Aequat. Math. 89 (2015), 725–744

0001-9054/15/030725-20
published online May 15, 2014
DOI 10.1007/s00010-014-0260-z

Aequationes Mathematicae

Beurling regular variation, Bloom dichotomy, and the Gołąb–Schinzel functional equation

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To Anatole Beck on his 84th birthday.

Abstract. The class of 'self-neglecting' functions at the heart of Beurling slow variation is expanded by permitting a positive asymptotic limit function $\lambda(t)$, in place of the usual limit 1, necessarily satisfying the following 'self-neglect' condition:

$$\lambda(x)\lambda(y) = \lambda(x + y\lambda(x)),$$

known as the *Gołąb–Schinzel functional equation*, a relative of the Cauchy equation (which is itself also central to Karamata regular variation). This equation, due independently to Aczél and Gołąb, occurring in the study of one-parameter subgroups, is here accessory to the λ -Uniform Convergence Theorem (λ-UCT) for the recent, flow-motivated, 'Beurling regular variation'. Positive solutions, when continuous, are known to be $\lambda(t) = 1 + at$ (below a new, 'flow', proof is given); $a = 0$ recovers the usual limit 1 for self-neglecting functions. The λ-UCT allows the inclusion of Karamata multiplicative regular variation in the Beurling theory of regular variation, with $\lambda(t) = 1 + t$ being the relevant case here, and generalizes Bloom's theorem concerning self-neglecting functions.

Mathematics Subject Classification (2000). 26A03; 33B99, 39B22, 34D05; 39A20.

Keywords. Beurling regular variation, Beurling's equation, self-neglecting functions, uniform convergence theorem, category-measure duality, Bloom dichotomy, Gołąb–Schinzel functional equation.

1. Regular variation, self-neglecting and Beurling functions

The Karamata theory of regular variation studies functions $f : (0, \infty) \to (0, \infty)$ with

$$f(tx)/f(x) \to g(t) \text{ as } x \to \infty \qquad \forall t, \tag{RV}$$

(and f is *slowly varying* if $g = 1$), or equivalently in isomorphic additive form

$$h(x + t) - h(x) \to k(t) \qquad \forall t, \tag{RV_+}$$

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for $h:\mathbb{R}\to\mathbb{R}$. Our reference for regular variation is [8] (BGT below). The Beurling theory of slow variation, originating in Beurling's generalization (for which see [41,45]—cf. [14]) of the Wiener Tauberian Theorem, studies functions f with

$$f(x+t\varphi(x))/f(x)\to 1 \qquad \forall t, \tag{BSV}$$

where φ is *positive* (on $\mathbb{R}_+:=[0,\infty)$) and itself satisfies (BSV) with φ for f, i.e.

$$\varphi(x+t\varphi(x))/\varphi(x)\to 1 \qquad \forall t; \tag{BSV_φ}$$

call such a φ 'Beurling-slow'. If convergence here is locally uniform in t, then φ is said to be *self-neglecting* (BGT §2.11; cf. [41,45]), i.e.

$$\varphi(x+u\varphi(x))/\varphi(x)\to 1,\ \text{locally uniformly in } u. \tag{SN}$$

Bloom [19] shows that a *continuous* Beurling-slow φ satisfies SN, and that $\varphi(x)=o(x)$. More generally, a Baire/measurable φ with a little more regularity (e.g. the Darboux property) satisfies SN; this may be viewed as a *Bloom dichotomy*: a Beurling-slow function is either self-neglecting or pathological—see [17] (or the more detailed [15] and [16], to which we refer below) or Sect. 5.

Although (BSV) includes via $\varphi=1$ the Karamata *additive* slow version (i.e. RV_+ with $k=0$), it excludes $\varphi(x)=x$ and the *multiplicative* Karamata format (RV), which, but for $\varphi(x)=o(x)$, it would capture. So one aim here is to expand the notion of self-neglect to allow direct specialization to the multiplicative Karamata form; our approach is motivated by recent work extending Beurling slow variation to Beurling regular variation. We recall from [16] that f is φ-*regularly varying* if as $x\to\infty$

$$f(x+t\varphi(x))/f(x)\to g(t), \qquad \forall t, \tag{BRV}$$

and φ, the *auxiliary function,* is *self-neglecting.* In Theorem 2 below we show that the multiplicative Karamata theory can be incorporated in a Beurling framework, but only if one replaces the limit 1 occurring above in (BSV_φ) with a more general limit $\lambda(t)$—yielding what we call *self-equivarying* functions with limit λ (definition below); exactly as with its Beurling analogue, Karamata multiplicative theory then takes its uniformity from the uniformity possessed by $\varphi(x)=x$. The case $\varphi(x)=1$ specializes to the Uniform Convergence Theorem of Karamata additive theory (UCT)—see BGT §1.2.

The recent Beurling theory of regular variation was established using the affine combinatorics of [15], where SN was deduced for Baire/measurable φ under various side-conditions including the Darboux property, more general than Bloom's continuity (as above) and more natural, since it implies continuous orbits for the underlying differential flows of Beurling variation in the measure case—defined by $\dot{x}(t)=\varphi(x(t))$. (See also its natural occurrence in [31].) In [16] it is shown that the uniformity in (SN) passes 'out' to uniformity in (BRV) and noted that conversely if $\varphi(x)=o(x)$, then the assumption of uniformity, but only in (BRV), passes 'in' the uniformity to the auxiliary function, when both are measurable or both have the Baire property (briefly: are Baire)—see BGT §3.10 for the 'φ monotonic' paradigm. Our methods focus on the in-out transfer of uniformity by considering a natural context of asymptotic equivalence, one that includes the Karamata multiplicative theory directly.

Definition. We say that f and g are *Beurling φ-equivarying*, or f is *Beurling φ -equivarying with g*, if

$$f(x+t\varphi(x))/g(x) \to 1 \text{ as } x \to \infty, \text{ for all } t>0. \qquad (BE_\varphi)$$

We call f, g *uniformly Beurling φ-equivarying* if

$$f(x+t\varphi(x))/g(x) \to 1 \text{ as } x \to \infty, \text{ on compact sets of } t>0. \qquad (UBE_\varphi)$$

For appropriate φ (as below), these actually yield equivalence relations on functions satisfying (BSV); indeed transitivity follows from

$$\frac{f(x+t\varphi(x))}{h(x)} = \frac{f(x+t\varphi(x))}{g(x)} \cdot \frac{g(x)}{g(x+t\varphi(x))} \cdot \frac{g(x+t\varphi(x))}{h(x)}. \qquad (1)$$

Proceeding as in (1) justifies the preferred *symmetric* terminology in an apparently asymmetric context; we omit the routine details, save to assert:

Proposition 1. (Symmetry) *For f, g satisfying (BSV):*

(i) *if f is Beurling φ -equivarying with g, then g is Beurling φ -equivarying with f;*

(ii) *similarly for f uniformly Beurling φ-equivarying with g.*

The two equivalence relations call for a study of 'self-equivalence' in Beurling regular variation terms—henceforth termed *equivariation*, or *equivariance.*

