Aequationes Math. 77 (2009) 107–118
0001-9054/09/010107-12
DOI 10.1007/s00010-008-2930-1

Aequationes Mathematicae

On the general solution of a generalization of the Gołąb–Schinzel equation

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Summary. Let X be a linear space over a commutative field K. Under some additional assumptions we determine a description of the general solution of the equation

$$f(x + M(f(x))y) = f(x) \circ f(y),$$

where $f : X \to K$, $M : K \to K$ and $\circ : K^2 \to K$ are unknown functions.

Mathematics Subject Classification (2000). 39B52.

Keywords. Gołąb–Schinzel functional equation.

The paper concerns a description of the general solution of the equation

$$f(x + M(f(x))y) = f(x) \circ f(y), \tag{1}$$

where $f : X \to K$, $M : K \to K$ and $\circ : K^2 \to K$. We also give some applications of this description. Equation (1) has been studied first by J. Brzdęk (cf. [9]–[10]). It is a generalization of the well known Gołąb–Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y), \tag{2}$$

which has been considered by many authors in various classes of functions. For the details concerning (2), its generalizations and applications, we refer e.g. to [1]–[5], [11]–[15] and [17]–[21].

J. Brzdęk [6, 8] has studied the following generalization of the Gołąb–Schinzel equation

$$f(x + f(x)^n y) = f(x)f(y). \tag{3}$$

In his survey paper (see [11]), J. Brzdęk posed a question which result obtained for (3) can be carried over to the case of (1). We give a partial solution to this problem, namely it is easily seen that there are similarities between the description of the general solution of equation (3) and the description presented in this paper.

We consider equation (1) assuming that:

(A1) $f : X \to K$, $M : K \to K$ and $\circ : K^2 \to K$;
(A2) $M^{-1}(\{0\}) = \{0\}$;
(A3) $\circ : K^2 \to K$ is commutative.

The inverse of an element x of K is denoted by $\frac{1}{x}$.
We start with some lemmas.

Lemma 1. *Assume that conditions* (A1)–(A3) *are valid,* $f, M, \circ$ *satisfy equation* (1) *and put* $e = f(0)$. *Then we have*

(i) $f(x) = f(x) \circ e$ *for* $x \in X$;

(ii) $f(z) = f(x) \circ f\big(\frac{z-x}{M(f(x))}\big)$ *for* $z, x \in X$, $f(x) \neq 0$;

(iii) $e = f(x) \circ f\big(\frac{-x}{M(f(x))}\big)$ *for* $x \in X$, $f(x) \neq 0$;

(iv) $f(x) = f(M(e)x)$ *for* $x \in X$;

(v) *if* $e = 0$, *then* $f \equiv 0$.

Proof. (i) Setting $y = 0$ in equation (1), we have

$$f(x) = f(x + M(f(x))0) = f(x) \circ f(0) = f(x) \circ e \quad \text{for} \quad x \in X.$$

(ii) It suffices to put $y = \frac{z-x}{M(f(x))}$ in (1).

(iii) It is enough to apply (ii) with $z = 0$.

(iv) Fix $x \in X$. By (1), (A3) and (i) we get

$$f(x) = f(x) \circ e = e \circ f(x) = f(0) \circ f(x) = f(0 + M(f(0))x) = f(M(e)x).$$

(v) Assume that $e = 0$. Then, on account of (A2), we have by (iv),

$$f(x) = f(0) = 0 \quad \text{for } x \in X. \qquad \square$$

Lemma 2. *Assume that conditions* (A1)–(A3) *hold,* $f, M, \circ$ *satisfy equation* (1) *and* $f \not\equiv 0$. *Let* $e := f(0)$, $T := f^{-1}(\{e\})$ *and* $W := f(X) \setminus \{0\}$. *Then we have*

(i) $T \setminus \{0\}$ *is the set of periods of* f;

(ii) T *is an additive subgroup of* X;

(iii) $M(a)T \subset T$ *for* $a \in W$;

(iv) $y - x \in T$ *for every* $x, y \in X$ *with* $f(x) = f(y) \neq 0$.

