Aequat. Math. 88 (2014), 183–198

0001-9054/14/010183-16
published online October 11, 2013
DOI 10.1007/s00010-013-0228-4

Aequationes Mathematicae

Semigroup-valued solutions of some composite equations

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Abstract. Let X be a linear space over the field K of real or complex numbers and $(S, \circ)$ be a semigroup. We determine all solutions of the functional equation

$$f(x + g(x)y) = f(x) \circ f(y) \quad \text{for} \quad x, y \in X$$

in the class of pairs of functions (f,g) such that $f : X \to S$ and $g : X \to K$ satisfies some regularity assumptions. Several consequences of this result are presented.

Mathematics Subject Classification (2000). 39B12, 39B52.

Keywords. Gołąb–Schinzel equation, semigroup-valued solution, algebraically interior point.

1. Introduction

Let X be a linear space over the field K of real or complex numbers. The solutions $f : X \to K$ of the Gołąb-Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y) \quad \text{for} \quad x, y \in X, \tag{1}$$

have been intensively studied in the last half-century. Equation (1) is one of the most important equations of a composite type and plays a prominent role e.g. in the determination of substructures of various algebraical structures [1, pp. 311–319], [3,4]. The solutions of (1) and its further generalizations, namely

$$f(x + f(x)^n y) = tf(x)f(y) \quad \text{for} \quad x, y \in X, \tag{2}$$

where n is a nonnegative integer and t is a nonzero real number; and

$$f(x + M(f(x))y) = f(x)f(y) \quad \text{for} \quad x, y \in X, \tag{3}$$

where $M : K \to K$, have been considered under various regularity assumptions e.g. in [2–4] and [19–21]. In the real case the functional equation

$$f(x + M(f(x))y) = f(x) \circ f(y) \quad \text{for} \quad x, y \in X, \tag{4}$$

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where $\circ$ is a binary operation on $\mathbb{R}$ satisfying some additional conditions (commutativity, associativity etc.), was studied in [5,7,16,25] and [26]. For more information concerning (1)–(4) and their further applications (e.g. to mathematical meteorology and fluid dynamics) we refer to the survey paper [6]. Various aspects of stability problems for the Gołąb-Schinzel functional equations were considered in [8–13,15,17] and [18]. In the case where $(S, \circ)$ is an arbitrary semigroup, the general solution of the equation

$$f(x + g(x)y) = f(x) \circ f(y) \quad \text{for} \quad x, y \in \mathbb{R}$$

in the class of pairs (f, g) such that $f : \mathbb{R} \to S$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous, was determined in [14]. The functional equation

$$f(x + g(x)y) = f(x)f(y) \quad \text{for} \quad x, y \in X$$

was considered in [22] under the assumption that f and g, mapping a real linear space X into $\mathbb{R}$, are continuous on rays.

In the present paper we generalize substantially the results from [14] and [22] in various directions. Namely, we determine the general solution of the equation

$$f(x + g(x)y) = f(x) \circ f(y) \quad \text{for} \quad x, y \in X \tag{5}$$

in the case where X is a linear space over the field K of real or complex numbers, $(S, \circ)$ is an arbitrary semigroup, $f : X \to S$ and $g : X \to K$ satisfies some regularity assumptions. Several consequences of this result are presented, as well. In particular, applying our main result and using a natural correspondence between (5) and the pexiderized version of the Gołąb-Schinzel equation, that is

$$F(x + G(x)y)) = H(x) \circ K(y) \quad \text{for} \quad x, y \in X, \tag{6}$$

we obtain a generalization of the results in [23].

In what follows $B(x, r)$ denotes the open ball (in K) with a center at $x \in K$ and a radius $r > 0$. Let us recall [24, p. 596] that given a nonempty subset A of X, we say that $a \in A$ is an algebraically interior point of A, provided, for every $x \in X \setminus \{0\}$, there is $r_x > 0$ such that $a + B(0, r_x)x = \{a + bx : b \in B(0, r_x)\} \subset A$. By $\mathrm{int}_a A$ we denote a set of all algebraically interior points of A. If $f : X \to \mathbb{R}$ and $x \in X$ then a function $f_x : \mathbb{R} \to \mathbb{R}$ is given by $f_x(t) = f(tx)$ for $t \in \mathbb{R}$. Furthermore, given a nonempty subset S_0 of S, we put $Z_L(S_0) := \{s \in S : s \circ a = s \ \text{ for } \ a \in S_0\}$ and $Z(S_0) := \{s \in S : s \circ a = a \circ s = s \ \text{ for } \ a \in S_0\}$.

2. Preliminary results

Remark 1. Let X be a linear space over the field K of real or complex numbers and $(S, \circ)$ be a semigroup. Equation (5) has a solution if and only if $E(S) :=$

$\{s \in S : s \circ s = s\} \neq \emptyset$. In fact, if (f, g) is a solution of (5), then $f(0) = f(0) \circ f(0)$, so $f(0) \in E(S)$. Conversely, if $s \in E(S)$, then the pair (f, g), where g is an arbitrary function and $f \equiv s \in E(S)$, is a solution of (5).

Next, we present a result describing degenerate solutions of (5), i.e. such solutions (f, g) that either f or g is constant.

