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Aequationes Mathematicae

A variant of Wigner's functional equation

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Abstract. We characterize mappings between inner product spaces satisfying a certain pair of functional equations. As a consequence a short proof of Wigner's theorem for real, complex or quaternionic inner spaces is presented.

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1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be inner product spaces (not necessarily complete) over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, where $\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers and $\mathbb{H}$ is the field of quaternions. Let us say that a mapping $f : \mathcal{H} \to \mathcal{K}$ satisfies condition (W) if

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \qquad (x, y \in \mathcal{H})$$

and it satisfies condition (R) if

$$\mathrm{Re}\langle f(x), f(y)\rangle = \mathrm{Re}\langle x, y\rangle \qquad (x, y \in \mathcal{H}).$$

Suppose for a moment that $\mathcal{H}$ and $\mathcal{K}$ are complex inner product spaces and $f : \mathcal{H} \to \mathcal{K}$ is a mapping satisfying $\langle f(x), f(y)\rangle = \langle x, y\rangle$, $x, y \in \mathcal{H}$, or $\langle f(x), f(y)\rangle = \langle y, x\rangle$, $x, y \in \mathcal{H}$. Then a mapping f certainly satisfies both conditions (W) and (R). Can we say something more about the mapping f? It follows easily from $\langle f(x), f(y)\rangle = \langle x, y\rangle$, $x, y \in \mathcal{H}$, that f is a linear isometry. Similarly, it follows from $\langle f(x), f(y)\rangle = \langle y, x\rangle$, $x, y \in \mathcal{H}$, that f is an anti-linear isometry, where anti-linear means $f(\lambda x + \mu y) = \lambda^* f(x) + \mu^* f(y)$, $\lambda, \mu \in \mathbb{C}$, and the star denotes complex conjugation. So, the natural question is whether linear or anti-linear isometries are the only solutions of the pair

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of functional equations (W) and (R). In Theorem 2.2 we show that this is indeed the case. Recall that the identity and conjugation are the only continuous automorphisms $\mathbb{C} \to \mathbb{C}$, hence linear isometries or anti-linear isometries, that are solutions of (W) and (R), are precisely all mappings $f : \mathcal{H} \to \mathcal{K}$ satisfying $\langle f(x), f(y)\rangle = \varphi(\langle x, y\rangle)$, $x, y \in \mathcal{H}$, where $\varphi : \mathbb{C} \to \mathbb{C}$ is a continuous automorphism. This is precisely what happens also in the quaternionic case. In Theorem 2.3 we prove that solutions of (W) and (R) in the quaternionic setting are mappings f such that $\langle f(x), f(y)\rangle = \varphi(\langle x, y\rangle)$, $x, y \in \mathcal{H}$, where $\varphi : \mathbb{H} \to \mathbb{H}$ is an automorphism.

The famous Wigner's theorem says that in the case of inner product spaces over $\mathbb{C}$ the solutions of functional equation (W) are mappings f of the form $f(x) = \sigma(x)Ux$, $x \in \mathcal{H}$, where $U : \mathcal{H} \to \mathcal{K}$ is either a linear isometry or an anti-linear isometry and $\sigma : \mathcal{H} \to \mathbb{C}$ is a so called phase function, which means that its values are of modulus one. This celebrated result plays a very important role in quantum mechanics and in representation theory in physics.

There are several proofs of this result, see [2,4–7,9,11,13] to list just some of them. The quaternionic version of Wigner's theorem is proved in [12] and in [10] Uhlhorn's generalization of Wigner's theorem, see [14], it is proved in quaternionic indefinite inner product spaces. For generalizations to Hilbert C^* -modules see [1,8].

As a consequence of Theorem 2.2 and Theorem 2.3 we are able to give a short, elementary and unified proof of Wigner's theorem for real, complex or quaternionic inner spaces. Our approach is not new, we follow the ideas of Bargmann, Sharma and Almeida, see [2,11,12].

Let us introduce some notations and basic facts about quaternions. Recall that the field of quaternions $\mathbb{H} = \{\xi_0 + \xi_1 i + \xi_2 j + \xi_3 k : \xi_0, \xi_1, \xi_2, \xi_3 \in \mathbb{R}\}$. For $\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$, ξ^* is defined by $\xi^* = \xi_0 - \xi_1 i - \xi_2 j - \xi_3 k$ and $|\xi|$ by $|\xi| = \sqrt{\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2}$. When $\xi \in \mathbb{C}$, then ξ^* denotes a conjugate complex number and when $\xi \in \mathbb{R}$, then ξ^* is just ξ.

