $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$ (Homogeneity).
- $x \perp y$ implies $y \perp x$ for all $x, y \in X$ (Symmetry).

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- $x \perp y$ and $x \perp z$ imply $x \perp (y+z)$ for all $x, y, z \in X$ (Right additivity).
- $y \perp x$ and $z \perp x$ imply $y + z \perp x$ for all $x, y, z \in X$ (Left additivity).
- For any $x, y \in X$ linearly independent, there exists at most one real number a such that $y \perp x + ay$ $(x + ay \perp y)$ (Right (left) uniqueness).

The existence property is the most important, since either the right existence or the left existence property can keep the concept of orthogonality from being vacuous.

We state two known orthogonalities introduced in normed linear spaces. In a normed linear space X, a vector x is said to be orthogonal to y in the sense of Singer [1] if the following relation holds:

$$x \perp_S y \qquad \text{whenever} \;\; \|x\|\|y\| = 0 \;\; \text{or} \;\; \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|.$$

Alsina et al. [4] introduced the following orthogonality relation:

$$x \perp_w y \qquad \text{whenever} \quad \|x\|\|y\| = 0 \;\; \text{or} \quad \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = \sqrt{2}.$$

Some other known orthogonalities in normed linear spaces can be found in [1,3,6,14] and references therein. The above orthogonalities are based on the concept of angular distance between nonzero vectors x and y in a normed linear space $(X, \|.\|)$ which was defined as $\alpha[x, y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$. There are interesting characterizations of inner product spaces connected with the concept of angular distance [4,5,7,13]. Another way to obtain characterizations of inner product spaces and other geometric properties of the space such as strict convexity and smoothness is to force some of the generalized orthogonalities to fulfill some properties of orthogonality such as homogeneity, symmetry, additivity and uniqueness.

In this paper, we present a new orthogonality in a normed linear space which is based on an angular distance inequality. Some properties of this orthogonality are discussed. We also find a new approach to the Singer orthogonality in terms of an angular distance inequality. Some related geometric properties of normed linear spaces are discussed. Finally a characterization of inner product spaces is obtained. In this paper $(X, \|.\|)$ always denotes a real normed linear space and S_X is the corresponding unit sphere.

2. Orthogonality and angular distance

In this section, we present a new orthogonality in a normed linear space $(X, \|.\|)$ which is based on an angular distance inequality. Some properties of this orthogonality are also discussed.

Definition 1. Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$. We say that x is orthogonal to y and we denote it by $x \perp_+ y$, if $\|x\|\|y\| = 0$ or the following two statements hold:

(i) $\{x, y\}$ is linearly independent,

(ii) $\alpha[x + ty, y] + \alpha[x + ty, -y] \leq \alpha[x, y] + \alpha[x, -y] \quad$ for all $t \in \mathbb{R}$. (1)

We note that if $\{x, y\}$ is an independent set, then $x + ty \neq 0$ for all $t \in \mathbb{R}$ and so $\alpha[x + ty, y]$ and $\alpha[x + ty, -y]$ in inequality (1) are well defined. Let

$$g_{x,y}(t) := \alpha[x + ty, y] + \alpha[x + ty, -y].$$

Then two independent vectors x and y are orthogonal if and only if

$$g_{x,y}(t) \leq g_{x,y}(0), \quad \text{for all} \quad t \in \mathbb{R}. \tag{2}$$

It is not difficult to check that this notion of orthogonality is not symmetric in general.

Lemma 1. *Let X be an inner product space, $x, y \in X$ be two linearly independent vectors and $t \in \mathbb{R}$ be arbitrary. Then the following two inequalities are equivalent.*

(i) $\alpha[x + ty, y] + \alpha[x + ty, -y] \leq \alpha[x, y] + \alpha[x, -y]$,

(ii) $\alpha[x + ty, y]\alpha[x + ty, -y] \leq \alpha[x, y]\alpha[x, -y]$.

Proof. Inequality (i) holds if and only if

$$\left\| \frac{x + ty}{\|x + ty\|} - \frac{y}{\|y\|} \right\|^2 + \left\| \frac{x + ty}{\|x + ty\|} + \frac{y}{\|y\|} \right\|^2 + 2\alpha[x + ty, y]\alpha[x + ty, -y]$$
$$\leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 + \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^2 + 2\alpha[x, y]\alpha[x, -y].$$

Equivalently, we obtain the following inequality

$$2 - \frac{2\langle x + ty, y\rangle}{\|x + ty\|\|y\|} + 2 + \frac{2\langle x + ty, y\rangle}{\|x + ty\|\|y\|} + 2\alpha[x + ty, y]\alpha[x + ty, -y]$$
$$\leq 2 - \frac{2\langle x, y\rangle}{\|x\|\|y\|} + 2 + \frac{2\langle x, y\rangle}{\|x\|\|y\|} + 2\alpha[x, y]\alpha[x, -y]$$

and this is true if and only if inequality (ii) holds. □

In the next theorem we show that the orthogonality $\perp_+$ is equivalent to the standard definition of orthogonality when the underlying space is an inner product space.

Theorem 2.1. *Let X be an inner product space and $x, y \in X$. Then $x \perp_+ y$ if and only if $\langle x, y\rangle = 0$.*

Proof. If x and y are linearly dependent, then the proof is obvious. Let $x, y \in X$ be linearly independent and $t \in \mathbb{R}$. The following relations hold:

$$\begin{aligned}
&\alpha[x+ty, y] + \alpha[x+ty, -y] \leq \alpha[x, y] + \alpha[x, -y] \\
\Leftrightarrow\ &\alpha[x+ty, y]\alpha[x+ty, -y] \leq \alpha[x, y]\alpha[x, -y] \qquad \text{(by Lemma 1)} \\
\Leftrightarrow\ &\left(4 - 4\langle \frac{x+ty}{\|x+ty\|}, \frac{y}{\|y\|}\rangle^2\right)^{\frac{1}{2}} \leq \left(4 - 4\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\rangle^2\right)^{\frac{1}{2}} \\
\Leftrightarrow\ &\frac{\langle x, y\rangle^2}{\|x\|^2\|y\|^2} \leq \frac{\langle x+ty, y\rangle^2}{\|x+ty\|^2\|y\|^2} \\
\Leftrightarrow\ &\|x+ty\||\langle x, y\rangle| \leq \|x\||\langle x+ty, y\rangle|. \qquad (3)
\end{aligned}$$

Let $x \perp_+ y$. Putting $t = \frac{-\langle x,y\rangle}{\|y\|^2}$ in (3) we obtain $\langle x, y\rangle = 0$. Conversely, if $\langle x, y\rangle = 0$, then (3) holds and so $g_{x,y}(t) \leq g_{x,y}(0)$ for all $t \in \mathbb{R}$ and hence $x \perp_+ y$. □

It is obvious that the orthogonality $\perp_+$ satisfies non-degeneracy, continuity and simplification. In the following we state an example in which the orthogonality is not right existent in general.

