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**Aequationes Mathematicae**

# On a functional equation related to competition

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**Abstract.** The functional equation

$$f\left(\frac{x+y}{1-xy}\right)=\frac{f(x)+f(y)}{1+f(x)f(y)},\quad xy<1,$$

(introduced by the first author in a competition model) is considered. The main result says that a function $f:\mathbb{R}\to\mathbb{R}$ satisfies this equation if, and only if, $f=\tanh\circ\alpha\circ\tan^{-1}$, where $\alpha:\mathbb{R}\to\mathbb{R}$ is an additive function.

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## 1. Introduction

Motivated by a model of competition coming from cloud physics, the first-named author [1,2] introduced the following functional equation

$$f\left(\frac{x+y}{1-xy}\right)=\frac{f(x)+f(y)}{1+f(x)f(y)},\quad (x,y)\in\mathbb{R}^2,\quad xy\neq 1. \tag{1}$$

Applying a uniqueness result [3] for a related equation in a single variable, the form of solutions under some special regularity conditions was established (cf. Remark 5).

In Sect. 2 we present properties of solutions of this equation which in a natural way lead to the consideration of Eq. (1) with the domain restricted to the set $\left\{(x,y)\in\mathbb{R}^2: xy<1\right\}$. We prove, among other things, that if $f:\mathbb{R}\to\mathbb{R}$ satisfies this equation and $f(y_0)=1$ or $f(y_0)=-1$ for some $y_0\in\mathbb{R}$, then $f$ is a constant function (Proposition 1). Moreover $f(0)$ is either 0 or $-1$ or 1. If $f(0)=0$ then $f$ is an odd function. In Sect. 3 we prove that the function $f=\tanh\circ\alpha\circ\tan^{-1}$, where $\alpha:\mathbb{R}\to\mathbb{R}$ is an arbitrary additive function, is the general solution. As a corollary we obtain that, under some weak regularity

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conditions, every solution must be of the form $f(x) = \tanh\left(c\tan^{-1}(x)\right)$ $(x \in \mathbb{R})$ for some $c \in \mathbb{R}$.

## 2. Properties of solutions of the functional equation

Since the function $(x, y) \to \frac{x+y}{1-xy}$ occurring in Eq. (1) is not defined on the set $\left\{(x, y) \in \mathbb{R}^2 : xy = 1\right\}$, instead of Eq. (1), it is natural to consider the following two functional equations on a restricted domain:

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x)+f(y)}{1+f(x)f(y)}, \quad (x, y) \in \mathbb{R}^2,\ xy < 1, \tag{2}$$

and

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x)+f(y)}{1+f(x)f(y)}, \quad (x, y) \in \mathbb{R}^2,\ xy > 1. \tag{3}$$

*Remark* 1. Since $\{x \in \mathbb{R} : xy < 1 \text{ for some } y \in \mathbb{R}\} = \mathbb{R}$, it is reasonable to ask for solutions of the type $f : \mathbb{R} \to \mathbb{R}$ of Eq. (2), that are defined on the whole $\mathbb{R}$.

Note that this problem makes no sense in the case of Eq. (3), as no point $(x, y)$ with $x = 0$ satisfies the condition $xy > 1$. Moreover, the domain of Eq. (3), the set $D := \{(x, y) : xy > 1\}$ is the sum of two disjoint open connected sets $D_+ := \left\{(x, y) : x > 0 \wedge y > \frac{1}{x}\right\}$ and $D_- := \left\{(x, y) : x < 0 \wedge y < \frac{1}{x}\right\}$. Since

$$\frac{x+y}{1-xy} < 0 \quad \text{for all } (x, y) \in D_+ \quad \text{and} \quad \frac{x+y}{1-xy} > 0 \quad \text{for all } (x, y) \in D_-,$$

neither the problem to find a solution $f : (0, \infty) \to \mathbb{R}$ nor the problem to find a solution $f : (-\infty, 0) \to \mathbb{R}$ make sense in the case of Eq. (3).