Proposition 1. *A pair of functions (f, g) is a degenerate solution of (5) if and only if one of the following conditions is valid:*

- (i) *there is an $s \in E(S)$ such that $f \equiv s$;*
- (ii) *$g \equiv 0$ and there exists a subsemigroup S_0 of S such that $u \circ v = u$ for $u, v \in S_0$ and $f(X) \subset S_0$;*
- (iii) *$g \equiv 1$ and f is a homomorphism of the additive group of K into $(S, \circ)$.*

Proof. It is clear that if one of the conditions $(i) - (iii)$ holds, then (f, g) is a degenerate solution of (5). So, assume that (f, g) is a degenerate solution of (5). If f is constant, then according to Remark 1, we get (i) with $s := f(0)$. Now, assume that f is nonconstant and g is constant, say $g \equiv c$. If $c = 0$, then (ii) holds with $S_0 := f(X)$. The case where $c = 1$ leads to (iii). Suppose that $c \notin \{0, 1\}$. Then, in view of (5), we obtain $f(x+cy) = f(x) \circ f(y)$ for $x, y \in X$, whence $f(cy) = f(0) \circ f(y)$ for $y \in X$. Therefore, for every $x, y \in X$, we get

$$\begin{aligned} f(y) &= f\left(c\frac{cx-y}{c^2-c} + c^2\frac{y-x}{c^2-c}\right) = f(0) \circ f\left(\frac{cx-y}{c^2-c} + c\frac{y-x}{c^2-c}\right) \\ &= f(0) \circ f\left(\frac{cx-y}{c^2-c}\right) \circ f\left(\frac{y-x}{c^2-c}\right) = f\left(c\frac{cx-y}{c^2-c}\right) \circ f\left(\frac{y-x}{c^2-c}\right) \\ &= f\left(c\frac{cx-y}{c^2-c} + c\frac{y-x}{c^2-c}\right) = f(x). \end{aligned}$$

This yields a contradiction, because f is nonconstant. □

From now on we will deal only with the non-degenerate solutions of (5), i.e. with such solutions (f, g) that neither f nor g is constant.

The following result plays a crucial role in our considerations.

Proposition 2. *Let X be a linear space over the field K of real or complex numbers, $(S, \circ)$ be a semigroup, $f : X \to S$ and $g : X \to K$. Assume that (f, g) is a non-degenerate solution of (5). Then each of the following regularity conditions:*

(C_1) *$0 \in g(X)$ and $int_a\{x \in X | g(x) \neq 0\} \neq \emptyset$;*
(C_2) *$int_a\{x \in X | g(x) \notin \{0, 1\}\} \neq \emptyset$;*
(C_3) *g is continuous on rays*

implies that either there exists a nontrivial K-linear functional $L : X \to K$ such that

$$g(x) = L(x) + 1 \quad \textit{for} \quad x \in X, \tag{7}$$

or there exists a nontrivial $\mathbb{R}$*-linear functional* $L : X \to \mathbb{R}$ *such that*

$$g(x) = \max\{L(x) + 1, 0\} \quad \textit{for} \quad x \in X. \tag{8}$$

Proof. In view of (5), for every $x, y, z \in X$, we have

$$(f(x) \circ f(y)) \circ f(z) = f(x + g(x)y) \circ f(z) = f(x + g(x)y + g(x + g(x)y)z)$$

and

$$f(x) \circ (f(y) \circ f(z)) = f(x) \circ f(y + g(y)z) = f(x + g(x)y + g(x)g(y)z).$$

Thus, as $\circ$ is associative, we get

$$f(x + g(x)y + g(x + g(x)y)z) = f(x + g(x)y + g(x)g(y)z) \ \text{ for } \ x, y, z \in X.$$

Therefore, if $g(x + g(x)y) = 0$ for some $x, y \in X$ then

$$f(x + g(x)y) = f(x + g(x)y + g(x)g(y)z) \ \text{ for } \ z \in X$$

and so, as f is nonconstant, we get $g(x)g(y) = 0$. Similarly if, for some $x, y \in X$, $g(x)g(y) = 0$ then

$$f(x + g(x)y + g(x + g(x)y)z) = f(x + g(x)y) \ \text{ for } \ z \in X.$$

Since f is nonconstant, this means that $g(x + g(x)y) = 0$. In this way we have proved that, for every $x, y \in X$, it holds that

$$g(x + g(x)y) = 0 \quad \text{if and only if} \quad g(x)g(y) = 0. \tag{9}$$

So, if (C_1) is valid, according to [17, Theorem 1], we obtain that

$$g(x + g(x)y) = g(x)g(y) \quad \text{for} \quad x, y \in X. \tag{10}$$

Hence, applying [4, Theorem 3], we get the assertion.

Now, assume that (C_2) holds. We show that $0 \in g(X)$. Suppose that $0 \notin g(X)$. Then, by (5), we get

$$f(0) = f\left(x + g(x)\left(-\frac{x}{g(x)}\right)\right) = f(x) \circ f\left(-\frac{x}{g(x)}\right) \quad \text{for} \quad x \in X. \tag{11}$$

Therefore, taking $x \in G_1 := \{x \in X | g(x) \neq 1\}$ and $y = \frac{x}{1-g(x)}$, we have $x + g(x)y = y$, so by (5) and (11), we obtain

$$\begin{aligned} f(x) = f(x) \circ f(0) &= f(x) \circ f(y) \circ f\left(-\frac{y}{g(y)}\right) \\ &= f(x + g(x)y) \circ f\left(-\frac{y}{g(y)}\right) = f(y) \circ f\left(-\frac{y}{g(y)}\right) = f(0). \end{aligned}$$