Example 1. Let X be a Minkowski plane with the l_∞ norm and let $x = (0, 1)$ and $y = (1, 0)$. We show that $x \not\perp_+ ax + y$ for all $a \in \mathbb{R}$. We consider four cases.

(i) If $a > 1$, then by taking $t \in \left(-\frac{1}{a}, \frac{1}{-1-a}\right)$ we have $x \not\perp_+ ax + y$.

(ii) If $a < -1$, then by taking $t \in \left(\frac{1}{1-a}, \frac{-1}{a}\right)$ we have $x \not\perp_+ ax + y$.

(iii) If $0 \leq a \leq 1$, then by taking $t \in \left(-\frac{1}{a}, \frac{-1}{a+1}\right)$ we have $x \not\perp_+ ax + y$.

(iv) If $-1 \leq a < 0$, then by taking $t \in \left(0, \frac{1}{1-a}\right)$ we have $x \not\perp_+ ax + y$.

Now we discuss the homogeneity and left existence properties of the orthogonality $\perp_+$. First we state the following lemma which can be proved by using the same method as in [8, Lemma 2.1].

Lemma 2. *Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$ be two independent vectors. Then*

(i) $\lim_{t\to\pm\infty} \alpha[x+ty, y] + \alpha[x+ty, -y] = 2$,

(ii) $\lim_{t\to\pm\infty} \alpha[x+ty, -y] - \alpha[x+ty, y] = \pm 2$.

Theorem 2.2. *The orthogonality $\perp_+$ is left existent and homogenous.*

Proof. Let $x, y \in X$ be two linearly independent vectors. We shall show that there exists $t_0 \in \mathbb{R}$ such that $x + t_0 y \perp_+ y$. From (2) it is enough to prove that there exists $t_0 \in \mathbb{R}$ such that the function $g_{x+t_0y,y}(t)$ takes a maximum at $t = 0$. We note that

$$g_{x+t_0y,y}(t) \leq g_{x+t_0y,y}(0) \quad \text{if and only if} \quad g_{x,y}(t_0 + t) \leq g_{x,y}(t_0). \qquad (4)$$

The function $g_{x,y}(t)$ is continuous and $\lim_{t\to\pm\infty} g_{x,y}(t) = 2$ from Lemma 2. Moreover, $2 \le g_{x,y}(t) \le 4$ for all $t \in \mathbb{R}$ by the triangle inequality. Thus $g_{x,y}(t)$ takes a maximum at some $t_0 \in \mathbb{R}$, i.e. $g_{x,y}(t) \le g_{x,y}(t_0)$ for all $t \in \mathbb{R}$. Now the result follows from (4).

For the proof of the homogeneity property of orthogonality, we may assume that $x, y \in X$ are linearly independent vectors and μ and λ are two nonzero real numbers. The following implications hold:

$$\begin{aligned}
x \perp_+ y &\Leftrightarrow \alpha[x+ty, y] + \alpha[x+ty, -y] \le \alpha[x,y] + \alpha[x,-y] \qquad \forall\, t \in \mathbb{R}\\
&\Leftrightarrow \alpha\left[\mu\left(x + \frac{t\lambda}{\mu}y\right), y\right] + \alpha\left[\mu\left(x + \frac{t\lambda}{\mu}y\right), y\right] \le \alpha[x,y] + \alpha[x,-y]\\
&\qquad \forall\, t \in \mathbb{R}\\
&\Leftrightarrow \alpha[\mu x + t\lambda y, \lambda y] + \alpha[\mu x + t\lambda y, -\lambda y] \le \alpha[\mu x, \lambda y] + \alpha[\mu x, -\lambda y]\\
&\qquad \forall\, t \in \mathbb{R}\\
&\Leftrightarrow \mu x \perp_+ \lambda y.
\end{aligned}$$

□

3. Singer orthogonality by an angular distance inequality

In this section we express an orthogonality relation in terms of an angular distance inequality. We show that this notion of orthogonality is equivalent to the Singer orthogonality. We also define the concept of acute and obtuse angles in normed linear spaces based on the equivalent definition of the Singer orthogonality.

Definition 2. Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$. We say that x is orthogonal to y and we denote it by $x \perp_- y$ if $\|x\|\|y\| = 0$ or the following two statements hold:

(i) $\{x, y\}$ is linearly independent.

(ii) $|\alpha[x,-y] - \alpha[x,y]| \le |\alpha[x+ty,-y] - \alpha[x+ty,y]| \quad$ *for all* $t \in \mathbb{R}$. (5)

We note that if $\{x, y\}$ is an independent set, then $x + ty \ne 0$ for all $t \in \mathbb{R}$ and so $\alpha[x+ty, y]$ and $\alpha[x+ty,-y]$ in the inequality (5) are well defined. Let

$$h(t) := \alpha[x+ty,-y] - \alpha[x+ty,y].$$

Then two independent vectors x and y are orthogonal if and only if

$$|h(0)| \le |h(t)|, \quad \text{for all} \;\; t \in \mathbb{R}. \tag{6}$$

Theorem 3.1. *Let $(X, \|.\|)$ be a normed linear space. Then the orthogonality $\perp_-$ and the Singer orthogonality are equivalent.*

Proof. Let x, y be two nonzero vectors in X. If $x \perp_- y$, then $|h(0)| \leq |h(t)|$ for all $t \in \mathbb{R}$. On the other hand since $-2 \leq h(t) \leq 2$ and by Lemma 2

$$\lim_{t \to \pm\infty} \alpha[x+ty, -y] - \alpha[x+ty, y] = \pm 2,$$

so by the continuity of $h(t)$, there exists a real number t_0 such that $h(t_0) = 0$ and $|h(0)| \leq |h(t_0)| = 0$ and hence $|h(0)| = 0$ which means $x \perp_S y$. The converse is obvious. □

Remark 1. Let $x, y \in X$ be two linearly independent vectors. Then there exists a unique $t_0 \in \mathbb{R}$ such that

$$|h(t_0)| \leq |h(t)|, \quad \text{for all} \quad t \in \mathbb{R}.$$

As it was shown in Theorem 3.1, there exists a number $t_0 \in \mathbb{R}$ such that $|h(t_0)| = 0$. Hence $\left\| \frac{x+t_0 y}{\|x+t_0 y\|} + \frac{y}{\|y\|} \right\| = \left\| \frac{x+t_0 y}{\|x+t_0 y\|} - \frac{y}{\|y\|} \right\|$. Now the uniqueness property of the Singer orthogonality [15, Theorem 2] implies that the above t_0 is unique.