(These facts show that in the case of Eq. (3) one could look for solutions $f : [(-\infty, 0) \cup (0, \infty)] \to \mathbb{R}$.)

Note also that if $f : \mathbb{R} \to \mathbb{R}$ satisfies (1) then, clearly, it satisfies (2), and its restriction to $\mathbb{R} \setminus \{0\}$ satisfies (3).

We begin with the following:

**Proposition 1.** *Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq. (1). Then*

(i) *the function $-f$ satisfies Eq. (1);*
(ii) *if $f(y_0) = 1$ for some $y_0 \in \mathbb{R}$ then $f(x) = 1$ for all $x \in \mathbb{R}$;*
(iii) *if $f(y_0) = -1$ for some $y_0 \in \mathbb{R}$ then $f(x) = -1$ for all $x \in \mathbb{R}$.*

*Proof.* The result (i) is obvious.

To prove (ii) assume that $f(y_0) = 1$ for some $y_0 \in \mathbb{R}$. If $y_0 = 0$, setting $y = y_0 = 0$ in (1), we get

$$f(x) = \frac{f(x)+1}{1+f(x)} = 1, \quad x \in \mathbb{R}.$$

If $y_0 > 0$ we get

$$f\left(\frac{x+y_0}{1-xy_0}\right) = \frac{f(x)+1}{1+f(x)} = 1, \quad x < \frac{1}{y_0}.$$

Since the range of the function

$$\left(-\infty, \frac{1}{y_0}\right) \ni x \longmapsto \frac{x+y_0}{1-xy_0}$$

is the interval $\left(-\frac{1}{y_0}, \infty\right)$, we get

$$f(x) = 1, \quad x \in \left(-\frac{1}{y_0}, \infty\right),$$

in particular $f(x) = 1$ for all $x \in (0, \infty)$. Taking $y_0$ arbitrarily close to 0 from the right, we obtain

$$f(x) = 1, \quad x \in (-\infty, \infty),$$

which was to be shown. If $y_0 < 0$, the argument is similar.

We omit an analogous proof of (iii). □

**Proposition 2.** *Neither Eq.* (2) *nor Eq.* (3) *has a solution that is continuous at a point* $x_0$ *and unbounded in a vicinity of* 0.

*Proof.* Assume that $f$ satisfies Eq. (2) or Eq. (3), is continuous at the point $x_0$, and there exists a sequence $(y_n)$ such that $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} |f(y_n)| = \infty$. Of course $f(x_0) \neq 0$. Then

$$\begin{aligned} f(x_0) &= \lim_{n\to\infty} f\left(\frac{x_0+y_n}{1-x_0y_n}\right) = \lim_{n\to\infty} \frac{f(x_0)+f(y_n)}{1+f(x_0)f(y_n)} \\ &= \lim_{n\to\infty} \frac{\frac{f(x_0)}{f(y_n)}+1}{\frac{1}{f(y_n)}+f(x_0)} = \frac{1}{f(x_0)}, \end{aligned}$$

which implies that $f(x_0) = 1$ or $f(x_0) = -1$. In view of Proposition 1 the function $f$ would be constant, contradicting the assumption. □

In the sequel we shall deal with the functional equation (2).

*Remark* 2. If $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq. (2), then either $f(0) = 0$ or $f(0) = 1$ or $f(0) = -1$.

Indeed, setting $x = y = 0$ in (2) we get

$$f(0)\left([f(0)]^2 - 1\right) = 0.$$

Hence, making use of Proposition 1, we obtain:

**Corollary 1.** *If* $f : \mathbb{R} \to \mathbb{R}$ *satisfies Eq.* (2) *then either* $f(0) = 0$ *or* $f$ *is a constant function of the value 1 or* $-1$.

Therefore in the sequel we are mainly interested in the solutions $f : \mathbb{R} \to \mathbb{R}$ of Eq. (2) such that $f(0) = 0$.