Definition. (i) For $\varphi(x) = O(x)$, we say that $\varphi > 0$ is *self-equivarying, $\varphi \in SE$*, with *limit* λ, if uniformly

$$\varphi(x+t\varphi(x))/\varphi(x) \to \lambda(t) \text{ as } x \to \infty, \text{ on compact sets of } t>0. \qquad (SE_\lambda)$$

(ii) For $\varphi(x) = O(x)$, we say that $\varphi > 0$ is *weakly self-equivarying, $\varphi \in WSE$ with limit* λ, if pointwise

$$\varphi(x+t\varphi(x))/\varphi(x) \to \lambda(t) \text{ as } x \to \infty, \text{ for all } t>0. \qquad (WSE)$$

(iii) For (positive) $\varphi \in WSE$, we set for $t>0$

$$\lambda_\varphi(t) := \lim_{x\to\infty} \frac{\varphi(x+t\varphi(x))}{\varphi(x)}, \text{ and } \lambda_\varphi(0) := 1. \qquad (2)$$

Preservation of SE and SN under equivariance (see Th. 5), and characterizing the limit λ_φ above for self-equivarying φ (see Th. 0) thus call for attention. The latter is linked to the *Cauchy functional equation* for additive functions (for which see [3,37]), which already plays a key role in determining the index theory of Karamata regular variation—see [9]. Here, for Beurling regular variation, there is an analogous functional equation satisfied by the limit functions λ_φ, namely the *Gołąb-Schinzel equation*

$$\lambda(x)\lambda(y) = \lambda(x + y\lambda(x)) \ (\forall x, y), \tag{GS}$$

first considered by Aczél [1] in work on geometric objects and independently by Gołąb in the study of 3-parameter affine subgroups of the plane. We refer to it as *Beurling's functional equation of self-neglect* and solutions that are positive on $\mathbb{R}_+$ as *Beurling functions.* Its additive equivalent for $\kappa = \kappa_\lambda := \log \lambda$ is

$$\kappa(x + y\lambda(x)) - \kappa(x) = \kappa(y), \text{ or } \Delta_y^\lambda \kappa(x) = \kappa(y) \tag{3}$$

(in mixed form), where

$$\Delta_y^\lambda \kappa(x) := \kappa(x + y\lambda(x)) - \kappa(x), \tag{4}$$

stresses the underlying 'Beurling difference-operator'. Viewing inputs as time, λ represents a *local time-change* – for connections here to the theory of flows see [6, Ch. 4]; cf. [16], the earlier [9,44]. Aczél originally observed in 1957 that the non-zero differentiable solutions of an equivalent form of (GS) take the form $1 + ax$; independently, a general analysis of its solutions was undertaken by Gołąb in collaboration with Schinzel in 1959 ([30]) and was amplified in 1965 by Popa's semi-group perspective via $x *_\lambda y := x + y\lambda(x)$ [47] (see also [34,35] and [21, Th. 1(ii)2^o]), surprisingly consonant with Beurling's 'generalized' convolution approach to the Wiener Tauberian theorem. Popa [47] also characterized real measurable solutions of (GS), but a description of the general solution had to wait till Javor [34] and Wołodźko [49], both in 1968; [49] also studied the continuous complex-variable case (complemented by Baron [5] in 1989). This was reviewed in [2] in 1970, but the complex-variable task was not completed until 1977 by Plaumann and Strambach [46]; for a recent text-book account see [3, Ch. 19] or the more recent survey [24] or [33], which includes generalizations of (GS) and a discussion of applications in algebra, meteorology and fluid mechanics—see for instance [36]. The key concept in this literature is micro-periodicity of solution functions (i.e. whether functions have arbitrarily small periods, and so a dense set of periods), an idea due to Burstin in 1915 [26] and Łomnicki in 1918 [40] (a measurable micro-periodic function is constant modulo a null set—see e.g. [25, Prop. 2]). Of interest is Theorem A below due to Popa, based on the Steinhaus subgroup theorem applied to the set of periods (an additive subgroup). Though the proof is given in the measure case, the category case is similar. Recall that 'quasi everywhere' means 'off a negligible set', be it meagre or null.

Theorem A. ([47, Th. 2]) *measure case,* [23] *Baire case;* [22] cf. [32] *Christensen-measurable case). Every measurable/Baire solution of the Gołąb–Schinzel equation is either continuous or quasi everywhere zero.*

Our interest is only in solutions that are positive on $\mathbb{R}_+$, so when they are Baire or measurable (as will be the case for λ_φ for Baire/measurable φ), Theorem A implies that the Beurling functions are necessarily continuous and of the form $1 + ax$ with $a \geq 0$. More is true: if $f : \mathbb{R} \to \mathbb{R}$ solves (GS) and is positive on some (non-trivial) interval, then f is necessarily continuous, by a result of Brzdęk [21, Cor 3]—cf. the analysis of 'local boundedness' in [23]; it follows that any Beurling function $\lambda : \mathbb{R}_+ \to \mathbb{R}$, since it may be extended to a solution of (GS) over $\mathbb{R}$ (see [25, Th. 1]), is continuous, as it is positive on $\mathbb{R}_+$. See also [33] and [42]. In view of their importance here, we give a new analysis (in §6) of this affine representation via the topological dynamics approach that underpins regular variation.

A brief comparison with Cauchy's exponential equation:

$$f(x)f(y) = f(x+y)\ (\forall x, y), \tag{CFE}$$

is helpful here; just as its continuous solutions are indeed the exponentials e^{at}, those of (GS) are the linear part[1] of the same exponential: $1 + at$. Recall also that additive functions if continuous are linear and so differentiable; they are continuous if Baire (Banach [4, Ch. I, § 3, Th. 4]), if measurable (Fréchet), if bounded on a non-null measurable set (Ostrowski's Theorem, refining Darboux's result for intervals), or on a non-meagre Baire set (Mehdi [39]); see [37], or the more recent account in [13]. Such automatic continuity results are mirrored in Beck's 'algebraic flows' in a metric space, which when bounded by a monotone function of the flow's distance from some set K are continuous at points of K ([6, Th. 1.65]). The latter approach motivates a new proof of the Aczél-Gołąb–Schinzel representation (in §6) and perhaps explains why (GS) has analogues, though not exact replicates, and possesses similarly to (CFE) unbounded discontinuous solutions, granted the existence of a Hamel basis (see [30]). [2] notes that $\mathbf{1}_{\mathbb{Q}}$, the indicator of the rationals (Dirichlet's function), is a measurable, bounded discontinuous solution to (GS)—a contrast to Ostrowski's Theorem, but see [23].