Proof. (i) If w is a period of f, then $e = f(0) = f(0 + w) = f(w)$, so $w \in T$.
Now take $w \in T$. On account of Lemma 1 (i), (iv) and (A3), we get

$$\begin{aligned} f(w + M(e)x) &= f(w + M(f(w))x) = f(w) \circ f(x) \\ &= e \circ f(x) = f(x) = f(M(e)x) \quad \text{for} \quad x \in X. \end{aligned} \tag{4}$$

Moreover, Lemma 1 (v) and (A2) imply $M(e) \neq 0$, thus for every $y \in X$ there exists $x \in X$ such that $y = M(e)x$. Consequently, in view of (4), $f(y+w) = f(y)$ for $y \in X$.

(ii) This follows from (i) and the definition of T.

(iii) Fix $a \in W$. There exists $x_0 \in X$ with $a = f(x_0) \neq 0$. For every $z \in T$, by Lemma 1 (i), we obtain

$$f(x_0) = f(x_0) \circ e = f(x_0) \circ f(z) = f(x_0 + M(f(x_0))z).$$

This implies

$$\begin{aligned} f(M(f(x_0))z) &= f\left(x_0 + M(f(x_0))z + M(f(x_0 + M(f(x_0))z))\left(\frac{-x_0}{M(f(x_0))}\right)\right) \\ &= f(x_0 + M(f(x_0))z) \circ f\left(\frac{-x_0}{M(f(x_0))}\right) \\ &= f(x_0) \circ f\left(\frac{-x_0}{M(f(x_0))}\right) \quad \text{for } \ z \in T. \end{aligned}$$

Now it follows from Lemma 1 (iii) that $f(M(f(x_0))z) = e$ for $z \in T$.

(iv) Take $x, y \in X$ with $f(x) = f(y) \neq 0$. Let us note that by (A2), $M(f(x)) = M(f(y)) \neq 0$. Next, in view of (1) and Lemma 1 (iii), we have

$$\begin{aligned} f(y-x) &= f\left(y + M(f(x))\frac{-x}{M(f(x))}\right) = f\left(y + M(f(y))\frac{-x}{M(f(x))}\right) \\ &= f(y) \circ f\left(\frac{-x}{M(f(x))}\right) = f(x) \circ f\left(\frac{-x}{M(f(x))}\right) = e. \end{aligned}$$

This means that $y - x \in T$. □

Lemma 3. *Let conditions* (A1)–(A3) *be fulfilled,* $f, M, \circ$ *satisfy* (1)*,* $f \not\equiv 0$*, and the operation* $\circ$ *be associative. Put* $e := f(0)$*,* $T := f^{-1}(\{e\})$ *and* $W := f(X) \setminus \{0\}$*. Then we have*

$$M(a)T = T \quad \textit{for} \ \ a \in W.$$

Proof. Fix $a \in W$. There exists $x_0 \in X$ with $a = f(x_0) \neq 0$, by (A2) also $M(f(x_0)) \neq 0$. In view of Lemma 2 (iii), it suffices to show that $T \subset M(a)T$. Let $z \in T$. From Lemma 2 (i) we derive

$$f(x_0) = f(x_0 + z) = f\left(x_0 + M(f(x_0))\frac{z}{M(f(x_0))}\right) = f(x_0) \circ f\left(\frac{z}{M(f(x_0))}\right).$$

Thus, in view of Lemma 1 (iii), we get

$$e = f(x_0) \circ f\left(\frac{-x_0}{M((f(x_0)))}\right) = \left(f(x_0) \circ f\left(\frac{z}{M(f(x_0))}\right)\right) \circ f\left(\frac{-x_0}{M(f(x_0))}\right).$$

Consequently, from Lemma 1 (i), (iii) we derive

$$e = f\left(\frac{z}{M(f(x_0))}\right) \circ \left(f(x_0) \circ f\left(\frac{-x_0}{M(f(x_0))}\right)\right)$$

$$= f\left(\frac{z}{M(f(x_0))}\right) \circ e = f\left(\frac{z}{M(f(x_0))}\right).$$

This means that $\frac{z}{M(f(x_0))} \in T$, and so $z = M(f(x_0))\frac{z}{M(f(x_0))} \in M(f(x_0))T$. □

Lemma 4. *Assume that conditions* (A1)–(A3) *are valid,* $f, M, \circ$ *satisfy* (1) *and the operation* $\circ$ *is associative. Then*

$$f(x) \circ f(y) = 0 \Leftrightarrow f(x)f(y) = 0 \quad \textit{for} \ \ x, y \in X. \tag{5}$$

Proof. Let $x, y \in X$ and $f(x)f(y) = 0$. Since $\circ$ is commutative, it is enough to consider the case where $f(x) = 0$. Then $M(f(x)) = 0$, whence $f(x) \circ f(y) = f(x + M(f(x))y) = f(x) = 0$.