*Remark* 3. If $f : \mathbb{R} \to \mathbb{R}$ satisfies equation (2) and $f(0) = 0$ then $f$ is an odd function.

Indeed, for $y = -x$ we have $xy = -x^2 < 1$ and, setting $y = -x$ in Eq. (2), we obtain

$$0 = f(0) = \frac{f(x) + f(-x)}{1 + f(x) f(-x)},$$

whence $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

*Remark* 4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq. (2). If $f(y_0) = 0$ for some $y_0 \neq 0$, then $f(0) = 0$ and

$$f\left(\frac{x + y_0}{1 - xy_0}\right) = f(x), \quad x \in \mathbb{R}.$$

Indeed, in view of Corollary 1, we have $f(0) = 0$. The remaining part of this remark one gets immediately by setting $y = y_0$ in (2).

From Proposition 1 and Corollary 1 we obtain the following:

**Corollary 2.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of Eq. (2). Then the following conditions are pairwise equivalent:*

(i) *there exist $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) \neq 1$and $f(x_2) \neq -1$;*
(ii) *there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \in (-1, 1)$;*
(iii) $f(0) = 0$;
(iv) $|f(x)| < 1$ *for all $x \in \mathbb{R}$.*

*Remark* 5. Setting $y = x$ in Eq. (2) we obtain the following functional equation in a single variable

$$f\left(\frac{2x}{1 - x^2}\right) = \frac{2f(x)}{1 + [f(x)]^2}, \quad |x| < 1,$$

which is used in [1,2].

Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq. (1). Replacing $y$ in (1) by $\frac{y+z}{1-yz}$ we obtain

$$f\left(\frac{x + y + z - xyz}{1 - xy - xz - yz}\right) = \frac{f(x) + f(y) + f(z) + f(xyz)}{1 + f(xy) + f(xz) + f(yz)}, \quad xy + xz + yz \neq 1,$$

whence, setting $z = y = x$, we obtain the following functional equation in a single variable

$$f\left(\frac{3x - x^3}{1 - 3x^2}\right) = \frac{3f(x) + f\left(x^3\right)}{1 + 3f\left(x^2\right)}, \quad 3x^2 < 1.$$

By induction, this procedure and Eq. (1) lead to the following infinite system of functional equations of $n$ variables $x_1,\ldots,x_n$,

$$f\left(\frac{\sum_{j=1}^{\left[\frac{n+1}{2}\right]}(-1)^j\sum_{i_1<\ldots<i_{2j-1}}\prod_{k=1}^{2j-1}x_{i_k}}{1-\sum_{j=1}^{\left[\frac{n+1}{2}\right]}(-1)^j\sum_{i_1<\ldots<i_{2j}}\prod_{k=1}^{2j}x_{i_k}}\right)=\frac{\sum_{j=1}^{\left[\frac{n+1}{2}\right]}\sum_{i_1<\ldots<i_{2j-1}}\prod_{k=1}^{2j-1}f\left(x_{i_k}\right)}{1+\sum_{j=1}^{\left[\frac{n+1}{2}\right]}\sum_{i_1<\ldots<i_{2j}}\prod_{k=1}^{2j}f\left(x_{i_k}\right)},$$

where $n\in\mathbb{N}, n\geq 2$, and $x_1,\ldots,x_n\in\mathbb{R}$ are such that

$$\sum_{j=1}^{\left[\frac{n+1}{2}\right]}(-1)^j\sum_{i_1<\ldots<i_{2j}}\prod_{k=1}^{2j}x_{i_k}\neq 1$$