Remarks

1. If φ is Baire, then λ_φ is Baire, being the limit of functions $\varphi_n(t) := \varphi(n + t\varphi(n))/\varphi(n)$, for $n \in \mathbb{N}$, which are Baire as each $t \to n + t\varphi(n)$ is a homeomorphism, since $\varphi > 0$. Similarly for measurability.
2. If λ_φ is continuous in (SE), then for $\varepsilon > 0$ and $t_n \to t$

$$|\varphi(x + t_n\varphi(x))/\varphi(x) - \lambda_\varphi(t)| < \varepsilon, \text{ for large enough } n \text{ and } x. \tag{SSE}$$

[1] Interestingly, (GS) implies a self-differential property:

$$\frac{d}{du}\lambda(u\lambda(t) + t) = \lambda(u\lambda(t) + t) \cdot \frac{\lambda'(u)}{\lambda(u)}.$$

This *strong self-equivariance* condition (SSE) could be adopted in place of (SE_λ), with continuity of λ immediate—motive enough to study $\varphi \in SE$.

3. For $\lambda(t) \equiv 1$, (SE_λ) differs from (SN) in requiring $O(x)$ rather than $o(x)$.
4. If $\varphi(x) = ax$ with $a > 0$, then $\varphi(x) = O(x)$ and we have an *affine* form:[2]

$$\varphi(x + t\varphi(x))/\varphi(x) = a(x + atx)/ax = 1 + at = \lambda_\varphi(t).$$

We now establish the significance of (GS) for Beurling regular variation.

Theorem 0. (A Characterization Theorem). *For Baire/measurable $\varphi \in SE$ the limit function λ_φ satisfies (GS), so is continuous, and if positive has the form $\lambda(t) = 1 + at$.*

Furthermore, $a \geq 0$ is required for λ_φ when φ satisfies the order condition $\varphi(x) = O(x)$. Also, up to re-scaling, there are only the two limits λ_φ: small-order limit $\lambda(t) \equiv 1$ and large-order limit $\lambda(t) \equiv 1 + t$.

Proof. Suppose that $\varphi \in SE$; writing $y := x + u\varphi(x)$ and $s = v\lambda_\varphi(u)$ note that

$$\frac{\varphi(x + (u + s)\varphi(x))}{\varphi(x)} = \frac{\varphi(y + s\frac{\varphi(x)}{\varphi(x+u\varphi(x))}\varphi(y))}{\varphi(y)} \cdot \frac{\varphi(x + u\varphi(x))}{\varphi(x)}. \tag{5}$$

The left-most and right-most terms tend to $\lambda_\varphi(u + s)$ and $\lambda_\varphi(u)$ respectively. Now $s\varphi(x)/\varphi(x + u\varphi(x)) \to s/\lambda_\varphi(u) = v$. Let $x \to \infty$ to get

$$\lambda_\varphi(u + s)/\lambda_\varphi(u) = \lambda_\varphi\left(v\right),$$

as required. For φ Baire/measurable, λ_φ is Baire/measurable and satisfies (GS) so, by Theorem A, is continuous. By the above results of Gołąb and Schinzel, and Wołodźko (or see [3, Ch. 19 Prop.1]), we conclude that

$$\lambda_\varphi(t) = 1 + at.$$

The condition $\varphi(t) = O(t)$ yields $a \geq 0$.

Given $\varphi \in WSE$, re-scaling to $\psi(t) = \varphi(t)/b$ with $b > 0$ yields

$$\lambda_\varphi(t) = \lim_x \varphi(x + bt\varphi(x)/b)/\varphi(x) = \lim_x \psi(x + bt\psi(x))/\psi(x) = \lambda_\psi(bt),$$

i.e. $\lambda_{\varphi/b}(bt) = \lambda_\varphi(t)$. So if $\lambda_\varphi(t) = 1 + at$, taking $b = 1/a$ yields $\lambda_{a\varphi}(t/a) = 1 + t$. □

Remark We note for completeness of §6 that, for $\lambda > 0$ and differentiable, differentiating (GS) w.r.t. y yields $\lambda'(y) = \lambda'(x + y\lambda(x))$ and in particular $\lambda'(x) = \lambda'(0)$, whence $\lambda(x) = 1 + ax$, as $\lambda(0) = 1$.

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[2] Affine functions $f : \mathbb{R}^d \to \mathbb{R}$ are termed *linear* in [37, § 7.7]. This usage sits well with the context of $\mathbb{R}$ as a field over $\mathbb{Q}$, to which the Beurling equation seems less suited.

Corollary. (Representation for SE) *For Baire/measurable $\varphi \in SE$ with positive limit λ_φ the function $\psi(x) := \varphi(x)/\lambda_\varphi(x)$ is self-neglecting and so*

$$\varphi(x) \sim \lambda_\varphi(x) \int_0^x e(u)du \text{ for some continuous } e \text{ with } e \to 0.$$

Proof. By Theorem 0, we may assume that $\lambda_\varphi(x) = 1 + ax$ for some $a > 0$, otherwise there is nothing to prove. So $\psi(x) = O(1)$, as $\varphi(x) = O(x)$. Fix $t > 0$; then $s_x := t\psi(x)/\varphi(x) \to 0$. Now $\lambda_\varphi(x)/\lambda_\varphi(x + t\psi(x)) \to 1$, so

$$\psi(x + t\psi(x))/\psi(x) = \varphi(x + s_x\varphi(x))/\varphi(x) \cdot \lambda_\varphi(x)/\lambda_\varphi(x + t\psi(x)) \to \lambda_\varphi(0) = 1.$$

So $\psi \in SN$ and the representation follows from a result of Bloom and Shea (see [19]; cf. [16]). □

2. Combinatorial preliminaries

We summarize from [15] the combinatorial framework needed here: Baire and measurable cases are handled together by working bi-topologically, using the Euclidean topology in the Baire case (the primary case) and the density topology in the measure case; see [10,12,13]. We work in the affine group $\mathcal{A}ff$ acting on $(\mathbb{R}, +)$ using the notation

$$\gamma_n(t) = c_n t + z_n,$$

where $c_n \to c_0 = c > 0$ and $z_n \to 0$ as $n \to \infty$, as in Theorem B below. These are to be viewed as (self-) homeomorphisms of $\mathbb{R}$ under either the Euclidean topology, or the density topology. We recall the following result from [15].

Theorem B. (Affine Two-sets Theorem) *For $c_n \to c > 0$ and $z_n \to 0$, if $cB \subseteq A$ for A, B non-negligible (measurable/Baire), then for quasi all $b \in B$ there exists an infinite set $\mathbb{M} = \mathbb{M}_b \subseteq \mathbb{N}$ such that*

$$\{\gamma_m(b) = c_m b + z_m : m \in \mathbb{M}\} \subseteq A.$$

As in [16], Theorem 1 below needs only the case $c = 1$*;* however, Theorem 3 needs the case $c \neq 1$.