The concept of angle between two vectors in a normed linear space has been introduced in different ways [8,12], so that they coincide with the standard definition of angle in inner product spaces. Now inspired by the new approach to the Singer orthogonality we define acute and obtuse angles in normed linear spaces as follows:

Definition 3. Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$ be two independent vectors. The angle between x and y is called an acute (obtuse) angle if there exists a unique number $t_0 \in (-\infty, 0)$ $\big(t_0 \in (0, \infty)\big)$ such that

$$|h(t_0)| \leq |h(t)| \quad \text{for all } t \in \mathbb{R}. \tag{7}$$

Moreover, the angle between x and y is right if

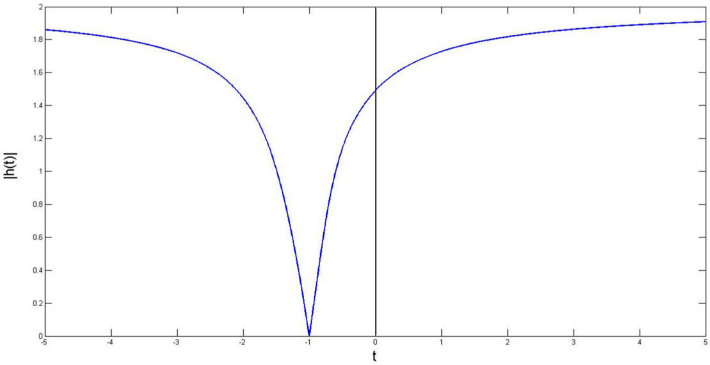
$$|h(0)| \leq |h(t)| \quad \text{for all } t \in \mathbb{R}. \tag{8}$$

Example 2. Consider the space $\mathbb{R}^2$ with the l_3 norm and let $x = (5, 0), y = (4, 2), z = (-4, 2)$ and $w = (0, 5)$. Then the angle between x and y is acute, the angle between x and z is obtuse and the angle between x and w is right (see Fig. 1).

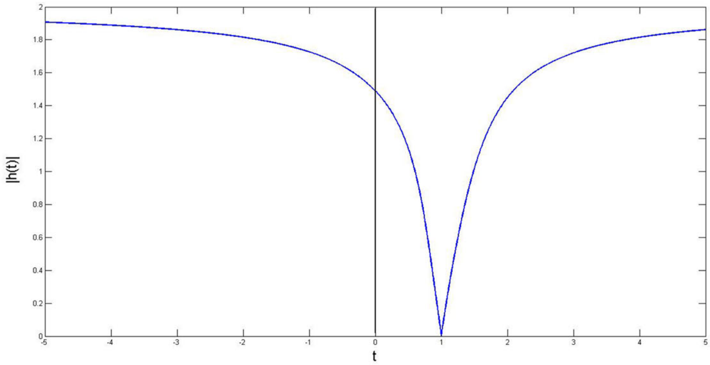
In the following theorem we show that in an inner product space the definition of acute and obtuse angles is equivalent to their standard definition. First we need the following lemma which can be proved with the same method as in Lemma 1.

Lemma 3. *Let X be an inner product space, $x, y \in X$ be independent vectors and $t \in \mathbb{R}$. Then the following two inequalities are equivalent.*

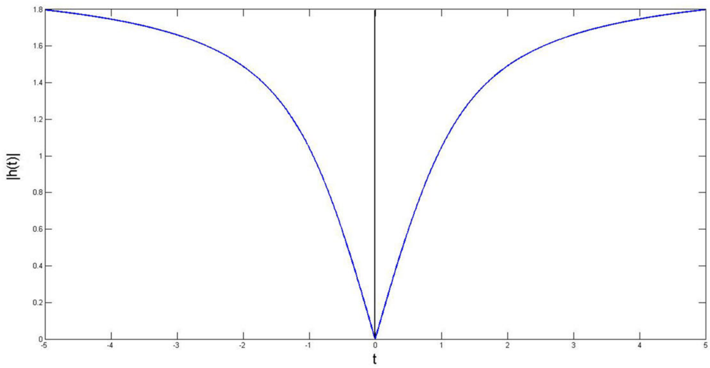
(i) $|\alpha[x, -y] - \alpha[x, y]| \leq |\alpha[x+ty, -y] - \alpha[x+ty, y]|$,
(ii) $\alpha[x+ty, -y]\alpha[x+ty, y] \leq \alpha[x, -y]\alpha[x, y]$.



(a) $x = (5, 0),\ y = (4, 2)$



(b) $x = (5, 0),\ z = (-4, 2)$



(c) $x = (5, 0), w = (0, 5)$

FIGURE 1. Example of acute, obtuse and right angles. **a** $x = (5, 0),\ y = (4, 2)$, **b** $x = (5, 0),\ z = (-4, 2)$, **c** $x = (5, 0), w = (0, 5)$

Theorem 3.2. *Let X be an inner product space and $x, y \in X$ be two independent vectors. Then*

(i) *The angle between x and y is acute if and only if*

$$0 < \cos^{-1} \frac{\langle x, y\rangle}{\|x\|\|y\|} < \pi/2.$$

(ii) *The angle between x and y is obtuse if and only if*

$$\pi/2 < \cos^{-1} \frac{\langle x, y\rangle}{\|x\|\|y\|} < \pi.$$

Proof. Let $t \in \mathbb{R}$. The following relations hold:

$$\begin{aligned} &|\alpha[x, -y] - \alpha[x, y]| \leq |\alpha[x + ty, -y] - \alpha[x + ty, y]| \\ &\Leftrightarrow \alpha[x + ty, -y]\alpha[x + ty, y] \leq \alpha[x, -y]\alpha[x, y] && (\textit{by Lemma } 3) \\ &\Leftrightarrow \|x + ty\||\langle x, y\rangle| \leq \|x\||\langle x + ty, y\rangle|. && (\textit{by } (3)) \end{aligned} \tag{9}$$

If $\langle x, y\rangle \geq 0$, then for all $t \geq 0$,

$$\begin{aligned} \|x + ty\||\langle x, y\rangle| &\leq (\|x\| + t\|y\|)|\langle x, y\rangle| \\ &\leq \|x\|\langle x, y\rangle + t\|y\|^2\|x\| \\ &= \|x\||\langle x + ty, y\rangle|. \end{aligned} \tag{10}$$