We will say that ξ_0 is the real part of ξ. Quaternions $\xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$ can be identified with ordered pairs $(\xi_0, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^3$, where $\boldsymbol{\xi} = \xi_1 \mathbf{i} + \xi_2 \mathbf{j} + \xi_3 \mathbf{k}$ and the triple $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is the standard orthonormal basis of $\mathbb{R}^3$.

Let $\mathcal{H}$ be a (left) vector space over $\mathbb{F}$. An inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a vector space together with the inner product $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ satisfying

(i) $\langle x, y\rangle = \langle y, x\rangle^*$
(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$
(iii) $\langle x, \alpha y + \beta z\rangle = \langle x, y\rangle\alpha^* + \langle x, z\rangle\beta^*$
(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$

for all $\alpha, \beta \in \mathbb{F}$ and all $x, y, z \in \mathcal{H}$. If $\langle\cdot,\cdot\rangle$ is an inner product on $\mathcal{H}$, then $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on $\mathcal{H}$. The geometry of quaternionic inner product spaces is similar to that of complex inner product spaces. For example, the

Cauchy–Schwarz inequality $|\langle x, y\rangle| \le \|x\|\|y\|, x, y \in \mathcal{H}$, holds and the notion of orthogonal complement is as in complex inner product spaces. Two mappings $f, g : \mathcal{H} \to \mathcal{K}$ are phase equivalent if $f(x) = \sigma(x)g(x), x \in \mathcal{H}$, where $\sigma : \mathcal{H} \to \mathbb{F}, |\sigma(x)| = 1, x \in \mathcal{H}$, is a phase function.

2. Results

The next proposition shows that a mapping satisfying functional equation (W) has a property close to linearity. See also [11, Lemma 5] for a different proof in the case of complex inner product spaces.

Proposition 2.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be inner product spaces over $\mathbb{F}$ and suppose that $f : \mathcal{H} \to \mathcal{K}$ satisfies (W).*

(1) *Let $x \in \mathcal{H}$ and $\lambda \in \mathbb{F}$. Then $f(\lambda x) = \lambda' f(x)$, where $\lambda' \in \mathbb{F}$, and $|\lambda'| = |\lambda|$.*
(2) *Let x and y be nonzero orthogonal vectors. Then $f(x+y) = \|x\|^{-2}\langle f(x+y), f(x)\rangle f(x) + \|y\|^{-2}\langle f(x+y), f(y)\rangle f(y)$.*

Proof. (1) $|\langle f(\lambda x), f(x)\rangle| = |\langle \lambda x, x\rangle| = \|\lambda x\|\|x\| = \|f(\lambda x)\|\|f(x)\|$. By the equality condition in the Cauchy–Schwarz inequality it follows that $f(\lambda x)$ and $f(x)$ are linearly dependent. Thus $f(\lambda x) = \lambda' f(x)$ for some $\lambda' \in \mathbb{F}$. Since f preserves the length of vectors it follows that $|\lambda'| = |\lambda|$.

(2) Let x and y be nonzero orthogonal vectors and denote $\alpha = \|x\|^{-2}\langle f(x+y), f(x)\rangle$ and $\beta = \|y\|^{-2}\langle f(x+y), f(y)\rangle$. Note that $|\alpha| = |\beta| = 1$ and that $\langle f(x+y), f(x)\rangle = \alpha\|x\|^2, \langle f(x+y), f(y)\rangle = \beta\|y\|^2$.

Then

$$\begin{aligned}\|f(x+y) - \alpha f(x) - \beta f(y)\|^2 &= \|f(x+y)\|^2 + \|\alpha f(x)\|^2 + \|\beta f(y)\|^2 \\ &\quad - 2\mathrm{Re}\langle f(x+y), \alpha f(x)\rangle - 2\mathrm{Re}\langle f(x+y), \beta f(y)\rangle \\ &= \|x+y\|^2 + \|x\|^2 + \|y\|^2 - 2\mathrm{Re}\langle f(x+y), f(x)\rangle\alpha^* - 2\mathrm{Re}\langle f(x+y), f(y)\rangle\beta^* \\ &= 2\|x\|^2 + 2\|y\|^2 - 2\mathrm{Re}\,\alpha\alpha^*\|x\|^2 - 2\mathrm{Re}\,\beta\beta^*\|y\|^2 = 0.\end{aligned}$$

□

Remark 2.1. Note that λ' in the previous proposition in general depends on λ and x, that is $\lambda' = \lambda'(\lambda, x)$.