3. Uniform convergence theorem

This section closely mirrors [16, § 4] in verifying the generalization needed here; some care is needed to distinguish SE from SN, likewise UBE_φ involving 1 as limit—from WSE involving a general limit λ. Our convention is to write $f_{\mathrm{N}} := f$ and $f_{\mathrm{D}} = g$, ("N for numerator, D for denomintor") and also

$$h := \log \varphi \qquad h_{\mathrm{N}} := \log f_{\mathrm{N}} \text{ and } h_{\mathrm{D}} := \log f_{\mathrm{D}}.$$

Definition. (i) For $\varphi \in SE$ we call $\{u_n\}$ with limit u a *1-witness sequence at* u (for non-uniformity in f_{N} *over* f_{D}) if there are $\varepsilon_0 > 0$ and a divergent sequence $x_n \to +\infty$ with

$$|h_{\mathrm{N}}(x_n + u_n\varphi(x_n)) - h_{\mathrm{D}}(x_n)| > \varepsilon_0 \qquad \forall\, n \in \mathbb{N}. \tag{6}$$

(ii) For $\varphi \in WSE$ we call $\{u_n\}$ with limit u a WSE *-witness sequence at* u (for non-uniformity in φ) if there are $\varepsilon_0 > 0$ and a divergent sequence x_n with

$$|h(x_n + u_n\varphi(x_n)) - h(x_n) - \kappa(u)| > \varepsilon_0 \qquad \forall\, n \in \mathbb{N}. \tag{7}$$

We call $\{u_n\}$ with limit u a *divergent* WSE*-witness sequence* if also

$$h(x_n + u_n\varphi(x_n)) - h(x_n) \to \pm\infty.$$

So this divergence gives a special type of WSE-witness sequence.

Below, *uniform near* a point u means 'uniformly on sequences converging to u'—equivalent to local uniformity at u (on compact neighbourhoods of u).

Lemma 1. (Shift Lemma: uniformity preservation under shift)

(i) *Let* $\varphi \in SE$. *For any* u, *convergence in* (BE_φ) *is uniform near* $t = 0$ *iff it is uniform near* $t = u$.

(ii) *Let* $\varphi \in WSE$ *with limit* λ_φ*: for any* u, *convergence in* (WSE) *is uniform near* $t = 0$ *iff it is uniform near* $t = u$.

Proof. Since in case (i) $h_{\mathrm{N}}(x_n + u\varphi(x_n)) - h_{\mathrm{D}}(x_n) \to 0$ and in case (ii) $h(x_n + u\varphi(x_n)) - h(x_n) - \kappa(u) \to 0$ we argue routinely, as in [15]. □

Theorem 1 follows from the argument presented in [16, Th. 2] with minimal amendments, so a sketch suffices; the detailed proof of Theorem 3 below (responding to the presence of λ in WSE) is a paradigm for the SE case here.

Theorem 1. (λ-Uniform convergence theorem, λ-UCT) *For* $\varphi \in SE$ *with limit* $\lambda = \lambda_\varphi$, *if* f, g, φ *have the Baire property (are measurable) and satisfy* (BE_φ)*, then they satisfy* (UBE_φ)*.*

Proof. Suppose otherwise. By Theorem A the limit λ_φ is continuous. Now we begin as in [16, Th. 2]; let u_n be a 1-witness sequence for the non-uniformity of f over g. For some $x_n \to \infty$ and $\varepsilon_0 > 0$ one has (6). By the Shift Lemma (i), we may assume that $u = 0$. So we will write z_n for u_n. As φ is self-equivarying for any $\varepsilon > 0$ and with $K := \{z_n : n = 0, 1, 2, ..\}$ (compact) for large enough n

$$|h(x_n + z_n\varphi(x_n)) - h(x_n) - \kappa(z_n)| \le \varepsilon \qquad \forall\, n \in \mathbb{N}.$$

But κ is continuous, so that $\kappa(z_n) \to \log \lambda(0) = 0$, and so

$$c_n := \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \longrightarrow 1 = \lambda_\varphi(0). \tag{8}$$

Write $y_n := x_n + z_n\varphi(x_n)$. Then $y_n = x_n(1 + z_n\varphi(x_n)/x_n) \to \infty$, and

$$|h_{\mathrm{N}}(y_n) - h_{\mathrm{D}}(x_n)| \ge \varepsilon_0.$$

Continue verbatim as in [16], applying Theorem B to $\gamma_n(s) := c_n s + z_n$ to derive a contradiction to (6). □

As an immediate corollary we have:

Theorem 2. (Beurling and Karamata UCT) *For $\varphi \in SN$, if f, φ have the Baire property (are measurable) and satisfy (BRV), then they satisfy (BRV) locally uniformly.*

For $\varphi(x) = x$, if f has the Baire property (is measurable) and satisfies (RV), then f satisfies

$$f(tx)/f(x) \to g(t), \text{ as } x \to \infty \text{ locally uniformly in } t. \qquad (RV)$$

Proof. In Theorem 1, take $g = f$. □

Theorem 1 invites an extension of Beurling regular variation based on $\varphi \in SE$, i.e. beyond SN. That extension yields only multiplicative Karamata regular variation—because, by Theorem 0, up to rescaling ("in t"), there is only one 'canonical' alternative beyond SN, namely $\lambda_\varphi(t) = 1 + t$, occurring e.g. for $\varphi(x) = x$. Here one has $f(x + t\varphi(x))/f(x) = f(x(1 + t))/f(x)$ so the unit shift on t below is inevitable.

Theorem 1′. (Extended regular variation) *For $\varphi \in SE$ if f, φ have the Baire property (are measurable), $\lambda_\varphi(t) = 1 + t$, and f satisfies, for $t > 0$,*

$$f(x + t\varphi(x))/f(x) \to \gamma(t),$$

then $\gamma(t) = (1 + t)^\rho$ for some $\rho \in \mathbb{R}$.

Proof. In Theorem 1, again with $g = f$, (UBE_φ) holds. So for $\gamma(t) := \lim f(x + t\varphi(x))/f(x)$, writing $y = x + s\varphi(x)$ and noting that $t\varphi(x)/\varphi(y) \to v := t/\lambda_\varphi(s)$, by (UBE_φ) one has

$$\begin{aligned}\gamma(s + t) &= \lim \frac{f(x + (s + t)\varphi(x))}{f(x)} \\ &= \lim \frac{f(y + [t\varphi(x)/\varphi(y)]\varphi(y))}{f(y)} \cdot \frac{f(x + s\varphi(x))}{f(x)} \\ &= \gamma(v)\gamma(s)\end{aligned}$$

(as $y \to \infty$ when $x \to \infty$), or, with u for s,

$$\gamma(u + v\lambda_\varphi(u)) = \gamma(u)\gamma(v),$$

where $\lambda_\varphi(t) = 1 + t$. Putting $G(t) = \gamma(t - 1)$, $x = 1 + u, y := 1 + v$, one has

$$G(xy) = \gamma(u + v + uv) = \gamma(u)\gamma(v) = G(x)G(y).$$

As G is Baire/measurable, $G(x) = x^\rho$ for some ρ (see [3, Ch. 3]), so $\gamma(t) = G(1 + t) = (1 + t)^\rho$. □

4. Stability properties of Beurling functions

There is a literature surrounding (GS) and its generalizations devoted to stability properties in the sense of Hyers-Ulam—for the general context and the literature concerned with (GS), initiated by Ger and his collaborators, see for example [27], cf. [28], and the more recent [20,30–32,34]. We pursue a related agenda, but motivated by the regular variation view of the interplay between $\varphi \in WSE$ and λ_φ. We begin with a rigidity property noting first a formula, an instance of which is the *doubling formula* $\lambda(2t) = \lambda(t/\lambda(t))\lambda(t)$. We omit the routine proof.