If $\langle x, y\rangle \leq 0$, then for all $t \leq 0$,

$$\begin{aligned} \|x + ty\||\langle x, y\rangle| &\leq (\|x\| + |t|\|y\|)|\langle x, y\rangle| \\ &\leq -(\|x\|\langle x, y\rangle + t\|y\|^2\|x\|) \\ &= \|x\||\langle x + ty, y\rangle|. \end{aligned} \tag{11}$$

(i) Let the angle between x and y be acute in terms of Definition 3. In general, $0 \leq \cos^{-1} \frac{\langle x,y\rangle}{\|x\|\|y\|} \leq \pi$ but since x and y are linearly independent vectors, from the conditions for the Cauchy–Schwarz inequality to be an equality we conclude that $\frac{|\langle x,y\rangle|}{\|x\|\|y\|} \neq 1$ and $0 < \cos^{-1} \frac{\langle x,y\rangle}{\|x\|\|y\|} < \pi$. Assume if possible that $\cos^{-1} \frac{\langle x,y\rangle}{\|x\|\|y\|} \geq \pi/2$. Then $\langle x, y\rangle \leq 0$. By using (9) and (11) we have $|h(0)| \leq |h(t)|$ for all $t \leq 0$, which is impossible by the definition of acute angle.

Conversely, assume that $0 < \cos^{-1} \frac{\langle x,y\rangle}{\|x\|\|y\|} < \pi/2$. We will show that the angle between x and y is acute in terms of Definition 3. Assume that there exists a unique $t_0 \in [0, \infty)$ such that $|h(t_0)| \leq |h(t)|$ for all $t \in \mathbb{R}$. Therefore $|h(0)| \leq |h(t)|$ for all $t \in (-\infty, 0)$ and so (9) holds for all $t \leq 0$ and by putting $t = \frac{-\langle x,y\rangle}{\|y\|^2}$ in the last inequality of (9), we have $\|x + ty\||\langle x, y\rangle| = 0$. But since x and y are linearly independent vectors, $\|x + ty\| \neq 0$ for all $t \in \mathbb{R}$. So $|\langle x, y\rangle| = 0$ and therefore $\cos^{-1} \frac{\langle x,y\rangle}{\|x\|\|y\|} = \pi/2$, which is a contradiction.

(ii) The proof is similar to the proof of part (i). □

In the next theorem we show that the notion of acute and obtuse angles between two independent vectors x and y is symmetric. First we state the following lemma which can be proved by [8, Theorems 2.4 and 2.5].

Lemma 4. *Let $(X.\|.\|)$ be a normed linear space and $x, y \in X$ be two independent vectors. Define the functions $h_+(t) := \left\| \frac{x+ty}{\|x+ty\|} + \frac{y}{\|y\|} \right\|$ and $h_-(t) := \left\| \frac{x+ty}{\|x+ty\|} - \frac{y}{\|y\|} \right\|$. The functions $h_+(t)$ and $h_-(t)$ are increasing and decreasing respectively. If the space $(X, \|.\|)$ is strictly convex, then the functions $h_+(t)$ and $h_-(t)$ are strictly increasing and decreasing respectively.*

Proposition 1. *The angle between two linearly independent vectors x and y is acute (obtuse) if and only if the angle between y and x is acute (obtuse).*

Proof. Let x and y be two independent vectors in X. Then by the uniqueness property of the Singer orthogonality there exist two real numbers t_1 and t_2 such that, $x+t_1y \perp_S y$ and $y+t_2x \perp_S x$. It is enough to show that t_1 and t_2 have the same sign. If we denote h(t) by more precise notation $h_{x,y}(t)$, then the functions $h_{x,y}(t) = \alpha[x+ty, -y] - \alpha[x+ty, y]$ and $h_{y,x}(t) = \alpha[y+tx, -x] - \alpha[y+tx, x]$ are increasing by Lemma 4 and since $h_{x,y}(t_1) = h_{y,x}(t_2) = 0$ and $h_{x,y}(0) = h_{y,x}(0)$, obviously, t_1 and t_2 have the same sign. □

Although in an inner product space the orthogonality relations $\perp_+$ and $\perp_-$ (Singer orthogonality) coincide with each other, the following remark shows that these orthogonalities are different in a general normed linear space.

Remark 2. Let X be a Minkowski plane with the l_∞ norm and let $x = (1, 0)$ and $y = (0, 1)$. Then $x \perp_- y$, since $\alpha[x, y] = \alpha[x, -y] = 1$ but $x \not\perp_+ y$, since

$$\alpha[x \pm y, y] + \alpha[x \pm y, -y] = 3 \nleq 2 = \alpha[x, y] + \alpha[x, -y].$$

On the other hand if we consider the vectors $x = (1, 1)$ and $y = (0, 1)$ we will see that $x \perp_+ y$ but $x \not\perp_- y$.

4. Orthogonality and some related constants

There are a lot of quantitative descriptions of geometrical properties of normed linear spaces which can give a better understanding about the shape of their unit balls. One way to provide these descriptions is to define geometric constants in normed linear spaces. For example the following constant was defined by Gao and Lau [9].

$$g(X) = \inf\{\alpha(y) : \ y \in S_X\}, \quad G(X) = \sup\{\alpha(y) : \ y \in S_X\}, \qquad (12)$$

where $\alpha(y) = \inf\{\max\{\|x - y\|, \|x + y\|\}, \ \ x \in S_X\}$, for all $y \in S_X$. They also defined

$$j(X) = \inf\{\beta(y) : y \in S_X\}, \quad J(X) = \sup\{\beta(y) : y \in S_X\}, \qquad (13)$$

where $\beta(y) = \sup\{\min\{\|x-y\|, \|x+y\|\},\ \ x \in S_X\}$, for all $y \in S_X$.