As already mentioned in the introduction, if $\mathcal{H}$ and $\mathcal{K}$ are complex inner product spaces and $f : \mathcal{H} \to \mathcal{K}$ is a linear or anti-linear isometry, then f satisfies conditions (W) and (R). The next theorem shows that the converse is true.

Theorem 2.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be inner product spaces over $\mathbb{C}$ and suppose that $f : \mathcal{H} \to \mathcal{K}$ satisfies conditions (W) and (R). Then f is either a linear isometry or an anti-linear isometry.*

Proof. Let $0 \ne x \in \mathcal{H}$. By Proposition 2.1 $f(ix) = \lambda(x)f(x)$, where $|\lambda(x)| = 1$. Then from

$$\mathrm{Re}\langle\lambda(x)f(x), f(x)\rangle = \mathrm{Re}\langle f(ix), f(x)\rangle = \mathrm{Re}\langle ix, x\rangle = 0,$$

it follows that $\mathrm{Re}\,\lambda(x) = 0$, hence $\lambda(x) = \pm i, x \in \mathcal{H}$. Suppose that $f(ix) = if(x)$ for some $0 \neq x \in \mathcal{H}$. Then

$$\begin{aligned}-\mathrm{Im}\langle f(x), f(y)\rangle &= \mathrm{Re}\, i\langle f(x), f(y)\rangle = \mathrm{Re}\langle f(ix), f(y)\rangle = \mathrm{Re}\langle ix, y\rangle\\ &= \mathrm{Re}\, i\langle x, y\rangle = -\mathrm{Im}\langle x, y\rangle.\end{aligned}$$

Hence $\langle f(x), f(y)\rangle = \langle x, y\rangle$ for all $y \in \mathcal{H}$. If $f(ix) = if(x)$ for all $x \in \mathcal{H}$ this shows that $\langle f(x), f(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$. But then f is a linear isometry.

Similarly, f is an anti-linear isometry in the case $f(ix) = -if(x)$ for all $x \in \mathcal{H}$. It remains to show that the third possibility, that is $f(ix) = if(x)$ and $f(ix') = -if(x')$ for some nonzero $x, x' \in \mathcal{H}$, leads to a contradiction. Indeed, since $f(ix) = if(x)$, it follows by the previous consideration that

$$\langle f(x), f(ix')\rangle = \langle x, ix'\rangle = -i\langle x, x'\rangle.$$

On the other hand, since $f(ix') = -if(x')$, we get

$$\langle f(x), f(ix')\rangle = \langle f(x), -if(x')\rangle = i\langle f(x), f(x')\rangle = i\langle x, x'\rangle.$$

If $\dim \mathcal{H} = 1$ this is a contradiction. If $\dim \mathcal{H} \geq 2$, then x and x' must be orthogonal and furthermore, for any two non orthogonal vectors, say u and v, either $f(iu) = if(u)$ and $f(iv) = if(v)$ or $f(iu) = -if(u)$ and $f(iv) = -if(v)$. Now choose $z \neq 0$ which is neither orthogonal to x nor to x', say $z = x + x'$. Then $f(iz) = if(z)$ since z and x are not orthogonal and $f(iz) = -if(z)$ since z and x' are not orthogonal. This is a contradiction and the proof is completed. □

The next theorem is a quaternionic analogue of the previous one.

Theorem 2.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be quaternionic inner product spaces with $\dim \mathcal{H} \geq 2$ and $f : \mathcal{H} \to \mathcal{K}$ be a mapping satisfying (W) and (R). Then there is a unit quaternion ξ such that ξf is a linear isometry.*

