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Aequationes Mathematicae

Stability of a conditional Cauchy equation

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Abstract. Let $\mathbb{R}$ be the set of real numbers, $f : \mathbb{R} \to \mathbb{R}$, $\epsilon \geq 0$ and $d > 0$. We denote by $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$ a countable dense subset of $\mathbb{R}^2$ and let

$$U_d := \bigcup_{j=1}^{\infty} \{(x, y) \in \mathbb{R}^2 : |x| + |y| > d, \, |x - x_j| < 1, \, |y - y_j| < 2^{-j}\}.$$

We consider the Hyers-Ulam stability of the conditional Cauchy functional inequality

$$|f(x + y) - f(x) - f(y)| \leq \epsilon$$

for all $(x, y) \in U_d$.

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1. Introduction

Mikusinski [1, p. 75] introduced the conditional Cauchy functional equation

$$f(x + y) \neq 0 \Rightarrow f(x + y) = f(x) + f(y), \quad x, y \in \mathbb{R} \tag{1.1}$$

which arises when he gives a different proof of the fundamental theorem of affine geometry, characterizing bijective mappings $T : \mathbb{R}^2 \to \mathbb{R}^2$ which map straight lines to straight lines. As a result, he proves that if f satisfies (1.1), then f is an *additive function*, i.e., f satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{1.2}$$

for all $x, y \in \mathbb{R}$. Likewise, it is a frequent situation to get a functional equation with a restricted condition. We refer the reader to [2–7,9,14] for some interesting results on conditional functional equations. Let $f : \mathbb{R} \to \mathbb{R}$ and $U \subset \mathbb{R}^2$. Then we call f an U-*additive function* provided that f satisfies equation (1.2) for all $(x, y) \in U$. In this paper, we are interested in a set U such that every U-additive function f is an additive function. Recently, Skof [15] considered the

Hyers–Ulam stability problem [16] of a conditional Cauchy functional inequality. In particular, the result can be stated as follows: If $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditional Cauchy functional inequality

$$|f(x+y) - f(x) - f(y)| \le \epsilon \tag{1.3}$$

for all $x, y \in \mathbb{R}$ with $|x|+|y| \ge d$, then f satisfies inequality (1.3) for all $x, y \in \mathbb{R}$ with the quantity ϵ replaced by 9ϵ. Some related results can be found in [12,13]. Regarding the problem the question arises if there exists a set $U \subset \mathbb{R}^2$ of measure zero(or of finite Lebesgue outer measure as a weaker question) such that every U-additive function f is an additive function. For functions $f : \mathbb{R}^n \to \mathbb{R}^m$ with $n \ge 2$, as a direct consequence of the results in [2,9] the answer for the corresponding question is affirmative with $U = \{(x, y) \in \mathbb{R}^{2n} : \|x\| = \|y\|\}$ or with some other sets. In this paper, for a given δ we find a set $U_\delta \subset \mathbb{R}^2$ satisfying $m(U_\delta) \le \delta$ such that if f satisfies (1.3) for all $(x, y) \in U_\delta$, then f satisfies (1.3) for all $(x, y) \in \mathbb{R}$ with ϵ replaced by 3ϵ and that there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f(x) - A(x)| \le 3\epsilon$$

for all $x \in \mathbb{R}$. As a consequence we prove that if

$$\sup_{(x,y)\in U_d} |f(x+y) - f(x) - f(y)| \to 0$$

as $d \to \infty$, then f is an additive function. It is still open whether there exists a set $U \subset \mathbb{R}^2$ of measure zero such that every U-additive function $f : \mathbb{R} \to \mathbb{R}$ is an additive function, or if for $U \subset \mathbb{R}^2$, every U-additive function $f : \mathbb{R} \to \mathbb{R}$ is an additive function, then the Lebesgue outer measure $m^*(U)$ must be positive.

2. Main theorems

As a consequence of the Hyers–Ulam stability theorem [10,11] we have the following.

Theorem A. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$, $\epsilon \ge 0$, *and*

$$|f(x+y) - f(x) - f(y)| \le \epsilon \tag{2.1}$$

for all $x, y \in \mathbb{R}$. *Then there exists a unique additive function* $A : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$|f(x) - A(x)| \le \epsilon$$

for all $x \in \mathbb{R}$.

Let $K := \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$ be a countable dense subset of $\mathbb{R}^2$. For each $j = 1, 2, 3, \ldots,$ we denote by

$$R_j = \{(x, y) \in \mathbb{R}^2 : |x - x_j| < 1,\ |y - y_j| < 2^{-j}\}$$

the rectangle in $\mathbb{R}^2$ with center (x_j, y_j) and let $U = \bigcup_{j=1}^{\infty} R_j$. It is easy to see that the Lebesgue measure $m(U)$ of U satisfies $m(U) \le 1$. Now for $d > 0$, let

$$U_d = U \cap \{(x, y) \in \mathbb{R}^2 : |x| + |y| > d\}.$$

Then for a given $\delta > 0$ we can choose $d > 0$ such that $m(U_d) \le \delta$.