In this section we introduce new constants in normed linear spaces. The relationship of the new constants and the above constants is extensively studied. We also indicate some relations between these constants and the orthogonalities mentioned in the previous sections. Let $(X, \|.\|)$ be a normed linear space. We define the following constants:

$$\begin{aligned} \tilde{g}(X) &:= \inf\{\tilde{\alpha}(x,y) :\ x, y \text{ are linearly independent}\}, \\ \tilde{G}(X) &:= \sup\{\tilde{\alpha}(x,y) :\ x, y \text{ are linearly independent}\}, \end{aligned} \tag{14}$$

where $\tilde{\alpha}(x,y) := \inf\left\{\max\left\{\left\|\frac{x+ty}{\|x+ty\|} - \frac{y}{\|y\|}\right\|, \left\|\frac{x+ty}{\|x+ty\|} + \frac{y}{\|y\|}\right\|\right\},\ t \in \mathbb{R}\right\}$. We also define

$$\begin{aligned} \tilde{j}(X) &:= \inf\{\tilde{\beta}(x,y) :\ x, y \text{ are linearly independent}\}, \\ \tilde{J}(X) &:= \sup\{\tilde{\beta}(x,y) :\ x, y \text{ are linearly independent}\}, \end{aligned} \tag{15}$$

where $\tilde{\beta}(x,y) := \sup\left\{\min\left\{\left\|\frac{x+ty}{\|x+ty\|} - \frac{y}{\|y\|}\right\|, \left\|\frac{x+ty}{\|x+ty\|} + \frac{y}{\|y\|}\right\|\right\},\ t \in \mathbb{R}\right\}$.

In the following lemma we show that $\tilde{\alpha}(x,y)$ and $\tilde{\beta}(x,y)$ are equal.

Lemma 5. *Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$ be two independent vectors. Then there exists a unique $t_0 \in \mathbb{R}$, such that $h_+(t_0) = h_-(t_0)$ and also $\tilde{\alpha}(x,y) = \tilde{\beta}(x,y) = h_+(t_0) = h_-(t_0)$.*

Proof. By the uniqueness property of the Singer orthogonality [15] there exists a unique $t_0 \in \mathbb{R}$ such that $h_+(t_0) = h_-(t_0)$. First we show that $\tilde{\alpha}(x,y) = h_+(t_0)$. By Lemma 4, the functions $h_+(t)$ and $h_-(t)$ are increasing and decreasing respectively. Let

$$q(t) := \max\{h_+(t), h_-(t)\} = \begin{cases} h_-(t) & t \le t_0 \\ h_+(t) & t \ge t_0 \end{cases}.$$

Clearly $q(t) \ge h_+(t_0)$, for all $t \in \mathbb{R}$. So $h_+(t_0)$ is a lower bound for the set $\{q(t),\ t \in \mathbb{R}\}$. Since $q(t_0) = h_+(t_0)$, we have $\tilde{\alpha}(x,y) = h_+(t_0)$. Similarly we can show that $\tilde{\beta}(x,y) = h_+(t_0)$ and so the result holds. □

Here we indicate a relation between the orthogonalities $\perp_+$ and $\perp_-$. Let $x, y \in X$ be two independent vectors. If we define $\gamma(x,y) := \max\{\alpha[x+ty, y] + \alpha[x+ty, -y],\ t \in \mathbb{R}\}$, then we have the following proposition.

Proposition 2. *Let $(X, \|.\|)$ be a normed linear space. Then*

(i) *If $\perp_+$ is equivalent to $\perp_-$, then $\gamma(x,y) = 2\tilde{\alpha}(x,y)$ (for all independent vectors x, y).*
(ii) *If $x \perp_- y$ and $\gamma(x,y) = 2\tilde{\alpha}(x,y)$, then $x \perp_+ y$ (for all independent vectors x, y).*

Proof. (i) Suppose that $\perp_+$ is equivalent to $\perp_-$. For every $x, y \in X$, there exists a unique $t_0 \in \mathbb{R}$ such that $x + t_0 y \perp_- y$ and by Lemma 5, $\alpha[x + t_0 y, y] = \alpha[x + t_0 y, -y] = \tilde{\alpha}(x, y)$. On the other hand since $\perp_+$ and $\perp_-$ are equivalent, $x + t_0 y \perp_+ y$, i.e. $g_{x,y}(t) \leq g_{x,y}(t_0)$ for all $t \in \mathbb{R}$ and so $\gamma(x, y) = \max_t g_{x,y}(t) = g_{x,y}(t_0) = h_+(t_0) + h_-(t_0) = 2\tilde{\alpha}(x, y)$.

(ii) If $x \perp_- y$, then $h_-(0) = h_+(0) = \tilde{\alpha}(x, y)$. Since $\gamma(x, y) = 2\tilde{\alpha}(x, y)$, $\max_t g_{x,y}(t) = \gamma(x, y) = 2\tilde{\alpha}(x, y) = h_-(0) + h_+(0) = g_{x,y}(0)$. So $x \perp_+ y$. □

Due to Lemma 5, $\tilde{g}(X) = \tilde{j}(X)$ and $\tilde{G}(X) = \tilde{J}(X)$. In the sequel we just apply the constants (14). Now we provide some relations between the constants (14) and the constants which were defined by Gao and Lau (12).

Proposition 3. *Let $(X, \|.\|)$ be a normed linear space and $x, y \in S_X$ be two independent vectors. Then*

(i) $\alpha(y) \leq \tilde{\alpha}(x, y)$.

(ii) *If* $\dim(X) = 2$, *then* $\alpha(y) = \tilde{\alpha}(x, y)$.

Proof. (i) Obviously the set $\left\{ \max\left\{ \left\| \frac{x+ty}{\|x+ty\|} - y \right\|, \left\| \frac{x+ty}{\|x+ty\|} + y \right\| \right\}, t \in \mathbb{R} \right\}$ is a subset of $\left\{ \max\{\|x - y\|, \|x + y\|\}, x \in S_X \right\}$. So $\alpha(y) \leq \tilde{\alpha}(x, y)$.

(ii) In order to prove that $\tilde{\alpha}(x, y) \leq \alpha(y)$,

it is sufficient to show that for all $u \in S_X$ there exists $t \in \mathbb{R}$, such that $\max\left\{ \left\| \frac{x+ty}{\|x+ty\|} - y \right\|, \left\| \frac{x+ty}{\|x+ty\|} + y \right\| \right\} \leq \max\{\|u - y\|, \|u + y\|\}$. Let $u \in S_X$, we can find $s_1, s_2 \in \mathbb{R}$ such that $u = s_1 x + s_2 y$. Since $\|u\| = 1$, $u = \frac{s_1 x + s_2 y}{\|s_1 x + s_2 y\|}$. If $s_1 \neq 0$, then $u = \frac{s_1\left(x + \frac{s_2}{s_1} y\right)}{|s_1| \|x + \frac{s_2}{s_1} y\|}$ and by putting $t := \frac{s_2}{s_1}$, we have

$$\max\{\|u - y\|, \|u + y\|\} = \max\left\{ \left\| \frac{x + ty}{\|x + ty\|} - y \right\|, \left\| \frac{x + ty}{\|x + ty\|} + y \right\| \right\}.$$

If $s_1 = 0$, then since $\|u\| = \|y\| = 1$, $u = \pm y$ and for all $t \in \mathbb{R}$,

$$\max\left\{ \left\| \frac{x + ty}{\|x + ty\|} - y \right\|, \left\| \frac{x + ty}{\|x + ty\|} + y \right\| \right\} \leq \max\{\|u - y\|, \|u + y\|\} = 2.$$

□

Corollary 1. *Let $(X, \|.\|)$ be a normed linear space, then*

(i) $g(X) \leq \tilde{g}(X)$ *and* $\tilde{G}(X) \leq G(X)$.