Lemma 2. (Internal time-change) *For λ satisfying (GS) the internal time-change $\mu(t) := \lambda(\beta t)$ with $\beta \neq 0$ yields a solution to (GS). Also one has*

$$\mu(t) = \lambda(\beta t) = \lambda(t)\lambda(\alpha t/\lambda(t)), \text{ with } \alpha := \beta - 1.$$

Proposition 2. (Slow time-changing) *For $\lambda \in SE$ and $w(.)$ Baire satisfying*

$$\lim_{x\to\infty} \frac{w(x+u\lambda(x))}{w(x)} = 1 \text{ and } \lim_{x\to\infty} w(x) = \beta := 1+\alpha, \qquad \forall u,$$

the time-changed function $\mu(x) := \lambda(x)w(x)$ is a solution of (GS) iff

$$w(t) = \lambda(\alpha t/\lambda(t)).$$

In particular, for $\beta = 1$, we have $w(t) = \lambda(0) = 1$.

Proof. Put $\mu(x) = \lambda(x)w(x)$; if μ is a solution of (GS), then $\mu(t) = \mu(x + t\mu(x))/\mu(x)$. Substituting into this identity,

$$\frac{\lambda(x+t\lambda(x)w(x))}{\lambda(x)}\frac{w(x+t\lambda(x)w(x))}{w(x)} = \lambda(t)w(t).$$

Using $\lambda(t) = \lambda(x+t\lambda(x))/\lambda(x)$ twice, we have

$$w(t) = \frac{\lambda(x+t\lambda(x)+t[w(x)-1]\cdot\frac{\lambda(x)}{\lambda(x+t\lambda(x))}\cdot[\lambda(x+t\lambda(x))])}{\lambda(x+t\lambda(x))} \cdot\frac{w(x+t\lambda(x)w(x))}{w(x)}.$$

Put $y := x + t\lambda(x)$ and $u(x) := t(w(x)-1)/\lambda(t)$; then

$$\frac{\lambda(y+u(x)\lambda(y))}{\lambda(y)}\frac{w(x+[tw(x)]\lambda(x))}{w(x)} = w(t),$$

or

$$\lambda(u(x))\frac{w(x+[tw(x)]\lambda(x))}{w(x)} = w(t).$$

As $\lambda \in SE$, if w is Baire and λ-slowly varying and bounded, then by λ-UCT

$$\frac{w(x+[tw(x)]\lambda(x))}{w(x)} \to 1.$$

So if $w(x) \to 1+\alpha$, then $u(x) \to t\alpha/\lambda(t)$, and so $w(t) = \lambda(\alpha t/\lambda(t))$.

For the converse, apply Lemma 2. □

Example Taking $\lambda(t) = 1+t$, we have $w(t) = 1+\alpha t/(1+t)$ and

$$\mu(t) = (1+t)\left(1+\frac{\alpha t}{1+t}\right) = (1+t)+\alpha t = 1+(1+\alpha)t.$$

Theorem 3 below enables an extension of Bloom's Theorem (see §1 and 5) with WSE replacing the original 'slow Beurling'. Analogous to the Divergence Theorem of [15], but more subtle, an extra twist calls here for a detailed proof. It should be borne in mind that λ_φ below is not known to satisfy (GS); that will be deduced later in Th. 4. The continuity assumption at 0 seems an inevitable 'connection' of the two parts of the definition (2).

Theorem 3. (Divergence Theorem—Baire/measurable) *For φ Baire/ measurable in WSE with limit λ_φ continuous at 0: if u_n with limit u is a WSE-witness sequence to the non-uniformity of φ over φ, then either u_n is a divergent witness sequence, or for some divergent sequence x_n*

$$\varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) \to \lambda_\varphi(u).$$

Proof. Begin as in the proof of Theorem 2, except that here $h_{\mathrm{N}} = h_{\mathrm{D}} = h = \log\varphi$. Let u_n with limit u be a WSE-witness sequence to the non-uniformity of φ over φ, with limit λ; for some $x_n \to \infty$ and $\varepsilon_0 > 0$ one has (7) with $\kappa = \log\lambda$. By the Shift Lemma (ii), we may assume that $u = 0$. So we will write z_n for u_n. That is, with $y_n := x_n + z_n\varphi(x_n)$,

$$|h(y_n) - h(x_n)| > \varepsilon_0.$$

Note that $y_n = x_n(1+z_n\varphi(x_n)/x_n)$ is divergent. Assume the non-divergence of $\{h(y_n)-h(x_n)\}$. Consider any convergent subsequence; we show its limit is 0, by contradiction. Working down a subsequence, suppose that

$$c_n := \varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) \longrightarrow c \in (0,\infty), \text{ with } c \neq 1. \tag{9}$$

As $|h(y_n) - h(x_n)| > \varepsilon_0$, passing to the limit we obtain

$$\log c \geq \varepsilon_0 > 0.$$

Choose η_0 with $0 < \eta_0 < \frac{1}{2}\log c$ and let $\eta = \eta_0/6$.

Suppose now that κ has the Baire property and is continuous on a co-meagre set S–see [43, Th. 8.1] or [38, § 28]. Take $T_0 := S$, set inductively $T_{n+1} := cT_n \cap T_n$ and $T_{-(n+1)} := c^{-1}T_{-n} \cap T_{-n}$, and put $T := \bigcap_{n=-\infty}^{+\infty} T_n$. Then $ct \in T$ and $c^{-1}t \in T$ for $t \in T$: each T_n and so T is co-meagre. So the restriction $\kappa|T$ is continuous on T.

By assumption there is $\delta_0 > 0$ such that for $s \in (0,\delta_0)$

$$|\kappa(c^{-1}s) - \kappa(s)| < \eta.$$

For $x = \{x_n\}$, working in T, put

$$V_n^x(\eta) := \{s \in T : |h(x_n + s\varphi(x_n)) - h(x_n) - \kappa(s)| \le \eta\},$$
$$H_k^x(\eta) := \bigcap\nolimits_{n \ge k} V_n^x(\eta),$$

and likewise for $y = \{y_n\}$. These are Baire sets, and

$$T = \bigcup\nolimits_k H_k^x(\eta) = \bigcup\nolimits_k H_k^y(\eta), \tag{10}$$

as $\varphi \in WSE$. The increasing sequence of sets $\{H_k^x(\eta)\}$ covers $T \cap (0, \delta_0)$. So for some k the set $H_k^x(\eta) \cap (0, \delta_0)$ is non-negligible. As $c^{-1}H_k^x(\eta)$ is non-negligible, so is $c^{-1}H_k^x(\eta) \cap T$ as well as $H_k^x(\eta) \cap cT$ and $H_k^x(\eta) \cap T$; by (10), for some l the set

$$B := c^{-1}[H_k^x(\eta) \cap (0, \delta_0)] \cap H_l^y(\eta)$$

is also non-negligible. Take $A := T \cap H_k^x(\eta)$; then $B \subseteq H_l^y(\eta)$ and $cB \subseteq A$ with A, B non-negligible. Applying Theorem B of §2 to the maps $\gamma_m(s) := c_n s + z_n$ with $c = \lim_n c_n$, there exist $b \in B$ and an infinite set $\mathbb{M}$ such that