(here $\left[\frac{n+1}{2}\right]$ denotes the largest integer not greater than $\frac{n+1}{2}$). Setting here $x_1,\ldots,x_n=x$ we obtain for $f$ the system of iterative functional equations

$$f\left(\frac{\sum_{j=1}^{\left[\frac{n+1}{2}\right]}(-1)^j\binom{n}{2j-1}x^{2j-1}}{1-\sum_{j=1}^{\left[\frac{n+1}{2}\right]}(-1)^j\binom{n}{2j}x^{2j}}\right)=\frac{\sum_{j=1}^{\left[\frac{n+1}{2}\right]}(-1)^j\binom{n}{2j-1}f\left(x\right)^{2j-1}}{1-\sum_{j=1}^{\left[\frac{n+1}{2}\right]}(-1)^j\binom{n}{2j}f\left(x\right)^{2j}},$$

for all $n\in\mathbb{N}, n\geq 2$, and $x\in\mathbb{R}$ such that $\sum_{j=1}^{\left[\frac{n+1}{2}\right]}(-1)^j\binom{n}{2j}x^{2j}\neq 1$.

## 3. Main result

**Theorem 1.** *A function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation (2) if, and only if, there exists an additive function $\alpha:\mathbb{R}\to\mathbb{R}$ such that*

$$f=\tanh\circ\alpha\circ\tan^{-1}.$$

*Proof.* We have the identity

$$\frac{x+y}{1-xy}=\tan\left(\tan^{-1}x+\tan^{-1}y\right),\quad xy<1.$$

Since

$$\frac{\tanh x+\tanh y}{1+(\tanh x)(\tanh y)}=\tanh\left(x+y\right),\quad x,y\in\mathbb{R},$$

we also have the identity

$$\frac{u+v}{1+uv}=\tanh\left(\tanh^{-1}(u)+\tanh^{-1}(v)\right),\quad u,v\in(-1,1).$$

Hence, assuming that $f:\mathbb{R}\to\mathbb{R}$ satisfies Eq. (2), we obtain

$$f\left(\tan\left(\tan^{-1}x+\tan^{-1}y\right)\right)=\tanh\left(\tanh^{-1}\left(f\left(x\right)\right)+\tanh^{-1}\left(f\left(y\right)\right)\right),\quad xy<1.$$

Setting here $u=\tan^{-1}x, v=\tan^{-1}y$, we obtain

$$\tanh^{-1}\circ f\circ\tan\left(u+v\right)=\tanh^{-1}\circ f\circ\tan\left(u\right)+\tanh^{-1}\circ f\circ\tan\left(v\right)$$

for all $u,v\in\mathbb{R}$ such that $\left(\tan u\right)\left(\tan v\right)<1$.

It follows that the function $\alpha:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$ defined by $\alpha:=\tanh^{-1}\circ f\circ\tan$ satisfies the Cauchy functional equation

$$\alpha\left(u+v\right)=\alpha\left(u\right)+\alpha\left(v\right),\qquad u,v\in\mathbb{R},\quad\left(\tan u\right)\left(\tan v\right)<1.$$

As the set $\left\{\ (u,v)\in\mathbb{R}^2:\left(\tan u\right)\left(\tan v\right)<1\right\}$ is an open connected set such that $(0,0)$ is its interior point, the function $\alpha$ has a unique additive extension defined on $\mathbb{R}$. Without any loss of generality, we can denote it also by $\alpha$. Thus we have shown that if $f:\mathbb{R}\to\mathbb{R}$ satisfies Eq. (2), then there exists an additive function $\alpha:\mathbb{R}\to\mathbb{R}$ such that

$$f=\tanh\circ\alpha\circ\tan^{-1}.$$

To prove the converse implication assume that $f:\mathbb{R}\to\mathbb{R}$ is of this form. Then, making use of the additivity of $\alpha$ and the properties of the functions tanh and tan, we have for all $x,y\in\mathbb{R}$ such that $xy<1$,