Thus

$$f(x) = f(0) \quad \text{for} \quad x \in G_1. \tag{12}$$

Next, as (f, g) is a non-degenerate solution of (5), there is $x \in X \setminus \{0\}$ such that f_x is a nonconstant function. Fix $a \in \mathrm{int}_a\{x \in X | g(x) \notin \{0,1\}\} = \mathrm{int}_a G_1$. Then, in view of (12), $f(a) = f(0)$. Furthermore, let $r_x > 0$ be such that

$$a + B(0, r_x)x \subset G_1. \tag{13}$$

We claim that there is $r > 0$ such that

$$f(B(0, r)x) = \{f(0)\}. \tag{14}$$

Suppose that (14) does not hold. Then there exists a sequence (t_n) of elements of K converging to 0 and such that $f(t_n x) \neq f(0)$ for $n \in \mathbb{N}$. Then, by (12), $g(t_n x) = 1$ for $n \in \mathbb{N}$ and so, in view of (5), for every $n \in \mathbb{N}$, we get

$$f(a + t_n x) = f(t_n x + g(t_n x)a) = f(t_n x) \circ f(a) = f(t_n x) \circ f(0) = f(t_n x) \neq f(0).$$

On the other hand, from (13) it follows that $a + t_n x \in G_1$ for sufficiently large $n \in \mathbb{N}$. Hence, by (12), $f(a + t_n x) = f(0)$ for sufficiently large $n \in \mathbb{N}$, which yields a contradiction. In this way we have proved (14). Since the function f_x is nonconstant, there is $k \in K$ with $f(kx) = f_x(k) \neq f_x(0) = f(0)$. Therefore, making use of (12), we get

$$g(kx) = 1. \tag{15}$$

Now, we show by induction that for every $b \in B(0, r)$ and $n \in \mathbb{N}$ it holds that

$$f((k + nb)x) \neq f(0). \tag{16}$$

Note that by (5), (14) and (15), for every $b \in B(0, r)$, we have

$$f((k + b)x) = f(kx + g(kx)bx) = f(kx) \circ f(bx) = f(kx) \circ f(0) = f(kx) \neq f(0).$$

Thus, (16) is valid for $n = 1$. Next, fix $n \in \mathbb{N}$ and assume that (16) holds for every $b \in B(0, r)$. Then, in view of (12), $g((k + bn)x) = 1$, so applying (5), (14) and (16), for every $b \in B(0, r)$, we obtain

$$\begin{aligned} f((k + (n+1)b)x) &= f((k + nb)x + bx) = f((k + nb)x + g((k + nb)x)bx) \\ &= f((k + nb)x) \circ f(bx) = f((k + nb)x) \circ f(0) \\ &= f((k + nb)x) \neq f(0). \end{aligned}$$

In this way we have proved that (16) holds for every $b \in B(0, r)$ and $n \in \mathbb{N}$. Note however that, taking $n_0 \in \mathbb{N}$ with $\frac{k}{n_0} \in B(0, r)$, we have

$$f\left(kx + n_0\left(-\frac{k}{n_0}\right)x\right) = f(0).$$

This yields a contradiction and shows that $0 \in g(X)$. Since by (C_2)

$$\emptyset \neq \mathrm{int}_a\{x \in X | g(x) \notin \{0,1\}\} \subset \mathrm{int}_a\{x \in X | g(x) \neq 0\},$$

applying [17, Theorem 1], from (9) we deduce (10). Thus, according to [4, Theorem 3], we get the assertion.

Finally, if (C_3) holds for n, as in the previous case, we get that the function f_x is nonconstant for some $x \in X \setminus \{0\}$. Moreover, by (C_3), g_x is continuous and, in view of (5),

$$f_x(t + g_x(t)s) = f(tx + g(tx)sx) = f(tx) \circ f(sx) = f_x(t) \circ f_x(s) \quad \text{for} \quad s, t \in \mathbb{R}.$$

So, applying [14, Lemma 1], we obtain that $g_x(t_0) = 0$ for some $t_0 \in \mathbb{R}$, that is $0 \in g_x(\mathbb{R}) \subset g(X)$. Note also that by [14, Lemma 2], we have $g(0) = g_x(0) = 1$. Thus, in view of (C_3), $0 \in \mathrm{int}_a\{x \in X | g(x) \neq 0\}$. Consequently, applying [17, Theorem 1], from (9) we derive (10). Therefore, according to [4, Theorem 3], we get the assertion. □

3. Main results

The next theorem is the main result of the paper.

Theorem 1. *Assume that X is a linear space over the field K of real or complex numbers, $(S, \circ)$ is a semigroup, $f : X \to S$, $g : X \to K$ and one of the conditions (C_1)-(C_3) holds. A pair of functions (f, g) is a non-degenerate solution of (5) if and only if one of the following two cases holds:*

(i) *there exist a nontrivial K-linear functional $L : X \to K$ and functions $a : X \to S$ and $\phi : K \to S$ satisfying the conditions*

$$a(x + y) = a(x) \circ a(y) \quad \text{for} \quad x, y \in X, \tag{17}$$

$$\phi(st) = \phi(s) \circ \phi(t) \quad \text{for} \quad s, t \in K, \tag{18}$$

and

$$\phi(t) \circ a(x) = a(tx) \circ \phi(t) \quad \text{for} \quad x \in X, t \in K \tag{19}$$

such that g is of the form (7) and

$$f(x) = a(x) \circ \phi(L(x) + 1) \quad \text{for} \quad x \in X; \tag{20}$$

(ii) *there exist a nontrivial $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$, functions $a : X \to S$ and $\phi : [0, \infty) \to S$ satisfying (17),*

$$\phi(st) = \phi(s) \circ \phi(t) \quad \text{for} \quad s, t \in [0, \infty), \tag{21}$$

$$a(tx) \circ \phi(t) = \phi(t) \circ a(x) \quad \text{for} \quad x \in X, t \in [0, \infty) \tag{22}$$

and $l \in Z_L(\phi([0, \infty))) \cap E(S)$ such that g is of the form (8) and

$$f(x) = \begin{cases} a(x) \circ \phi(L(x) + 1) & \text{whenever} \quad L(x) + 1 \geq 0 \\ a(x) \circ \phi(-(L(x) + 1)) \circ l & \text{otherwise.} \end{cases} \tag{23}$$