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A = H_k^x(\eta),$$

and as $bc \in (0, \delta_0)$

$$|\kappa(b) - \kappa(bc)| < \eta.$$

That is, as $B \subseteq H_l^y(\eta)$,there is $b \in H_l^y(\eta)$ and an infinite $\mathbb{M}_t$ such that

$$\{\gamma_m(b) := c_m b + z_m : m \in \mathbb{M}_t\} \subseteq H_k^x(\eta).$$

In particular, for this b and $m \in \mathbb{M}_b$ with $m > k, l$ one has

$$b \in V_m^y(\eta) \text{ and } \gamma_m(b) \in V_m^x(\eta).$$

As $t := cb \in T$ and $\gamma_m(b) \in T$, we have by the continuity of $\kappa|T$ at t, since $\gamma_m(b) \to cb$, that for all m large enough

$$|\kappa(t) - \kappa(\gamma_m(b))| \le \eta. \tag{11}$$

Fix such an m. As $\gamma_m(b) \in V_m^x(\eta)$,

$$|h(x_m + \gamma_m(b)\varphi(x_m)) - h(x_m) - \kappa(\gamma_m(b))| \le \eta. \tag{12}$$

But $\gamma_m(b) = c_m b + z_m = z_m + b\varphi(y_m)/\varphi(x_m)$, so

$$x_m + \gamma_m(b)\varphi(x_m) = x_m + z_m\varphi(x_m) + b\varphi(y_m) = y_m + b\varphi(y_m),$$

'absorbing' the affine shift component of $\gamma_m(b)$ into y. So, by (12),

$$|h(y_m + b\varphi(y_m)) - h(x_m) - \kappa(\gamma_m(b))| \le \eta.$$

But $b \in V_m^y(\eta)$, so

$$|h(y_m + b\varphi(y_m)) - h(y_m) - \kappa(b))| \le \eta.$$

Using the triangle inequality, and combining the last two inequalities with (11), we have

$$\begin{aligned}|h(y_m) - h(x_m)| &\leq |h(y_m + b\varphi(y_m)) - h(y_m) - \kappa(b)| \\ &\quad + |\kappa(b) - \kappa(cb)| + |\kappa(cb) - \kappa(\gamma_m(b))| \\ &\quad + |h(y_m + t\varphi(y_m)) - h(x_m) - \kappa(\gamma_m(b))| \\ &\leq 4\eta < \eta_0.\end{aligned}$$

For large m one has $\log c - \eta_0 < h(y_m) - h(x_m) < \log c + \eta_0$, so for any one such large m we have $\log c - \eta_0 < h(y_m) - h(x_m) < \eta_0$, that is, $\log c < 2\eta_0$ contradicting the choice of η_0. Thus $c = 1$.

Now suppose that κ is measurable. Proceed as before, but now apply Luzin's Theorem ([43], Ch. 8) to select $T \subseteq [c, 2c] \cup [1, 2]$ such that $|T \cap [1, 2]| > 2/3$ and $|T \cap [c, 2c]| > 3c/4$ with $\kappa|T$ continuous on T. As before, put

$$\begin{aligned}V_n^x(\eta) &:= \{s \in T : |h(x_n + s\varphi(x_n)) - h(x_n) - \kappa(s)| \leq \eta\}, \\ H_k^x(\eta) &:= \bigcap\nolimits_{n \geq k} V_n^x(\eta),\end{aligned}$$

and likewise for $y = \{y_n\}$. These are measurable sets, and

$$T = \bigcup\nolimits_k H_k^x(\eta) = \bigcup\nolimits_k H_k^y(\eta), \tag{13}$$

since $\varphi \in WSE$. The increasing sequence of sets $\{H_l^y(\eta)\}$ covers $T \cap [c, 2c]$. So $|(T \cap [c, 2c]) \cap H_k^x(\eta)| > 2|T \cap [c, 2c]|/3$ for some k. So in particular $H_k^x(\eta)$ is non-null, and furthermore, $|T \cap [c, 2c] \backslash H_k^x(\eta)| < |T \cap [c, 2c]|/3 < c/3$. So $|[1, 2] \backslash c^{-1} H_k^x(\eta)| < 1/3$; but $|T \cap [1, 2]| > 2/3$, so $|c^{-1} H_k^x(\eta) \cap [1, 2]| > 0$; by (13), for some l the set

$$B := c^{-1} H_k^x(\eta)) \cap H_l^y(\eta)$$

is also non-null. Taking $A := H_k^x(\eta)$, one has $B \subseteq H_l^y(\eta)$ and $cB \subseteq A$ with A, B non-null. From here continue as in the Baire argument. □

5. The extended Bloom dichotomy

The preceding section implies the *Bloom dichotomy*—that φ Beurling-slow (i.e. φ with $\lambda_\varphi = 1$) is either self-neglecting or pathological—extends to WSE: when $\varphi \in WSE$ either $\varphi \in SE$, or φ is 'pathological'. (For other occurrences of dichotomy in this area see [11–13].) Indeed, $\varphi \in WSE$ says merely that the limit function λ_φ is well-defined, but nothing about whether λ_φ satisfies (GS). However, if λ_φ is *continuous at the origin* and φ has just the kind of regularity considered in the Generalized Bloom Theorem of [15], then in fact $\varphi \in SE$, so that λ_φ satisfies (GS) and takes a simple form. This brings to mind, as an analogy, Lévy's Continuity (or Convergence) Theorem, see [50, Ch.18], or [29, 9.8.2], that if a sequence of characteristic functions converges pointwise to a limit function which is *continuous at the origin*, then that limit is itself a characteristic function; the continuity assumption is critical, as Bochner's theorem asserts the converse: a positive-definite function λ, normalized so that $\lambda(0) = 1$, and continuous at the origin is a characteristic function (cf. [48, 1.4.3]).

Theorem 4. (Bloom's Theorem for weak self-equivariance) *For $\varphi \in WSE$ with limit function λ_φ continuous at 0 and $\varphi(x) = O(x)$, if φ is Baire/measurable and has any of the following properties:*

(i) *φ has the Darboux property (in particular, φ is continuous),*
(ii) *$\varphi(x)$ has bounded range on $(0, \infty)$,*
(iii) *$\varphi(x)/x$ is bounded in $(0, \infty)$,*
(iv) *$\varphi(x)$ is increasing in $(0, \infty)$,*
— then $\varphi \in SE$ and so λ_φ is continuous.