(ii) *If* $\dim(X) = 2$, *then* $g(X) = \tilde{g}(X)$ *and* $\tilde{G}(X) = G(X)$.

Along with some results due to [9], we organize the following consequences which complete the comparison between the new constants and the constants defined by Gao and Lau.

Theorem 4.1. *Let* $(X, \|.\|)$ *be a normed linear space. Then* $1 \le \tilde{g}(X) \le \sqrt{2} \le \tilde{G}(X) \le 2$ *and* $\tilde{g}(X)\tilde{G}(X) = 2$.

Proof. Obviously $1 \le \tilde{g}(X)$ and $\tilde{G}(X) \le 2$. Let x and y be two independent vectors in S_X. By Lemma 5, there exists a unique $t_0 \in \mathbb{R}$ such that $\left\|\frac{x+t_0y}{\|x+t_0y\|} - y\right\| = \left\|\frac{x+t_0y}{\|x+t_0y\|} + y\right\| = \tilde{\alpha}(x,y)$, let $z = \frac{x+t_0y}{\|x+t_0y\|}$, $p = \frac{z+y}{\tilde{\alpha}(x,y)}$, $q = \frac{z-y}{\tilde{\alpha}(x,y)}$. By considering the triangles determined by $-y, y, z$ and $p, q, 0$, we have

$$\tilde{\alpha}(x,y) = \frac{\|z-y\|}{\|p\|} = \frac{2\|y\|}{\|p-q\|},$$

so $\|p-q\| = \frac{2}{\tilde{\alpha}(x,y)}$. Similarly we have $\|p+q\| = \frac{2}{\tilde{\alpha}(x,y)}$ by considering the triangles determined by $-z, z, y$ and $-p, q, 0$. So $\tilde{\alpha}(p,q) = \frac{2}{\tilde{\alpha}(x,y)}$ and we can assume that

$$\tilde{g}(X) \le \tilde{\alpha}(x,y) \le \sqrt{2} \le \tilde{\alpha}(p,q) \le \tilde{G}(X),$$

or

$$\tilde{g}(X) \le \tilde{\alpha}(p,q) \le \sqrt{2} \le \tilde{\alpha}(x,y) \le \tilde{G}(X).$$

Now we want to prove $\tilde{g}(X)\tilde{G}(X) = 2$. For all $n \in \mathbb{N}$ there exist x_n and y_n in X, such that $\tilde{\alpha}(x_n, y_n) < \tilde{g}(X) + 1/n$. There also exist p_n and q_n in X such that $\tilde{\alpha}(x_n, y_n)\tilde{\alpha}(p_n, q_n) = 2$. So for all $n \in \mathbb{N}$,

$$2 < \tilde{g}(X)\tilde{\alpha}(p_n, q_n) + \frac{\tilde{\alpha}(p_n, q_n)}{n} \le \tilde{g}(X)\tilde{G}(X) + \frac{\tilde{\alpha}(p_n, q_n)}{n}.$$

This implies that for sufficiently large values of n, $2 \le \tilde{g}(X)\tilde{G}(X)$. On the other hand for all $n \in \mathbb{N}$, we can choose $x_n{}', y_n{}' \in X$ such that $\tilde{G}(X) - 1/n < \tilde{\alpha}(x_n{}', y_n{}')$ and by the same argument as in the above we obtain that $\tilde{g}(X)\tilde{G}(X) \le 2$. □

A normed linear space is called uniformly convex if for any $0 < \epsilon \le 2$, there exists $\delta(\epsilon) > 0$ such that for $x, y \in S_X$ with $\|x-y\| \ge \epsilon$, we have $\|x+y\| < 2 - 2\delta(\epsilon)$. Let $\delta_0(\epsilon) = \inf\{1 - 1/2\|x+y\| : \ x, y \in S_X, \|x-y\| \ge \epsilon\}$ be the modulus of convexity of X. Clearly X is uniformly convex if and only if $\delta_0(\epsilon) > 0$. A relationship between $\delta_0(\epsilon)$ and the constant $J(X)$ was proved by Gao and Lau [9,10]. Now using these notions we prove that $\tilde{G}(X) = J(X)$ and $\tilde{g}(X) = g(X)$.

Theorem 4.2. *Let* $(X, \|.\|)$ *be a normed linear space. Then* $\tilde{G}(X) = \sup\{\epsilon : \epsilon < 2 - 2\delta_0(\epsilon)\}$. *Moreover,* $\tilde{G}(X) = J(X)$ *and* $\tilde{g}(X) = g(X)$.

Proof. Let $\epsilon_0 = \sup\{\epsilon : \ \epsilon < 2 - 2\delta_0(\epsilon)\}$, first we show that $\tilde{G}(X) \le \epsilon_0$. Since by Theorem 4.1 $\tilde{G}(X) \le 2$, the inequality is obvious if $\epsilon_0 = 2$. So we can assume that $\epsilon_0 < 2$. Let $\epsilon > \epsilon_0$ and $x, y \in X$. Then either $\left\|\frac{x+ty}{\|x+ty\|} - \frac{y}{\|y\|}\right\| \le \epsilon$, or

$\left\| \frac{x+ty}{\|x+ty\|} - \frac{y}{\|y\|} \right\| \geq \epsilon$. In the latter case, we have $\left\| \frac{x+ty}{\|x+ty\|} + \frac{y}{\|y\|} \right\| \leq 2 - 2\delta_0(\epsilon) \leq \epsilon$. Hence $\tilde{\beta}(x, y) \leq \epsilon$. Since x, y, ϵ are arbitrary, we have $\tilde{G}(X) \leq \epsilon_0$.