Proof. Assume that a pair (f, g) is a non-degenerate solution of (5). Then, according to Proposition 2, g is either of the form (7) or (8). In the first case, let us fix $x_0 \in X \setminus \ker L$ and define the functions $\Pi_1, \Pi_2 : X \to X$ by

$$\Pi_1(x) = x - \frac{L(x)}{L(x_0)}x_0 \quad \text{for} \quad x \in X \tag{24}$$

and

$$\Pi_2(x) = \frac{L(x)}{L(x_0)}x_0 \quad \text{for} \quad x \in X. \tag{25}$$

Then Π_1 and Π_2 are linear and $\Pi_1(x) + \Pi_2(x) = x$ for $x \in X$. Moreover $L(\Pi_1(x)) = 0$ for $x \in X$, which yields that

$$g(\Pi_1(x)) = 1 \quad \text{for} \quad x \in X. \tag{26}$$

Therefore, taking $a : X \to S$ of the form

$$a(x) = f(\Pi_1(x)) \quad \text{for} \quad x \in X \tag{27}$$

and $\phi : K \to S$ of the form

$$\phi(t) = f\left(\frac{t-1}{L(x_0)}x_0\right) \quad \text{for} \quad t \in K, \tag{28}$$

in view of (5), we obtain

$$\begin{aligned} a(x+y) &= f(\Pi_1(x+y)) = f(\Pi_1(x) + \Pi_1(y)) = f(\Pi_1(x) + g(\Pi_1(x))\Pi_1(y)) \\ &= f(\Pi_1(x)) \circ f(\Pi_1(y)) = a(x) \circ a(y) \quad \text{for} \quad x, y \in X \end{aligned}$$

and, since $st - 1 = s - 1 + ((s-1)+1)(t-1)$,

$$\begin{aligned} \phi(st) &= f\left(\frac{st-1}{L(x_0)}x_0\right) = f\left(\frac{s-1}{L(x_0)}x_0 + \left(L\left(\frac{s-1}{L(x_0)}x_0\right) + 1\right)\frac{t-1}{L(x_0)}x_0\right) \\ &= f\left(\frac{s-1}{L(x_0)}x_0 + g\left(\frac{s-1}{L(x_0)}x_0\right)\frac{t-1}{L(x_0)}x_0\right) \\ &= f\left(\frac{s-1}{L(x_0)}x_0\right) \circ f\left(\frac{t-1}{L(x_0)}x_0\right) \\ &= \phi(s) \circ \phi(t) \quad \text{for} \quad s, t \in K. \end{aligned}$$

Moreover, for every $x \in X$, we get

$$f(x) = f(\Pi_1(x) + \Pi_2(x)) = f(\Pi_1(x) + g(\Pi_1(x))\Pi_2(x)) = f(\Pi_1(x)) \circ f(\Pi_2(x)).$$

Hence by (25) and (27), we obtain

$$f(x) = a(x) \circ f\left(\frac{L(x)}{L(x_0)}x_0\right) \quad \text{for} \quad x \in X. \tag{29}$$

Thus, taking (28) into account, we conclude that (20) holds. It remains to show (19). To this end note that, in view of (26), for every $x \in X$ and $t \in K$, we have

$$\begin{aligned}\Pi_1(tx)+g(\Pi_1(tx))\frac{t-1}{L(x_0)}x_0 &= \frac{t-1}{L(x_0)}x_0+\left(\frac{t-1}{L(x_0)}L(x_0)+1\right)\Pi_1(x)\\ &=\frac{t-1}{L(x_0)}x_0+\left(L\left(\frac{t-1}{L(x_0)}x_0\right)+1\right)\Pi_1(x)\\ &=\frac{t-1}{L(x_0)}x_0+g\left(\frac{t-1}{L(x_0)}x_0\right)\Pi_1(x).\end{aligned}$$

Hence, by (5), we obtain

$$f(\Pi_1(tx))\circ f\left(\frac{t-1}{L(x_0)}x_0\right)=f\left(\frac{t-1}{L(x_0)}x_0\right)\circ f(\Pi_1(x))\quad \text{for} \quad x\in X, t\in K.$$

So, taking (27) and (28) into account, we get (19). In this way we have proved that (i) is valid.

Next consider the case where g is of the form (8). Let $x_0\in X\setminus\ker L$ and let $\Pi_1,\Pi_2: X\to X$ be given by (24) and (25), respectively. Furthermore, let $a: X\to S$ be of the form (27) and $\phi:[0,\infty)\to S$ be given by

$$\phi(t)=f\left(\frac{t-1}{L(x_0)}x_0\right)\quad \text{for} \quad t\in[0,\infty). \tag{30}$$