We first consider the stability of functional inequality (2.1) in the restricted domain U_d.

Theorem 2.1. *Let $d > 0$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$|f(x + y) - f(x) - f(y)| \le \epsilon \tag{2.2}$$

for all $(x, y) \in U_d$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x)| \le 3\epsilon \tag{2.3}$$

for all $x \in \mathbb{R}$.

Proof. For given $x, y \in \mathbb{R}$ we choose $p \in \mathbb{R}$ such that

$$|p| \ge d + |x| + |y| + 1. \tag{2.4}$$

We first choose $(x_{i_1}, y_{i_1}) \in K$ such that

$$| - p - x_{i_1}| + |p - y_{i_1}| < \frac{1}{4}, \tag{2.5}$$

and then we choose $(x_{i_2}, y_{i_2}) \in K$, $(x_{i_3}, y_{i_3}) \in K$ and $(x_{i_4}, y_{i_4}) \in K$ with $1 < i_1 < i_2 < i_3 < i_4$, step by step, satisfying

$$|x - y_{i_1} - x_{i_2}| + |y_{i_1} - y_{i_2}| < 2^{-i_1-1}, \tag{2.6}$$

$$|x - y_{i_2} - x_{i_3}| + |y + y_{i_2} - y_{i_3}| < 2^{-i_2-1}, \tag{2.7}$$

$$|y - y_{i_3} - x_{i_4}| + |y_{i_3} - y_{i_4}| < 2^{-i_3-1}. \tag{2.8}$$

Let

$$\begin{aligned} z_1 &= y_{i_1} - p, \\ z_2 &= y_{i_2} - y_{i_1}, \\ z_3 &= y_{i_3} - y_{i_2} - y, \\ z_4 &= y_{i_4} - y_{i_3}, \end{aligned}$$

and

$$z = z_1 + z_2 + z_3 + z_4.$$

Then from (2.5)–(2.8) we have

$$|z_1| < \frac{1}{4}, \quad |z_2| < 2^{-i_1-1}, \quad |z_3| < 2^{-i_2-1}, \quad |z_4| < 2^{-i_3-1}, \quad |z| < \frac{1}{2}. \tag{2.9}$$

Thus, from (2.4), (2.5) and (2.9) we have

$$\begin{aligned} |-p-z|+|p+z| &\geq 2(|p|-|z|) \geq 2\left(|p|-\frac{1}{2}\right) \\ &> 2d \geq d \end{aligned} \tag{2.10}$$

$$\begin{aligned} |-p-z-x_{i_1}| &\leq |-p-x_{i_1}|+|z| \\ &< \frac{1}{4}+\frac{1}{2} < 1, \end{aligned} \tag{2.11}$$

and

$$|p+z-y_{i_1}| = |z_2+z_3+z_4| < 2^{-i_1-1}+2^{-i_2-1}+2^{-i_3-1} < 2^{-i_1}. \tag{2.12}$$

Inequalities (2.10), (2.11) and (2.12) imply

$$(-p-z, p+z) \in U_d. \tag{2.13}$$

Also from the inequalities

$$|x-p-z|+|p+z| \geq 2\big(|p|-|x|-|z|\big) > 2\left(|p|-|x|-\frac{1}{2}\right) > d,$$

$$\begin{aligned} |x-p-z-x_{i_2}| &\leq |x-y_{i_1}-x_{i_2}|+|z_2|+|z_3|+|z_4| \\ &< \frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} < 1, \end{aligned}$$

and

$$|p+z-y_{i_2}| = |z_3+z_4| < 2^{-i_2-1}+2^{-i_3-1} < 2^{-i_2},$$

we have

$$(x-p-z, p+z) \in U_d. \tag{2.14}$$

Similarly, using the inequalities

$$\begin{aligned} &|x-p-z-x_{i_3}| \leq |x-y_{i_2}-x_{i_3}|+|z_3|+|z_4| < 1, \\ &|y+p+z-y_{i_3}| = |z_4| < 2^{-i_3}, \\ &|-p-z-x_{i_4}| \leq |y-y_{i_3}-x_{i_4}|+|z_4| < 1, \\ &|y+p+z-y_{i_4}| = 0, \end{aligned}$$

we have

$$(x-p-z, y+p+z),\ (-p-z, y+p+z) \in U_d. \tag{2.15}$$

Now it follows from (2.14) and (2.15) that

$$\begin{aligned} |f(x+y)-f(x)-f(y)| &\leq |-f(x)+f(x-p-z)+f(p+z)| \\ &\quad +|f(x+y)-f(x-p-z)-f(y+p+z)| \\ &\quad +|-f(y)+f(-p-z)+f(y+p+z)| \\ &\leq 3\epsilon. \end{aligned}$$

Using Theorem A we get the result. □

Now we obtain an asymptotic behavior of the Cauchy difference

$$C(f,x,y) := f(x+y) - f(x) - f(y) \tag{2.16}$$

on the set U_d as $d \to \infty$.