Proof. Let $x \neq 0$. Then by Proposition 2.1 $f(ix) = \lambda(x)f(x), f(jx) = \mu(x)$ $f(x), f(kx) = \nu(x)f(x)$, where $|\lambda(x)| = |\mu(x)| = |\nu(x)| = 1$. Next,

$$\mathrm{Re}\langle f(ix), f(x)\rangle = \mathrm{Re}\langle ix, x\rangle = 0$$

implies that $\mathrm{Re}\lambda(x) = 0$. Similarly we conclude that $\mathrm{Re}\mu(x) = \mathrm{Re}\nu(x) = 0$. From

$$\begin{aligned}\mathrm{Re}\lambda(x)\mu(x)^*\|f(x)\|^2 &= \mathrm{Re}\langle\lambda(x)f(x), \mu(x)f(x)\rangle = \mathrm{Re}\langle f(ix), f(jx)\rangle = \mathrm{Re}\langle ix, jx\rangle\\ &= \mathrm{Re}ij^*\|x\|^2 = 0,\end{aligned}$$

it follows that $\mathrm{Re}\lambda(x)\mu(x)^* = 0$. Similarly, $\mathrm{Re}\lambda(x)\nu(x)^* = \mathrm{Re}\mu(x)\nu(x)^* = 0$. Write

$$\lambda(x) = \lambda_1(x)i + \lambda_2(x)j + \lambda_3(x)k, \quad \mu(x) = \mu_1(x)i + \mu_2(x)j + \mu_3(x)k.$$

Then

$$\mathrm{Re}\lambda(x)\mu(x)^* = \lambda_1(x)\mu_1(x) + \lambda_2(x)\mu_2(x) + \lambda_3(x)\mu_3(x) = 0.$$

Let $\nu(x) = \nu_1(x)i + \nu_2(x)j + \nu_3(x)k$. Then $\mathrm{Re}\lambda(x)\nu(x)^* = \mathrm{Re}\mu(x)\nu(x)^* = 0$ implies that

$$\lambda_1(x)\nu_1(x) + \lambda_2(x)\nu_2(x) + \lambda_3(x)\nu_3(x) = \mu_1(x)\nu_1(x) + \mu_2(x)\nu_2(x) \\ +\mu_3(x)\nu_3(x) = 0.$$

Hence the matrix

$$Q(x) = \begin{bmatrix} \lambda_1(x) & \mu_1(x) & \nu_1(x) \\ \lambda_2(x) & \mu_2(x) & \nu_2(x) \\ \lambda_3(x) & \mu_3(x) & \nu_3(x) \end{bmatrix} \in M_3(\mathbb{R})$$

is orthogonal, that is $Q(x)Q(x)^T = Q(x)^T Q(x) = I$.

Let x and y be nonzero orthogonal vectors and by Proposition 2.1 write

$$f(x+y) = \alpha f(x) + \beta f(y), \quad |\alpha| = |\beta| = 1.$$

Then

$$\mathrm{Re}(\alpha\|x\|^2) = \mathrm{Re}\langle \alpha f(x) + \beta f(y), f(x)\rangle = \mathrm{Re}\langle f(x+y), \\ f(x)\rangle = \mathrm{Re}\langle x+y, x\rangle = \|x\|^2.$$

Hence $\alpha = 1$, similarly we get $\beta = 1$. Thus f is orthogonally additive, that is $f(x+y) = f(x) + f(y)$ whenever x and y are orthogonal. Now ix and iy are also orthogonal and

$$f(i(x+y)) = f(ix+iy) = f(ix) + f(iy) = \lambda(x)f(x) + \lambda(y)f(y). \tag{1}$$

On the other hand,

$$f(i(x+y)) = \lambda(x+y)f(x+y) = \lambda(x+y)f(x) + \lambda(x+y)f(y). \tag{2}$$

From (1) and (2) it follows that

$$\lambda(x) = \lambda(y) = \lambda(x+y), \tag{3}$$

whenever x and y are nonzero orthogonal vectors. Next we will show that λ is a constant function. Choose any $0 \neq u \in \mathcal{H}$. First we show that $\lambda(\gamma u) = \lambda(u), \gamma \in \mathbb{H}$. Let u' be orthogonal to u. Then u' is orthogonal also to γu, hence by (3) $\lambda(u) = \lambda(u') = \lambda(\gamma u)$. Now choose any $v \in \mathcal{H}$. If u and v are linearly dependent, that is $v = \gamma u$ for some $\gamma \in \mathbb{H}$, then we already know that $\lambda(v) = \lambda(u)$. If u and v are linearly independent, then write $u = \gamma x + \delta y, v = \gamma' x + \delta' y$, where x, y are orthogonal vectors, $\gamma, \gamma', \delta, \delta' \in \mathbb{H}$. By (3)

$$\lambda(u) = \lambda(\gamma x + \delta y) = \lambda(\gamma x) = \lambda(\gamma' x) = \lambda(\gamma' x + \delta' y) = \lambda(v).$$

Analogous reasoning shows that μ and ν are constant functions and so the matrix $Q(x) = Q$ is a constant matrix.