Proof. Apply Theorem 3 and use the Darboux property as in the Beurling-Darboux UCT of [15, Th. 4] to argue as with Bloom's Theorem that there are no divergent witness sequences; otherwise, proceed as in [15, Th. 3]. □

Theorem 5. (i) *For $\varphi \in SE$, if $\psi > 0$ is smooth, Beurling-slow and Beurling φ-equivarying with φ, then $\psi \in SE$ and φ is ψ-equivarying with ψ; and likewise for SN, mutatis mutandis, so in particular:*
(ii) *For $\varphi \in SN$, if $\psi > 0$ is smooth and Beurling φ-equivarying with φ, then $\psi \in SN$.*

Proof. Notice first that for any fixed $u > 0$, we have

$$\psi(x)/\varphi(x) = \psi(x)/\psi(x + u\varphi(x)) \cdot \psi(x + u\varphi(x))/\varphi(x) \to 1,$$

since ψ satisfies (BSV) and ψ is Beurling φ -equivarying with φ. So one has $\psi(x) = O(x)$ in the SE case and $\psi(x) = o(x)$ in the SN case. Since ψ is Beurling φ-equivarying with φ, by Theorem 1, as ψ is measurable

$$\psi(x + u\varphi(x))/\varphi(x) \to 1, \text{ loc unif. in } u.$$

In particular, since $t[\psi(x)/\varphi(x)] \to t$, one has as before

$$\psi(x + t\psi(x))/\psi(x) = \psi(x + t[\psi(x)/\varphi(x)]\varphi(x))/\varphi(x) \cdot \varphi(x)/\psi(x) \to 1.$$

So $\psi \in WSE$ with limit $\lambda = 1$. But ψ is continuous, so by Th. 4 $\psi \in SE$.

As to role reversal here, similarly to Prop. 1, both terms on the right below tend to 1 locally uniformly in t as $x \to \infty$:

$$\varphi(x + t\psi(x))/\psi(x) = \varphi(x + t[\psi(x)/\varphi(x)]\varphi(x))/\varphi(x) \cdot \varphi(x)/\psi(x) \to 1,$$

as $\varphi \in SE$ by the opening remark of the proof. □

Remark Above, if one assumed instead that $\psi \in WSE$ with limit λ_ψ and as before that ψ is Beurling φ-equivarying with φ, then for any fixed $u > 0$

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi(x)}{\psi(x+u\varphi(x))}\frac{\psi(x+u\varphi(x))}{\psi(x)} \to \lambda_\psi(u),$$

implying that $\lambda_\psi(u)$ is constant. From here continuing the proof as above yields $\psi \in SE$ with limit λ_ψ, so that $\lambda_\psi = 1$, i.e. ψ is Beurling-slow.

6. Continuous Beurling functions

In this section we offer a new proof that every continuous solution λ of (GS), in particular every Beurling function, is differentiable, assuming that λ satisfies $\lambda(t) > 1$ for arbitrarily small $t > 0$. In fact, the latter assumption already implies continuity (as then $\lambda \geq 1$, so a fortiori is positive—see Prop. 4 below), by results of Brzdęk [21] combined with [25], as noted after Theorem A (in §1) (See also 'Added in Proof' at end.). Our approach is via a discrete analogue of the obvious differentiation approach to solving (GS), using the constancy of $\Delta_u^\lambda \kappa(x)$. First we clarify the continuity and differentiability conditions of Theorem 0 (for an alternative see [21, Cor. 6 and 7]).

Lemma 3. *For λ satisfying (GS), if λ is continuous at some point t where $\lambda(t) \neq 0$, then it is continuous whenever it is non-zero. Similarly, if λ is differentiable at some point t where $\lambda(t) \neq 0$, then it is differentiable at all points.*

Proof. From (GS) for $u \neq 0$ and fixed t with $\lambda(t) \neq 0$ one has the 'Δ^λ-identity'

$$\frac{1}{\lambda(t)}\Delta_u^\lambda \lambda(t) = \frac{\lambda(t+u\lambda(t)) - \lambda(t)}{\lambda(t)} = \lambda(u) - 1. \qquad (\Delta^\lambda)$$

The linear monotonic map $y(u) := t + u\lambda(t)$ carries any open neighbourhood of $u = 0$ to an open neighbourhood of t, and likewise for its inverse. The equivalence of global continuity and continuity at $u = 0$ follows from this identity (since $\lambda(0) = 1$). As to differentiability, the argument is almost the same (upon division by $u \neq 0$). □

The following recurrence occurs in [30, Lemma 7], [6,19].

Definition. For $u > 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ define the *Beck φ-sequence* $t_n(u)$ by the recurrence $t_n := T_u^\varphi(t_{n-1}) = t_{n-1} + u\varphi(t_{n-1})$ with $t_0 = 0$. (Though we do not assume φ to be monotone, this generalizes the *Beck iteration* of $\gamma(x) := T_1^\varphi(x) = x + \varphi(x)$ via $\gamma_{n+1}(x) = \gamma_1(\gamma_n(x))$, used in bounding flows—see [6, 1.64]; cf. [19] or BGT §2.11 and [15, § 6]). Call the Beck sequence a *Bloom partition* if $t_n(u)$ diverges to $+\infty$, in which case define the *Beck u-step norm* of T (u-step distance from the origin) to be the integer $n = n_T(u)$ such that

$$t_n(u) \leq T < t_{n+1}(u).$$

Our first observation is motivated by summing the differences $\kappa(t_i) - \kappa(t_{i-1}) = \kappa(u)$.

Lemma 4. (cf. [25, Lemma 7]) *For λ, any solution to (GS), and $t_n = t_n(u)$ its Beck λ-sequence above,*

$$\lambda(t_n) = \lambda(u)^n.$$

Proof. From (GS) one has $\lambda(t_i)/\lambda(t_{i-1}) = \lambda(t_{i-1} + u\lambda(t_{i-1}))/\lambda(t_{i-1}) = \lambda(u)$; now take products for $i = 1, ..., n$ and use $\lambda(t_0) = 1$. □

The following, though quite distinct, resembles a result due to Beck [6, 1.69] and relies on (14), a formula noted also in [30, Lemma 8].

Proposition 4. (Bounding Formula) *For λ, any solution of (GS), and t_n its associated Beck sequence, defined by $t_n := t_{n-1} + u\lambda(t_{n-1})$, with $u > 0$, if $\lambda(u) \neq 1$, then*

$$t_n(u) = u\frac{\lambda(u)^n - 1}{\lambda(u) - 1} = (\lambda(u)^n - 1) \Bigg/ \frac{\lambda(u) - 1}{u}\ . \tag{14}$$

Suppose further that λ is continuous and in any neighbourhood of the origin there is $u > 0$ with $\lambda(u) > 1$; then $\lambda(T) \geq 1$ for all $T > 0$. Moreover, given $T, \varepsilon > 0$, for all small enough $u > 0$ with $\lambda(u) > 1$ and with $n = n_T(u)$, the Beck u-step norm of T:

$$\frac{(1-\varepsilon)\lambda(u)^n - 1}{\lambda(u)^{n+1} - 1}\frac{\lambda(u) - 1}{u} < \frac{\lambda(T) - 1}{T} < \frac{(1+\varepsilon)\lambda(u)^n - 1}{\lambda(u)^n - 1}\frac{\lambda(u) - 1}{u}. \tag{15}$$

Proof. As $t_i - t_{i-1} = u\lambda(t_{i-1}) = u\lambda(u)^{i-1}$, by Lemma 4, summation of the differences over $i = 1, ..., n$ yields the result (since $t_0 = 0$).