Then, arguing as in the previous case, we obtain (17), (21), (22) and (29). In particular, by (29) and (30), we have

$$f(x)=a(x)\circ\phi(L(x)+1)\quad \text{whenever} \quad L(x)+1\geq 0. \tag{31}$$

Fix $x\in X$ with $L(x)+1<0$. Then

$$L\left(\frac{-(L(x)+2)}{L(x_0)}x_0\right)+1=-(L(x)+1)>0, \tag{32}$$

so by (8), we get $g\left(\frac{-(L(x)+2)}{L(x_0)}x_0\right) = -(L(x)+1)$. Therefore, making use of (5), (31) and (32), we obtain

$$\begin{aligned}f\left(\frac{L(x)}{L(x_0)}x_0\right) &= f\left(\frac{-(L(x)+2)}{L(x_0)}x_0+g\left(\frac{-(L(x)+2)}{L(x_0)}x_0\right)\left(\frac{-2x_0}{L(x_0)}\right)\right)\\ &=f\left(\frac{-(L(x)+2)}{L(x_0)}x_0\right)\circ f\left(\frac{-2x_0}{L(x_0)}\right)\\ &=a\left(\frac{-(L(x)+2)}{L(x_0)}x_0\right)\circ\phi(-(L(x)+1))\circ f\left(\frac{-2x_0}{L(x_0)}\right).\end{aligned}$$

On the other hand, in view of (24), we get $\Pi_1(0)=0$ and $\Pi_1\left(\frac{-(L(x)+2)}{L(x_0)}x_0\right) = 0$, whence by (27), $a\left(\frac{-(L(x)+2)}{L(x_0)}x_0\right) = f(0) = a(0)$. Thus, for every $x\in X$ with $L(x)+1<0$, we have

$$f\left(\frac{L(x)}{L(x_0)}x_0\right)=a(0)\circ\phi(-(L(x)+1))\circ f\left(\frac{-2x_0}{L(x_0)}\right)$$

and so, by (17),

$$a(x) \circ f\left(\frac{L(x)}{L(x_0)}x_0\right) = a(x) \circ \phi(-(L(x)+1)) \circ f\left(\frac{-2x_0}{L(x_0)}\right).$$

Hence, taking (29) into account, we conclude that

$$f(x) = a(x) \circ \phi(-(L(x)+1)) \circ l \quad \text{whenever} \quad L(x)+1 < 0, \tag{33}$$

where $l := f\left(\frac{-2x_0}{L(x_0)}\right)$. Note also that $L\left(\frac{-2x_0}{L(x_0)}\right)+1 = -1 < 0$, so $g\left(\frac{-2x_0}{L(x_0)}\right) = 0$. Thus, in view of (5), for every $x \in X$, it holds that

$$l \circ f(x) = f\left(\frac{-2x_0}{L(x_0)}\right) \circ f(x) = f\left(\frac{-2x_0}{L(x_0)} + g\left(\frac{-2x_0}{L(x_0)}\right)x\right) = f\left(\frac{-2x_0}{L(x_0)}\right) = l.$$

Hence $l \in Z_L(f(X))$. Since $l \in f(X)$ and, by (30), $\phi([0,\infty)) \subset f(X)$, this implies that $l \in Z_L(\phi([0,\infty))) \cap E(S)$. Therefore, taking (31) and (33) into account, we get (23). Consequently (ii) holds.

Since the converse is easy to check, the proof is completed. □

In the case of a commutative semigroup, a description of the solutions of (5) is significantly simpler. Namely, we have the following result.

Proposition 3. *Let X be a linear space over the field K of real or complex numbers, $(S, \circ)$ be a commutative semigroup, $f : X \to S$, $g : X \to K$ and assume that one of the conditions (C_1)–(C_3) holds. Then a pair (f, g) is a non-degenerate solution of (5) if and only if one of the subsequent cases holds:*

(a) *there exist a nontrivial K-linear functional $L : X \to K$ and a nontrivial homomorphism ψ of the multiplicative semigroup of K into $(S, \circ)$ such that g is of the form (7) and*

$$f(x) = \psi(L(x)+1) \quad \textit{for} \quad x \in X; \tag{34}$$

(b) *there exist a nontrivial $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$, a homomorphism ψ of the multiplicative semigroup of nonnegative real numbers into $(S, \circ)$ and $z \in Z(\psi([0,\infty)))$ such that $\psi \neq z$, g is of the form (8) and*

$$f(x) = \begin{cases} \psi(L(x)+1) & \textit{whenever} \quad L(x)+1 \geq 0 \\ z & \textit{otherwise.} \end{cases} \tag{35}$$

Proof. Assume that a pair (f, g) is a non-degenerate solution of (5). Then one of the conditions (i), (ii) of Theorem 1 holds. In the case of (i), (19) and the commutativity of $\circ$ imply that $a(2x) \circ \phi(2) = a(x) \circ \phi(2)$ for $x \in X$, whence by (18), $a(2x) \circ \phi(1) = a(x) \circ \phi(1)$ for $x \in X$. Thus, in view of (17), for every $x \in X$, we get

$$a(x) \circ \phi(1) = a(-x) \circ a(2x) \circ \phi(1) = a(-x) \circ a(x) \circ \phi(1) = a(0) \circ \phi(1). \tag{36}$$

Therefore, taking $\psi := a(0) \circ \phi$, by (17), (18), (20) and (36), for every $x \in X$, we obtain

$$\begin{aligned} f(x) &= a(x) \circ \phi(L(x)+1) = a(x) \circ a(0) \circ \phi(1) \circ \phi(L(x)+1) \\ &= a(x) \circ \phi(1) \circ a(0) \circ \phi(L(x)+1) = a(0) \circ \phi(1) \circ a(0) \circ \phi(L(x)+1) \\ &= a(0) \circ \phi(L(x)+1). \\ &= \psi(L(x)+1). \end{aligned}$$

Moreover, by (17), $a(0) = a(0) \circ a(0)$, so using (18) one can easily check that ψ is a homomorphism of the multiplicative semigroup of K into $(S, \circ)$. Note also that, as f is nonconstant, ψ is nontrivial. In this way we have proved that (a) holds.