Theorem 2.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition*

$$\sup_{(x,y)\in U_d} |C(f,x,y)| \to 0 \tag{2.17}$$

as $d \to \infty$. Then f is an additive function.

Proof. By condition (2.17), for each $j \in \mathbb{N}$, there exists $d_j > 0$ such that

$$|f(x+y) - f(x) - f(y)| \le \frac{1}{j}$$

for all $(x,y) \in U_{d_j}$. By Theorem 2.1, there exists a unique additive function $A_j : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A_j(x)| \le \frac{3}{j} \tag{2.18}$$

for all $x \in \mathbb{R}$. From (2.18), using the triangle inequality we have

$$|A_j(x) - A_k(x)| \le \frac{3}{j} + \frac{3}{k} \le 6 \tag{2.19}$$

for all $x \in \mathbb{R}$ and all positive integers j,k. Now, inequality (2.19) implies $A_j = A_k$. Indeed, for all $x \in \mathbb{R}$ and all rational numbers $r > 0$ we have

$$|A_j(x) - A_k(x)| = \frac{1}{r}|A_j(rx) - A_k(rx)| \le \frac{6}{r}. \tag{2.20}$$

Letting $r \to \infty$ in (2.20) we have $A_j = A_k$. Thus, letting $j \to \infty$ in (2.18) we get the result. □

Let $\{(x_1,y_1),(x_2,y_2),(x_3,y_3),\ldots\}$ be defined as above. For each $j = 1,2,3,\ldots$, let

$$S_j = \{(x,y) : x,y \in \mathbb{R} : \ |x+y-x_j-y_j| < 1, \ |x-y-x_j+y_j| < 2^{-j}\}$$

and let $V = \bigcup_{j=1}^{\infty} S_j$. Then V satisfies $m(V) \le 1$. For a fixed $d > 0$, let

$$V_d = V \cap \{(x,y) \in \mathbb{R}^2 : |x| + |y| > d\}.$$

Using a similar method as in the proof of Theorem 2.1 we can show that V_d satisfies the following conditions(see [8]): For any points $P_1, P_2, \ldots, P_m \in \mathbb{R}^2$, there exist $u,v,w \in \mathbb{R}$ such that

$$\{P_1 + (-u,u),\ P_2 + (-u,u),\ \ldots, P_m + (-u,u)\} \subset V_d, \tag{2.21}$$

$$\{P_1 + (0,v),\ P_2 + (0,v),\ \ldots, P_m + (0,v)\} \subset V_d, \tag{2.22}$$

$$\{P_1 + (w,0),\ P_2 + (w,0),\ \ldots, P_m + (w,0)\} \subset V_d. \tag{2.23}$$

As a consequence of Theorem 2.6 of [8] we obtain the following.

Theorem 2.3. *Suppose that $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy*

$$|f(x+y) - g(x) - h(y)| \le \epsilon \tag{2.24}$$

for all $(x,y) \in V_d$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x) - f(0)| \le 4\epsilon,$$
$$|g(x) - A(x) - g(0)| \le 4\epsilon,$$
$$|h(x) - A(x) - h(0)| \le 4\epsilon,$$

for all $x \in \mathbb{R}$.

As direct consequences of the above result we have the followings.

Corollary 2.4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right| \le \epsilon \tag{2.25}$$

for all $(x,y) \in V_d$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x) - f(0)| \le 2\epsilon \tag{2.26}$$

for all $x \in \mathbb{R}$.

Corollary 2.5. *Suppose that $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy the condition*

$$\sup_{(x,y)\in V_d} |f(x+y) - g(x) - h(y)| \to 0 \tag{2.27}$$

as $d \to \infty$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = A(x) + f(0),$$
$$g(x) = A(x) + g(0),$$
$$h(x) = A(x) + h(0)$$

for all $x \in \mathbb{R}$.

Remark. Let X be a normed linear space with a countable dense subset D, Y be a Banach space and $f : X \to Y$. For each $j = 1, 2, 3, \ldots$ we denote by

$$S_j = \{(x,y) \in X^2 : \|x + y - x_j - y_j\| < 1,\ \|x - y - x_j + y_j\| < 2^{-j}\}$$

the rectangle with center (x_j, y_j) and let $V = \bigcup_{j=1}^{\infty} S_j$, where $D \times D := \{(x_1,y_1), (x_2,y_2), (x_3,y_3), \ldots\}$. For $d > 0$, let

$$V_d = V \cap \{(x,y) \in X^2 : \|x\| + \|y\| > d\}.$$

Then, replacing $|\cdot|$ by $\|\cdot\|$, all the above results are valid for the function $f : X \to Y$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $(x, y) \in V_d$, and for the functions $f, g, h : X \rightarrow Y$ satisfying

$$\|f(x+y) - g(x) - h(y)\| \leq \epsilon$$

for all $(x, y) \in V_d$, respectively.

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