Now fix T. As there are arbitrarily small $u > 0$ with $\lambda(u) > 1$, there are arbitrarily small $u > 0$ with $t_n(u)$ divergent, by (14), and with $\lambda(t_n(u)) > 1$, by Lemma 4. So by continuity $\lambda(T) \geq 1$.

Fix $\varepsilon > 0$. Again by continuity at T, there is $\delta_\varepsilon > 0$ such that for each t with $|t - T| < \delta_\varepsilon$ one has $\lambda(t) \neq 0$ and $|\lambda(T)/\lambda(t) - 1| < \varepsilon$. Consider $0 < u < \delta_\varepsilon$ with $\lambda(u) > 1$ and $t_{n+1} - t_n < \delta_\varepsilon$; the latter is possible, since by (14) and continuity, $u\lambda(u)^n < T(\lambda(u) - 1) + u \to 0$. For any such u, put $n := n_T(u)$. As $|t_n(u) - T| < \delta_\varepsilon$, and $\lambda(t_n(u)) = \lambda(u)^n$, by Lemma 4, one has

$$(1-\varepsilon)\lambda(u)^n < \lambda(T) < (1+\varepsilon)\lambda(u)^n$$
$$\frac{(1-\varepsilon)\lambda(u)^n - 1}{T} < \frac{\lambda(T) - 1}{T} < \frac{(1+\varepsilon)\lambda(u)^n - 1}{T}.$$

Approximating T from below and above by t_n and t_{n+1} gives (15). □

Theorem 6. *If λ is a continuous solution of (GS), with $\lambda(u) > 0$ for all $u > 0$, then λ is differentiable (and so of form $\lambda(t) = 1 + at$); in particular, this is so if there are arbitrarily small $u > 0$ with $\lambda(u) > 1$.*

Proof. Note first that if $0 < \lambda(u) < 1$ for some $u > 0$, then $t_n(u)$ is monotone increasing for such u and converges to $\tau = u/(1 - \lambda(u))$ by (14). Then, by continuity, $\lambda(\tau) = \lim_n \lambda(t_n) = \lim_n \lambda(u)^n = 0$, contradicting positivity. Thus positivity implies that $\lambda(u) \geq 1$ for all $u > 0$. The latter conclusion holds also if $\lambda(u) > 1$ for arbitrarily small $u > 0$, by Prop. 4.

We shall now prove that $(\lambda(u) - 1)/u$ has a limit as $u \to 0$, i.e. λ is differentiable at the origin and so everywhere, by Lemma 3. For the purposes of this proof only, call a sequence u_n *nice* if it is null (i.e. satisfies $u_n \to 0$), and $\lambda(u_n) \leq 2$ for all n. By the continuity of λ at 0, any null sequence may be assumed to be nice, and satisfy $\lambda(u_n) \to 1$.

We claim that for every nice sequence u_n the corresponding quotient sequence $(\lambda(u_n) - 1)/u_n$ is bounded. Otherwise, there is a nice sequence u_n with $\{(\lambda(u_n) - 1)/u_n\}$ unbounded. Take $T = 1$ and let $\varepsilon > 0$ be arbitrary. Choose δ_ε as in the proof of Proposition 4. Without loss of generality suppose that $(\lambda(u_n) - 1)/u_n > 2$, so that, in particular, $\lambda(u_n) > 1$ and Proposition 4 applies to $T = 1$ for all n.

For $m = m(n) = n_T(u_n)$, as $t_m(u_n) \leq T = 1 < t_{m+1}(u_n)$, by (14)

$$\lambda(u_n) \leq \lambda(u_n)^{m(n)} \leq 1 + \frac{\lambda(u_n) - 1}{u_n} \leq \lambda(u_n)^{m(n)+1}.$$

As $\{(\lambda(u_n) - 1)/u_n\}$ is unbounded, so is $\lambda(u_n)^{m(n)+1}$ and $\lambda(u_n)^{m(n)}$ (as $\lambda(u_n) < 2$). By Lemma 4, $\lambda(u_n)^{m(n)} = \lambda(t_m(u_n))$ and $|\lambda(t_m(u_n)) - \lambda(1)| < \delta_\varepsilon$ for all n with $u_n < \delta_\varepsilon$ so that $\lambda(u_n)^{m(n)} \to \lambda(1)$, a contradiction to the unboundedness assumption.

Now we may suppose, by passing to a subsequence if necessary, that for every nice sequence u_n the corresponding quotient sequence $(\lambda(u_n) - 1)/u_n$ is not only bounded but in fact convergent. If the limit of the quotient sequence is 0 for each nice sequence, then $\lambda'(0) = 0$, so by the Δ^λ-identity of Lemma 3, $\lambda'(t) = 0$ for all t; then $\lambda(t)$ is constant (and so equal to 1). If, however, the limit of the quotients is not always zero, then fix a nice sequence u_n with positive quotient limit ρ. Here again $\lambda(u_n) > 1$ for all n.

Next fix any $T > 0$ with $\lambda(T) > 1$ (possible as otherwise λ is again constant). Again take $m = m(n) = n_T(u_n)$. Then, as in the unbounded case above, $\lambda(u_n)^{m(n)} = \lambda(t_m(u_n)) \to \lambda(T) > 1$. Then, by (15),

$$(\lambda(T) - 1)/T = \lim_{n\to\infty}(\lambda(u_n) - 1)/u_n = \rho, \text{ i.e. } \lambda(T) = 1 + \rho T.$$

But this holds also in an interval around T, making λ differentiable with derivative ρ in an interval around T and so everywhere, including the origin, by Lemma 3. □

Remark By Proposition 3, $nu \leq t_n(u) \leq T$ for $n = n_T(u)$, so $u \leq T/n$. So if $\lambda(t) = 1 + at$, then $\lambda(u)^n \leq (1 + aT/n)^n \to e^{aT}$ as $u \to 0$, explaining why the unbounded case in the proof above does not arise.

Added in proof The thrust of Theorem 6 above was to explain why continuity entails differentiability here; this is a matter to which we will return elsewhere—with a recent perspective inspired by [7]—see [18]. As to our assumptions: (working in $\mathbb{R}_+$) if $f > 0$, then $f \geq 1$. Indeed, otherwise, suppose $f(u) < 1$ for some $u > 0$; then $v := u/(1 - f(u)) > 0$, and $0 < f(v) = f(u + vf(u)) = f(u)f(v)$, implying $f(u) = 1$, a contradiction. So $f(x + y) = f(x)f(y/f(x)) \geq f(x)$ for $x, y > 0$; so f is (weakly) increasing, and so continuous somewhere, and hence everywhere.

Acknowledgements

It is a pleasure to thank Nick Bingham for discussions concerning the connection between the current contribution and earlier joint work on Beurling regular variation, and to the Referee for a careful reading of the manuscript, and very helpful and scholarly comments.

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Received: May 22, 2013
Revised: January 21, 2014