Next assume that condition (ii) of Theorem 1 holds. Then, as in the previous case, we obtain that (36) is valid, $\psi := a(0) \circ \phi$ is a homomorphism of the multiplicative semigroup of $[0, \infty)$ into $(S, \circ)$ and $f(x) = \psi(L(x)+1)$ whenever $L(x) + 1 \geq 0$. Furthermore, the commutativity of $\circ$ implies that $z := a(0) \circ l \in Z(\psi([0, \infty)))$. Hence, as $(S, \circ)$ is commutative, by (17), (18), (23) and (36), for every $x \in X$ with $L(x) + 1 < 0$, we get

$$\begin{aligned} f(x) = a(x) \circ \phi(-(L(x)+1)) \circ l &= a(x) \circ a(0) \circ a(0) \circ \phi(1) \circ \phi(-(L(x)+1)) \circ l \\ &= a(x) \circ \phi(1) \circ a(0) \circ l \circ a(0) \circ \phi(-(L(x)+1)) \\ &= a(0) \circ \phi(1) \circ z \circ \psi(-(L(x)+1)) = \psi(1) \circ z = z. \end{aligned}$$

Thus f is of the form (35) and therefore (b) is valid.

The converse is easy to check, so the proof is completed. □

The next result concerns the case where $X = \mathbb{R}$.

Proposition 4. *Assume that $(S, \circ)$ is a semigroup, $f : \mathbb{R} \to S, g : \mathbb{R} \to \mathbb{R}$ are nonconstant functions and $int\{x \in \mathbb{R} | g(x) \notin \{0, 1\}\} \neq \emptyset$. Then a pair of functions (f, g) satisfies (5) if and only if one of the following two conditions holds:*

(a) *there exist a nontrivial homomorphism $\psi : \mathbb{R} \to S$ of the multiplicative semigroup of real numbers into $(S, \circ)$ and $c \in \mathbb{R} \setminus \{0\}$ such that*

$$\begin{aligned} g(x) &= cx + 1 \quad \textit{for} \quad x \in \mathbb{R}, \\ f(x) &= \psi(cx + 1) \quad \textit{for} \quad x \in \mathbb{R}; \end{aligned}$$

(b) *there exist a nontrivial homomorphism $\psi : [0, \infty) \to S$ of the multiplicative semigroup of nonnegative real numbers into $(S, \circ)$, $c \in \mathbb{R} \setminus \{0\}$ and $z \in Z(\psi([0, \infty))) \cap E(S)$ such that*

$$\begin{aligned} g(x) &= max\{cx + 1, 0\} \quad \textit{for} \quad x \in \mathbb{R}, \\ f(x) &= \begin{cases} \psi(cx+1) & \textit{whenever} \quad cx + 1 \geq 0 \\ z & \textit{otherwise.} \end{cases} \end{aligned}$$

Proof. Assume that a pair (f, g) satisfies (5). Since f and g are nonconstant and $int\{x \in X | g(x) \notin \{0,1\}\} \neq \emptyset$, the assumptions of Theorem 1 are satisfied with $X = \mathbb{R}$. So, one of the conditions (i) or (ii) of Theorem 1 holds. In the case of (i), we have $L(x) = cx$ for $x \in \mathbb{R}$ with some $c \in \mathbb{R} \setminus \{0\}$. Let

$$\psi(t) := a\left(-\frac{1}{c}\right) \circ \phi(t) \circ a\left(\frac{1}{c}\right) \quad \text{for} \quad t \in \mathbb{R}. \tag{37}$$

Since, by (19), $a(0) \circ \phi(t) = \phi(t) \circ a(0)$ for $t \in X$ and $\phi : \mathbb{R} \to S$ is a homomorphism of the multiplicative semigroup of real numbers into $(S, \circ)$, making use of (17), we obtain

$$\begin{aligned}\psi(st) &= a\left(-\frac{1}{c}\right) \circ \phi(st) \circ a\left(\frac{1}{c}\right) = a\left(-\frac{1}{c}\right) \circ \phi(s) \circ \phi(t) \circ a\left(\frac{1}{c}\right)\\ &= a\left(-\frac{1}{c}\right) \circ \phi(s) \circ \phi(t) \circ a(0) \circ a\left(\frac{1}{c}\right)\\ &= a\left(-\frac{1}{c}\right) \circ \phi(s) \circ a(0) \circ \phi(t) \circ a\left(\frac{1}{c}\right)\\ &= a\left(-\frac{1}{c}\right) \circ \phi(s) \circ a\left(\frac{1}{c}\right) \circ a\left(-\frac{1}{c}\right) \circ \phi(t) \circ a\left(\frac{1}{c}\right)\\ &= \psi(s) \circ \psi(t) \quad \text{for} \quad s, t \in \mathbb{R}.\end{aligned}$$

Hence ψ is a homomorphism of the multiplicative semigroup of real numbers into $(S, \circ)$. Moreover, by (17) and (37), we get $\phi(t) = a\left(\frac{1}{c}\right) \circ \psi(t) \circ a\left(-\frac{1}{c}\right)$ for $t \in \mathbb{R}$. Thus, as ϕ is nontrivial, so is ψ. Furthermore, considering (17), (19) and (20), we have

$$\begin{aligned}f(x) &= a(x) \circ \phi(cx+1) = a\left(-\frac{1}{c}\right) \circ a\left(\frac{1}{c}(cx+1)\right) \circ \phi(cx+1)\\ &= a\left(-\frac{1}{c}\right) \circ \phi(cx+1) \circ a\left(\frac{1}{c}\right) = \psi(cx+1) \quad \text{for} \quad x \in \mathbb{R}.\end{aligned}$$

In this way we have proved that (a) holds.