On the other hand, Let $0 < \eta < \epsilon_0/3$, and $\epsilon = \epsilon_0 - \eta$. Then there exist $x, y \in S_X$ such that $\|x - y\| > \epsilon$ and $\|x + y\| > 2 - 2\delta_0(\epsilon) - 2\eta$. So

$$\begin{aligned} \tilde{\beta}(x, y) &\geq \min\{\|y - x\|, \|y + x\|\} \\ &\geq \min\{\epsilon, 2 - 2\delta_0(\epsilon) - 2\eta\} \\ &\geq \min\{\epsilon, \epsilon - 2\eta\} \\ &\geq \epsilon_0 - 3\eta. \end{aligned}$$

Since η is arbitrary, we have $\tilde{G}(X) \geq \tilde{\beta}(x, y) \geq \epsilon_0$. Now by using [9, Theorem 2.5 and Theorem 3.3] and since by Theorem 4.1, $\tilde{g}(X)\tilde{G}(X) = 2$, we have $\tilde{G}(X) = J(X)$ and $\tilde{g}(X) = g(X)$. □

A normed linear space is called uniformly non square if there exists a $\delta > 0$ such that for $x, y \in S_X$, either

$$\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta \quad \text{or} \quad \left\| \frac{1}{2}(x - y) \right\| \leq 1 - \delta.$$

Using the the constant $\tilde{G}(X)$ and [9, Theorem 3.4] we obtain a necessary and sufficient condition for uniform non squareness in normed linear spaces.

Corollary 2. *Let $(X, \|.\|)$ be a normed linear space. Then X is uniformly non square if and only if $\tilde{G}(X) < 2$.*

4.1. Examples

In the following we compute the constants (14) for l_p and L_p spaces.

Example 3. Let p and q be two positive numbers such that $1/p + 1/q = 1$. Then

(i) For $2 \leq p < \infty$, $\tilde{g}(l_p) = 2^{1/p}$ and $\tilde{G}(l_p) = 2^{1/q}$.
(ii) For $1 \leq p < 2$, $\tilde{g}(l_p) = 2^{1/q}$ and $\tilde{G}(l_p) = 2^{1/p}$.
(iii) $\tilde{g}(l_\infty) = 1$ and $\tilde{G}(l_\infty) = 2$.

We recall the Clarkson inequality when $p \geq 2$, $x, y \in X$,

$$2(\|x\|^p + \|y\|^p) \leq \|x + y\|^p + \|x - y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p),$$

and when $1 < p \leq 2$ these inequalities hold in the reverse sense.

(i) By the Clarkson inequality we have $2^{1/p} \leq \tilde{\alpha}(x, y) \leq 2^{1/q}$ and so $2^{1/p} \leq \tilde{g}(l_P)$ but since $\tilde{\alpha}(e_1, e_2) = 2^{1/p}$, $\tilde{g}(l_p) = 2^{1/p}$ and $\tilde{G}(l_p) = 2^{1/q}$.
(ii) We have $2^{1/q} \leq \tilde{\alpha}(x, y) \leq 2^{1/p}$. So $\tilde{G}(l_p) \leq 2^{1/p}$, we also have $\tilde{\alpha}(e_1, e_2) = 2^{1/p}$. Hence $\tilde{G}(l_p) = 2^{1/p}$ and $\tilde{g}(l_p) = 2^{1/q}$.

(iii) Clearly $1 \geq \tilde{g}(l_\infty)$. Since $\tilde{\alpha}(e_1, e_2) = 1$, $\tilde{g}(l_\infty) = 1$ and $\tilde{G}(l_\infty) = 2$.

Example 4. Let p and q be two positive numbers such that $1/p + 1/q = 1$. Then

(i) For $2 \leq p < \infty$, $\tilde{g}(L_p) = 2^{1/p}$ and $\tilde{G}(L_p) = 2^{1/q}$.
(ii) For $1 \leq p < 2$, $\tilde{g}(L_p) = 2^{1/q}$ and $\tilde{G}(L_p) = 2^{1/p}$.
(iii) $\tilde{g}(L_\infty) = 1$ and $\tilde{G}(L_\infty) = 2$.

(i) By the Clarkson inequality we have $2^{1/p} \leq \tilde{\alpha}(x, y) \leq 2^{1/q}$ and $\tilde{G}(L_p) \leq 2^{1/q}$. If $x(t) = 1$ and

$$y(t) = \begin{cases} 1 & 0 \leq t < 1/2 \\ -1 & 1/2 \leq t \leq 1 \end{cases}$$

then $\tilde{\alpha}(x, y) = 2^{1/q}$. So $\tilde{G}(L_p) = 2^{1/q}$ and $\tilde{g}(L_p) = 2^{1/p}$.
(ii) For $1 \leq p < 2$ we have $2^{1/q} \leq \tilde{\alpha}(x, y) \leq 2^{1/p}$ and $2^{1/q} \leq \tilde{g}(L_p)$. If we take $x(t)$ and $y(t)$ as in part (i), then $\tilde{g}(L_p) = 2^{1/q}$ and $\tilde{G}(L_p) = 2^{1/p}$.
(iii) Clearly $\tilde{G}(L_\infty) \leq 2$. By taking $x(t)$ and $y(t)$ as in part (i), $\tilde{G}(L_\infty) = 2$ and as a result $\tilde{g}(L_\infty) = 1$.

Remark 3. Let $X = l_p$, $2 < p < \infty$. Then $\tilde{G}(l_p) \neq G(l_p)$. Since $\tilde{G}(l_p) = 2^{1/q}$ and as in [9, Theorem 3.1], $G(l_p) = 2^{1/p}$.

4.2. A characterization of inner product spaces

Let X be a normed linear space. Then

$$x \perp_w y \qquad \text{whenever} \qquad \|x\|\|y\| = 0 \ \text{ or } \ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = \sqrt{2}.$$

It is known that the orthogonality $\perp_w$ is non-degenerate, symmetric and existent [4]. Dimmine et al. [8] showed that this notion of orthogonality is not homogeneous in general. He also mentioned the problem of whether the additivity of the orthogonality characterizes inner product spaces or not. In the sequel we show that the answer to this problem is affirmative when $\dim(X) \geq 3$. First we need the following two lemmas.

Lemma 6. [2, Proposition 1] *Let X be a normed linear space, $\lambda > 0$, $0 < \epsilon < 2$ and $\delta(\epsilon)$ denotes the modulus of convexity of the space. Then the properties P_λ, Q_ϵ and R_ϵ are equivalent when $\lambda = \epsilon(4 - \epsilon^2)^{-1/2}$ and*

$P_\lambda :\ x, y \in S_X \ \ \|x + \lambda y\| = \|x - \lambda y\|$ implies $\|x + \lambda y\| = 1 + \lambda^2$,
$Q_\epsilon :\ x, y \in S_X \ \ \|x - y\| = \epsilon$ implies $\|x + y\|^2 = 4 - \epsilon^2$,
$R_\epsilon :\ \delta(\epsilon) = 1 - (1 - \epsilon^2/4)^{1/2}$.