Now let $x, y \in \mathcal{H}$ be arbitrary and write

$$\begin{aligned}\langle x, y\rangle &= \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k = (\alpha_0, \boldsymbol{\alpha}),\ \langle f(x), f(y)\rangle \\ &= \alpha_0 + \beta_1 i + \beta_2 j + \beta_3 k = (\alpha_0, \boldsymbol{\beta}).\end{aligned}$$

From $\mathrm{Re}\langle ix, y\rangle = \mathrm{Re}\langle f(ix), f(y)\rangle = \mathrm{Re}(\lambda\langle f(x), f(y)\rangle$ it follows that

$$\alpha_1 = \lambda_1\beta_1 + \lambda_2\beta_2 + \lambda_3\beta_3.$$

Similarly,

$$\alpha_2 = \mu_1\beta_1 + \mu_2\beta_2 + \mu_3\beta_3, \quad \alpha_3 = \nu_1\beta_1 + \nu_2\beta_2 + \nu_3\beta_3,$$

hence $\boldsymbol{\beta} = Q\boldsymbol{\alpha}$. This shows that

$$\langle f(x), f(y)\rangle = (\alpha_0, \boldsymbol{\beta}) = (\alpha_0, Q\boldsymbol{\alpha}) = \varphi((\alpha_0, \boldsymbol{\alpha})) = \varphi(\langle x, y\rangle),$$

where $\varphi : \mathbb{H} \to \mathbb{H}$ is defined by $\varphi((\xi_0, \boldsymbol{\xi})) = (\xi_0, Q\boldsymbol{\xi}), \xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k = (\xi_0, \boldsymbol{\xi})$. It is easy to check, see [10], that φ is an automorphism in the case $\det Q = 1$, and an anti-automorphism in the case $\det Q = -1$. Let us show that φ is an automorphism. Indeed, let $x \in \mathcal{H}$ be a unit vector and $\alpha \in \mathbb{H}$. Then by Proposition 2.1 $f(\alpha x) = \alpha' f(x)$, hence

$$\alpha' = \langle \alpha' f(x), f(x)\rangle = \langle f(\alpha x), f(x)\rangle = \varphi(\langle \alpha x, x\rangle) = \varphi(\alpha).$$

Now let $\alpha, \beta \in \mathbb{H}$ be arbitrary, $x \in \mathcal{H}$ be a unit vector and compute

$$\varphi(\alpha\beta^*) = \varphi(\langle \alpha x, \beta x\rangle) = \langle f(\alpha x), f(\beta x)\rangle = \langle \varphi(\alpha) f(x), \varphi(\beta) f(x)\rangle = \varphi(\alpha)\varphi(\beta^*).$$

Thus φ is indeed an automorphism. Since φ must take the center onto itself, this means that $\varphi(\mathbb{R}) = \mathbb{R}$. Then the Skolem–Noether theorem [3, p. 262] says that φ is an inner automorphism, that is $\varphi(\alpha) = \xi^*\alpha\xi$ for some unit quaternion ξ. Then from $\langle f(x), f(y)\rangle = \varphi(\langle x, y\rangle) = \xi^*\langle x, y\rangle\xi$ it follows that $\langle \xi f(x), \xi f(y)\rangle = \langle x, y\rangle, x, y \in \mathcal{H}$, from which it follows that ξf is a linear isometry. □

Now we are ready to prove Wigner's theorem. We follow [2,11,12].