In the case of (ii), similar arguments lead to (b). As the converse is easy to check, this completes the proof. □

Remark 2. Proposition 4 together with Proposition 1 generalizes [14, Theorem 1]. Note that in [14, Theorem 1] and [14, Corollary 1] in the formulae "$l \in Z_L(\phi([0,\infty))) \cap E(S)$" and "$l \in Z(\psi([0,\infty))) \cap E(S)$" the term "$\cap E(S)$" is missing.

In general, the function a acting in the assertion of Theorem 1 need not be constant.

Example 1. Let X be a real linear space of dimension at least 2 and let $S = \mathbb{R}^2$ be endowed with the following binary operation

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 y_1, x_1 y_2 + y_1) \quad \text{for} \quad (x_1, y_1), (x_2, y_2) \in S.$$

Define the functions $a : X \to S$ and $\phi : \mathbb{R} \to S$ by $a(x) = (1, x)$ for $x \in X$ and $\phi(t) = (t, 0)$ for $t \in \mathbb{R}$, respectively. Then an easy calculation shows that (17)–(19) hold. So, taking a nontrivial $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$ and applying Theorem 1(i), we conclude that a pair of functions (f, g), where $g : X \to \mathbb{R}$ is of the form (7) and $f : X \to S$ is given by

$$f(x) = a(x) \circ \phi(L(x) + 1) = (L(x) + 1, x) \quad \text{for} \quad x \in X,$$

satisfies (5). Note also that f is injective and, as *dim* $X \geq 2$, L is not. Thus, f can not be represented in the form $f(x) = \psi(L(x) + 1)$ for $x \in X$ with some $\psi : \mathbb{R} \to S$.

Next, we present the result concerning the solutions of (5) in the case where $(S, \circ)$ is a group.

Proposition 5. *Let X be a linear space over the field K of real or complex numbers and let $(S, \circ)$ be a group (with unit element e). Assume that $f : X \to S$, $g : X \to K$ and that one of the conditions (C_1)–(C_3) holds. Then a pair (f, g) satisfies (5) if and only if either $f = e$, or $g = 1$ and f is a homomorphism of an additive group $(X, +)$ into $(S, \circ)$.*

Proof. Suppose that (f, g) is a non-degenerate solution of (5). Then one of the conditions (i) or (ii) of Theorem 1 holds. In the first case, putting $t = 0$ in (18) and considering the fact that $(S, \circ)$ is a group, we conclude that ϕ is constant. Furthermore, applying (19) with $t = 0$, we get $\phi(0) \circ a(x) = a(0) \circ \phi(0)$ for $x \in X$, so also a is constant. Hence, in view of (20), f is constant, which yields a contradiction. In the case where condition (ii) of Theorem 1 holds, the same arguments yield a contradiction (note that in this case $\phi = e$, $Z_L(\phi([0, \infty))) \cap E(S) = \{e\}$ and so $l = e$).

In this way we have proved that Eq. (5) has only degenerate solutions. Furthermore, as $(S, \circ)$ is a group, we have $E(S) = \{e\}$. Moreover, the only subsemigroup S_0 of $(S, \circ)$ such that $u \circ v = u$ for $u, v \in S_0$, is $S_0 = \{e\}$. Therefore, applying Proposition 1, we obtain the assertion. □

We complete the paper with two results concerning (6), which generalize some results in [23].

Proposition 6. *Let X be a linear space over the field K of real or complex numbers and let $(S, \circ)$ be a commutative semigroup with unit element e. Assume that $F, H, K : X \to S$, $G : X \to K$, F and G are nonconstant and the set $F(X)$ contains at least one invertible element. Then a quadruple (F, G, H, K)*

satisfies equation (6) *if and only if there exist* $x_0 \in X$, $s,t \in S$, $k \in K \setminus \{0\}$ *and functions* $f : X \to S$, $g : X \to K$ *such that a pair* (f,g) *satisfies* (5) *and*

$$F(x) = s \circ t \circ f(x - x_0) \quad \textit{for} \quad x \in X, \tag{38}$$

$$G(x) = kg(x - x_0) \quad \textit{for} \quad x \in X, \tag{39}$$

$$H(x) = t \circ f(x - x_0) \quad \textit{for} \quad x \in X, \tag{40}$$

$$K(x) = s \circ f(kx) \quad \textit{for} \quad x \in X. \tag{41}$$

Proof. Assume that a quadruple (F,G,H,K) satisfies Eq. (6). By the assumption, there is $x_0 \in X$ such that $F(x_0) \circ p = p \circ F(x_0) = e$ for some $p \in S$. In (6) taking $y = 0$ and next $x = x_0$, we get

$$F(x) = H(x) \circ K(0) \quad \text{for} \quad x \in X \tag{42}$$

and

$$F(x_0 + G(x_0)y) = H(x_0) \circ K(y) \quad \text{for} \quad y \in X, \tag{43}$$

respectively. Since, in view of (42),

$$F(x_0) = H(x_0) \circ K(0), \tag{44}$$

using the commutativity of $\circ$, we derive that

$$H(x) = H(x) \circ e = H(x) \circ F(x_0) \circ p = H(x_0) \circ p \circ H(x) \circ K(0) \quad \text{for} \quad x \in X.$$