Lemma 7. [11, Theorem 5] *Let X be a normed linear space with* $\dim(X) \geq 3$. *If the Singer orthogonality is additive, then X is an inner product space.*

Theorem 4.3. *Let* $(X, \|.\|)$ *be a normed linear space and* $\dim(X) \geq 3$*. Then the following are equivalent.*

(i) *The orthogonality* $\perp_w$ *is additive,*
(ii) *The Singer orthogonality is additive,*
(iii) X *is an inner product space.*

Proof. Using Lemma 7, it suffices to show that (i) implies (ii). First we show that the following statements are equivalent:

(a) The orthogonality $\perp_w$ is homogeneous.
(b) $\tilde{g}(X) = \tilde{G}(X)$.
(c) The orthogonality $\perp_w$ is equivalent to the Singer orthogonality.

(a) $\Rightarrow$ (b): Let $x, y \in X$. By the existence and homogeneity properties of the orthogonality $\perp_w$, there exists $t_0 \in \mathbb{R}$ such that $\alpha[x+t_0y, y] = \alpha[x+t_0y, -y] = \sqrt{2}$. So $\tilde{\alpha}(x, y) = \sqrt{2}$ for all $x, y \in X$ and as a result $\tilde{g}(X) = \tilde{G}(X) = \sqrt{2}$.
(b) $\Rightarrow$ (c): Let $x, y \in X$ and $x \perp_w y$. Using Lemma 6 and the fact that $\tilde{g}(X) = \tilde{G}(X) = \sqrt{2}$, we have $x \perp_S y$. On the other hand if $x \perp_S y$, then $\alpha[x, y] = \alpha[x, -y]$. It is enough to show that $\alpha[x, y] = \sqrt{2}$. Since $\tilde{g}(X) = \tilde{G}(X) = \sqrt{2}$, we have $\alpha[x, y] = \sqrt{2}$ and therefore $x \perp_w y$.
(c)$\Rightarrow$ (a): It is obvious, since the Singer orthogonality is homogeneous.
(i) $\Rightarrow$ (ii) Suppose that the orthogonality $\perp_w$ is additive. We show that $\perp_w$ is homogeneous. For this purpose it suffices to show that $x \perp_w y$ implies $x \perp_w -y$. Let $x \perp_w y$ and $y \neq 0$, by the existence property of the orthogonality $\perp_w$, there exists $a \in \mathbb{R}$ such that $ay - x \perp_w y$. Using the additivity we have $x \perp_w -y$. So by the above equivalent statements, the Singer orthogonality is additive. □

In the following we show that the assumption $\dim(X) \geq 3$ in the previous theorem is essential. We show that in the two dimensional case there exists a strictly convex normed linear space which is not an inner product space, but the orthogonality $\perp_w$ is additive. In fact, by considering the example which was mentioned by Diminnie et al. [8], we have a strictly convex normed linear space which is not an inner product space, but the orthogonality $\perp_w$ is homogeneous (see [8, p. 203]). Now we can see that the orthogonality $\perp_w$ is also additive in this space. Let $x, y, z \in X$, $x \perp_w y$ and $z \perp_w y$. We show that $x + z \perp_w y$. We may assume that $x, y, z \neq 0$, clearly the sets $\{x, y\}$ and $\{y, z\}$ are linearly independent. Assume $z = \alpha x + \beta y$, for some real numbers $\alpha \neq 0$ and β. So $\alpha x + \beta y \perp_w y$ and by the homogeneity of the orthogonality we have $x + \frac{\beta}{\alpha} y \perp_w y$. On the other hand, by the hypothesis of strict convexity of the space and Lemma 4 the function $h_+(t) = \left\| \frac{x+ty}{\|x+ty\|} + \frac{y}{\|y\|} \right\|$ is strictly increasing and so there exists just one $t \in \mathbb{R}$ such that $x + ty \perp_w y$. Therefore $\beta = 0$ and $z = \alpha x$ and by the homogeneity of the orthogonality we have $x + z = (\alpha + 1)x \perp_w y$.

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References

[1] Alonso, J., Benitez, C.: Orthogonality in normed linear spaces: a Survey. II: Relations between main orthogonalities. Extr. Math. **4**, 121–131 (1989)
[2] Alonso, J., Benitez, C.: Some characteristic and non-characteristic properties of inner product spaces. J. Approx. Theory **55**, 318–325 (1988)
[3] Alonso, J., Martini, H., Wu, S.: On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces. Aequat. Math. **83**, 153–189 (2012)
[4] Alsina, C., Sikorska, J., Tomás, M.S.: Norm Derivatives and Characterizations of Inner Product Spaces. World Scientific, Singapore (2010)
[5] Amir, D.: Characterizations of Inner Product Spaces, Operator Theory: Advances and Applications, vol. 20. Birkhauser Verlag, Basel (1986)
[6] Chmieliński, J., Wójcik, P.: On a ρ-orthogonality. Aequat. Math. **80**, 45–55 (2010)
[7] Dadipour, F., Moslehian, M.S.: A characterization of inner product spaces related to the p-angular distance. J. Math. Anal. Appl. **371**(2), 677–681 (2010)
[8] Diminnie, C.R., Andalafte, E.Z., Freese, R.W.: Angles in normed linear spaces and a characterization of real inner product spaces. Math. Nachr. **129**, 197–204 (1986)
[9] Gao, J., Lau, K.S.: On the geometry of spheres in normed linear spaces. J. Aust. Math. Soc. (Ser A) **48**, 101–112 (1990)
[10] Gao, J., Lau, K.S.: On two classes of Banach spaces with uniform normal structure. Stud. Math. **99**(1), 40–56 (1991)
[11] Lin, P.K.: A remark on the Singer-orthogonality in normed linear spaces. Math. Nachr. **160**, 325–328 (1993)
[12] Milicic, P.M.: On the B-angle and g-angle in normed spaces. J. Inequal. Pure Appl. Math. **8**(3), 1–9 (2007)
[13] Moslehian, M.S., Dadipour, F., Rajić, R., Marić, A.: A glimpse at the Dunkle–Williams inequality. Banach J. Math. Anal. **5**(2), 138–151 (2011)
[14] Zamani, A., Moslehian, M.S.: Approximate Roberts orthogonality. Aequat. Math. (2013). doi:10.1007/s00010-013-0233-7
[15] Zheng, L., Yadong, Z.: Singer orthogonality and characterizations of inner product spaces. Arch. Math. **55**, 588–594 (1990)

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