Theorem 2.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be inner product spaces over $\mathbb{F}$ and $f : \mathcal{H} \to \mathcal{K}$ be a mapping satisfying*

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \qquad (x, y \in \mathcal{H}).$$

(i) *If $\mathbb{F} = \mathbb{R}$ and $\dim \mathcal{H} \geq 2$, then f is phase equivalent to a linear isometry.*
(ii) *If $\mathbb{F} = \mathbb{C}$ and $\dim \mathcal{H} \geq 2$, then f is phase equivalent to either a linear isometry or to an anti-linear isometry.*
(iii) *If $\mathbb{F} = \mathbb{H}$ and $\dim \mathcal{H} \geq 3$, then f is phase equivalent to a linear isometry.*

Proof. Choose and fix a unit vector $e \in \mathcal{H}$ and let $\mathcal{V} = \{e\}^\perp$. Let $0 \neq x \in \mathcal{V}$ be arbitrary and by Proposition 2.1 write

$$\begin{aligned}f(e + x) &= \langle f(e + x), f(e)\rangle f(e) + \|x\|^{-2}\langle f(e + x), f(x)\rangle f(x) \\ &= \alpha(x) f(e) + \beta(x) f(x).\end{aligned}$$

Then

$$\alpha(x)^* f(e+x) = f(e) + \alpha(x)^* \beta(x) f(x).$$

Define a mapping $g : (e+\mathcal{V}) \cup \mathcal{V} \to \mathcal{K}$ as follows:

$$g(e) = f(e), \quad g(x) = \alpha(x)^* \beta(x) f(x), \quad g(e+x) = g(e) + g(x),$$

where $0 \neq x \in \mathcal{V}$. Then g is phase equivalent to f, hence

$$|\langle g(x), g(y)\rangle| = |\langle x, y\rangle|, \tag{4}$$

and

$$|1 + \langle g(x), g(y)\rangle| = |\langle g(e+x), g(e+y)\rangle| = |\langle e+x, e+y\rangle| = |1 + \langle x, y\rangle| \tag{5}$$

for all $x, y \in \mathcal{V}$. From (4) and (5) it follows that $\mathrm{Re}\langle g(x), g(y)\rangle = \mathrm{Re}\langle x, y\rangle$ for all $x, y \in \mathcal{V}$. If $\mathbb{F} = \mathbb{R}$, then clearly $g|_{\mathcal{V}} : \mathcal{V} \to \mathcal{K}$ is a linear isometry. In the complex case Theorem 2.2 implies that $g|_{\mathcal{V}} : \mathcal{V} \to \mathcal{K}$ is either a linear isometry or an anti-linear isometry. In the quaternionic case we use Theorem 2.3 to conclude that $\xi g|_{\mathcal{V}} : \mathcal{V} \to \mathcal{K}$ is a linear isometry for some unit quaternion ξ. Let us finish the proof in the quaternionic case, the others being analogous. Extend $\xi g|_{\mathcal{V}}$ to a linear isometry $U : \mathcal{H} \to \mathcal{K}$, that is $Ue = \xi g(e), Ux = \xi g(x)$ if $x \in \mathcal{V}$ and $U(\lambda e + x) = \lambda Ue + Ux$. It remains to show that U and f are phase equivalent. If $\lambda = 0$ they are, so suppose that $\lambda \neq 0$. Then

$$\begin{aligned} f(\lambda e + x) &= f(\lambda(e + \lambda^{-1}x)) = \lambda' f(e + \lambda^{-1}x) = \lambda' \alpha(\lambda^{-1}x) g(e + \lambda^{-1}x) \\ &= \lambda' \alpha(\lambda^{-1}x) \xi^* U(e + \lambda^{-1}x) = \lambda' \alpha(\lambda^{-1}x) \xi^* \lambda^{-1} U(\lambda e + x). \end{aligned}$$

Since $|\lambda' \alpha(\lambda^{-1}x) \xi^* \lambda^{-1}| = 1, f$ and U are indeed phase equivalent and the proof is completed. $\square$

Remark 2.2. In dimension one any operator that preserves the modulus of the inner product is Wigner equivalent to the identity operator. Indeed, after a suitable identification, any such mapping f can be regarded as a mapping $f : \mathbb{F} \to \mathbb{F}$ and then $f(x) = \sigma(x)x$, where σ is a phase function defined by $\sigma(x) = f(x)x^{-1}$ if $x \neq 0$ and $\sigma(0) = 1$.

Remark 2.3. It is known that the quaternionic version of Wigner's theorem does not hold if $\dim \mathcal{H} = 2$. An easy counter example of Wigner's theorem in this case is the following. Let $x, y \in \mathcal{H}$ be unit orthogonal vectors, take any $z \in \mathcal{H}$ and write it as $z = \alpha x + \beta y$. If $\alpha = 0$, define $f(z) = \beta^* y$. If $\alpha \neq 0$, define $f(z) = \alpha^* x + \alpha^{-1} \beta^* \alpha y$. See [2,12] for the details.

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