So considering (42), we get

$$H(x) = H(x_0) \circ p \circ F(x) \quad \text{for} \quad x \in X. \tag{45}$$

In a similar way, using (43), we obtain

$$K(x) = K(0) \circ p \circ F(x_0 + G(x_0)x) \quad \text{for} \quad x \in X. \tag{46}$$

Furthermore, by (6) and (44), for every $x, y \in X$, we get

$$\begin{aligned} F(x + G(x)y)) &= F(x + G(x)y)) \circ e = F(x + G(x)y)) \circ F(x_0) \circ p \\ &= p \circ H(x) \circ K(0) \circ H(x_0) \circ K(y). \end{aligned}$$

Hence, in view of (42) and (43), we obtain

$$F(x + G(x)y)) = p \circ F(x) \circ F(x_0 + G(x_0)y) \quad \text{for} \quad x, y \in X. \tag{47}$$

Note also that $G(x_0) \neq 0$. Otherwise, as G is nonconstant, In (45) putting $y = y_1$ with $G(y_1) \neq 0$, we would have that F is constant, which yields a contradiction. Now, let

$$f(x) = p \circ F(x + x_0) \quad \text{for} \quad x \in X \tag{48}$$

and

$$g(x) = \frac{G(x + x_0)}{G(x_0)} \quad \text{for} \quad x \in X. \tag{49}$$

Then, making use of (47), we obtain

$$\begin{aligned} f(x+g(x)y) &= p\circ F\left(x_0+x+G(x+x_0)\frac{y}{G(x_0)}\right)\\ &= p\circ F(x_0+x)\circ p\circ F\left(x_0+G(x_0)\frac{y}{G(x_0)}\right)\\ &= f(x)\circ f(y) \quad \text{for} \quad x,y\in X, \end{aligned}$$

so a pair (f,g) satisfies (5). Moreover, considering (44)–(46), (48) and (49), we obtain (38)–(41) with $k:=G(x_0)$, $s:=K(0)$ and $t:=H(x_0)$.

Conversely, assume that (38)–(41) hold with some $k\in K\setminus\{0\}$, $x_0\in X$, $s,t\in S$ and functions $f:X\to S$, $g:X\to K$ such that the pair (f,g) satisfies (5). Then, for every $x,y\in X$, we have

$$\begin{aligned} F(x+G(x)y)) &= s\circ t\circ f(x-x_0+kg(x-x_0)y))\\ &= s\circ t\circ f(x-x_0)\circ f(ky)=H(x)\circ K(y). \end{aligned}$$

□

Proposition 7. *Let X be a linear space over the field K of real or complex numbers, $(S,\circ)$ be a commutative semigroup with unit element e and let $F,H,K: X\to S$, $G:X\to K$. Assume that F and G are nonconstant, G satisfies one of the conditions (C_1)–(C_3) and the set $F(X)$ contains at least one invertible element. Then the quadruple (F,G,H,K) satisfies Eq. (6) if and only if one of the following two cases holds:*

(i) *there exist a nontrivial K-linear functional $L:X\to K$ and a nontrivial homomorphism ψ of the multiplicative semigroup of K into $(S,\circ)$, $s,t \in S$, $k\in K\setminus\{0\}$ and $l\in K$ such that*

$$\begin{aligned} F(x) &= s\circ t\circ\psi(L(x)+l) \quad \textit{for} \quad x\in X,\\ G(x) &= k(L(x)+l) \quad \textit{for} \quad x\in X,\\ H(x) &= t\circ\psi(L(x)+l) \quad \textit{for} \quad x\in X,\\ K(x) &= s\circ\psi(L(kx)+1) \quad \textit{for} \quad x\in X; \end{aligned}$$

(ii) *there exist a nontrivial $\mathbb{R}$-linear functional $L:X\to\mathbb{R}$, a homomorphism ψ of the multiplicative semigroup of nonnegative real numbers into $(S,\circ)$, $z\in Z(\psi([0,\infty)))$, $s,t\in S$, $k\in K\setminus\{0\}$ and $l\in\mathbb{R}$ such that*

$$\begin{aligned} F(x) &= \begin{cases} s\circ t\circ\psi(L(x)+l) & \textit{whenever} \quad L(x)+l\geq 0\\ s\circ t\circ z & \textit{otherwise}, \end{cases}\\ G(x) &= k\max\{L(x)+l,0\} \quad \textit{for} \quad x\in X, \end{aligned}$$

$$H(x) = \begin{cases} t \circ \psi(L(x) + l) & \textit{whenever} \quad L(x) + l \geq 0 \\ t \circ z & \textit{otherwise,} \end{cases}$$

$$K(x) = \begin{cases} s \circ \psi(L(kx) + l) & \textit{whenever} \quad L(kx) + l \geq 0 \\ s \circ z & \textit{otherwise.} \end{cases}$$

Proof. Assume that a quadruple (F, G, H, K) satisfies equation (6). Then, according to Proposition 6 there exist $x_0 \in X$, $s, t \in S$, $k \in K \setminus \{0\}$ and functions $f : X \to S$, $g : X \to K$ such that the pair (f, g) satisfies (5) and (38)–(41) hold. Moreover, as G satisfies one of the conditions (C_1)–(C_3), so does g; and as F and G are nonconstant, so are f and g. Hence one of the conditions (a) or (b) of Proposition 3 holds. Therefore, taking $l := 1 - L(x_0)$, we obtain (i) or (ii), respectively.

Since the converse is easy to check, the proof is completed. □

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Received: May 17, 2013
Revised: July 29, 2013