$$\begin{aligned}
f\left(\frac{x+y}{1-xy}\right)&=\tanh\circ\alpha\circ\tan^{-1}\left(\frac{x+y}{1-xy}\right)\\
&=\tanh\left(\alpha\left[\tan^{-1}\left(\frac{\tan\left(\tan^{-1}x\right)+\tan\left(\tan^{-1}y\right)}{1-\tan\left(\tan^{-1}x\right)\cdot\tan\left(\tan^{-1}y\right)}\right)\right]\right)\\
&=\tanh\left(\alpha\left[\tan^{-1}\left(\tan\left(\tan^{-1}x+\tan^{-1}y\right)\right)\right]\right)\\
&=\tanh\left(\alpha\left[\tan^{-1}x+\tan^{-1}y\right]\right)=\tanh\left(\alpha\left(\tan^{-1}x\right)+\alpha\left(\tan^{-1}y\right)\right)\\
&=\frac{\tanh\left(\alpha\left(\tan^{-1}x\right)\right)+\tanh\left(\alpha\left(\tan^{-1}y\right)\right)}{1+\left(\tanh\left(\alpha\left(\tan^{-1}x\right)\right)\right)\left(\tanh\left(\alpha\left(\tan^{-1}y\right)\right)\right)}\\
&=\frac{\tanh\circ\alpha\circ\tan^{-1}\left(x\right)+\tanh\circ\alpha\circ\tan^{-1}\left(y\right)}{1+\left[\tanh\circ\alpha\circ\tan^{-1}\left(x\right)\right]\left[\tanh\circ\alpha\circ\tan^{-1}\left(y\right)\right]}=\frac{f\left(x\right)+f\left(y\right)}{1+f\left(x\right)f\left(y\right)}.
\end{aligned}$$

This completes the proof. □

*Remark* 6. The family of all solutions of Eq. (2) is extremely big in the following sense: for each point $(x_0,y_0)\in\mathbb{R}^2$ such that $x_0\neq0$ and $y_0\in(-1,1)$ there exists a continuum of different solutions of the form $f=\tanh\circ\alpha\circ\tan^{-1}$ with an additive function $\alpha$ such that $f\left(x_0\right)=y_0$.

**Corollary 3.** *Suppose that $f:\mathbb{R}\to\mathbb{R}$ satisfies one of the following conditions:*

(i) *$f$ is continuous at a point;*

(ii) *$f$ is measurable in the sense of Lebesgue;*

(iii) *f is bounded from above or bounded from below on a set of positive Lebesgue measure;*
(iv) *the graph of f is not dense in* $\mathbb{R}^2$*.*

*Then f satisfies the functional equation* (2),

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x)+f(y)}{1+f(x)\,f(y)}, \quad xy<1,$$

[*or Eq.* (1)] *if, and only if, there exists a constant* $c \in \mathbb{R}$ *such that*

$$f = \tanh\circ\left(c\,\tan^{-1}\right).$$

*Proof.* Since $\alpha := \tanh^{-1}\circ f\circ\tan$, it satisfies one of the conditions (i), (ii), (iii), (iv) and is an additive function. This implies (cf. M. Kuczma [4]) that there exists a constant $c \in \mathbb{R}$ such that $\alpha(u) = c\,u, \ \ u \in \mathbb{R}$. □

Hence, taking also into account Proposition 1, we obtain

*Remark* 7. The family of regular solutions of Eq. (2) [Eq. (1)] [i.e. satisfying one of conditions (i)–(iv)] form a one-parameter family of functions such that for each point $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0 \neq 0$ and $y_0 \in [-1, 1]$ there exists a unique solution $f : \mathbb{R} \to \mathbb{R}$ of Eq. (2) [Eq. (1)] such that $f(x_0) = y_0$. Moreover, if $y_0 \in (-1, 1)$ then

$$f = \tanh\circ\left(\frac{\tanh^{-1}(y_0)}{\tan^{-1}(x_0)}\tan^{-1}\right);$$

if $y_0 = -1$ then $f = -1$; if $y_0 = 1$ then $f = 1$.

In particular the sum of all graphs of this family of solutions is the set

$$((\mathbb{R}\setminus\{0\})\times[-1,1])\cup(\{0\}\times\{-1,0,1\}).$$

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