We have proved that

$$f(x)f(y) = 0 \Rightarrow f(x) \circ f(y) = 0 \quad \text{for} \ \ x, y \in X. \tag{6}$$

Now take $x, y \in X$ with $f(x) \circ f(y) = 0$ and $f(x) \neq 0$. Then, by (1), $f(x + M(f(x))y) = 0$, whence in view of (6), we get

$$(f(x) \circ f(y)) \circ f\left(\frac{-x}{M(f(x))}\right) = f(x + M(f(x))y) \circ f\left(\frac{-x}{M(f(x))}\right) = 0.$$

Thus, on account of Lemma 1 (iii), we obtain

$$0 = (f(x) \circ f(y)) \circ f\left(\frac{-x}{M(f(x))}\right) = f(y) \circ \left(f(x) \circ f\left(\frac{-x}{M(f(x))}\right)\right) = f(y) \circ e,$$

where $e := f(0)$. Hence, from Lemma 1 (i) we derive $f(y) = 0$. This completes the proof. □

In the sequel, we need the following definition.

Definition 1. *A commutative groupoid* $(A, \star)$ *with neutral element is called a loop if for every* $x, y \in A$ *there exists* $c \in A$ *such that* $x = c \star y$.

The following four theorems are the main results of the paper.

Theorem 1. *Assume that conditions* (A1)–(A3) *and* (5) *hold,* $f, M, \circ$ *satisfy equation* (1), $f \not\equiv 0$ *and put* $W := f(X) \setminus \{0\}$. *Then* $(W, \circ|_{W^2})$ *is a loop with neutral element* $f(0) \neq 0$. *Moreover, there exist an additive subgroup* T *of* X *and a function* $w : W \to X$ *such that*

$$M(a)T \subset T \quad \textit{for} \ \ a \in W; \tag{7}$$

$$w(a \circ b) - M(a)w(b) - w(a) \in T \quad \textit{for} \ \ a, b \in W; \tag{8}$$

$$w^{-1}(T) = \{f(0)\}; \tag{9}$$

$$f(x) = \begin{cases} a, \ \textit{if}\ x \in w(a) + T \ \textit{and}\ a \in W; \\ 0 \qquad\qquad \textit{otherwise} \end{cases} \quad \textit{for} \ \ x \in X. \tag{10}$$

Proof. From (1), (5) and Lemma 1 (i), (ii), (v), we see that $(W, \circ|_{W^2})$ is a loop with neutral element $f(0) \neq 0$. On account of Lemma 2 (ii), the set $T := f^{-1}(\{f(0)\})$ is an additive subgroup of X. By Lemma 2 (iii) we get (7).

The axiom of choice implies that there exists a function $w : W \to X$ such that $w(a) \in f^{-1}(\{a\})$ for every $a \in W$. It is easily seen that condition (9) is valid. In virtue of Lemma 2 (i), (iv), we have $f^{-1}(\{a\}) = w(a) + T$ for $a \in W$, so we get (10).

It remains to prove that (8) holds. So, let us fix $a, b \in W$. On account of (1) and (5) we obtain $f(w(a) + M(a)w(b)) = a \circ b \neq 0$. On the other hand we have $f(w(a \circ b)) = a \circ b$. Now from Lemma 2 (iv) we infer that $w(a \circ b) - M(a)w(b) - w(a) \in T$. This completes the proof. □

If we replace the inclusion in condition (7) by equality, we can get a result converse to the above one.

Theorem 2. *Let conditions* (A1)–(A3) *and* (5) *be fulfilled and suppose that there exists a set $W \subset K \setminus \{0\}$ such that $(W, \circ|_{W^2})$ is a loop with neutral element e. Moreover, assume that there exist an additive subgroup T of X, and a function $w : W \to X$ such that conditions* (8), (10) *hold and*

$$M(a)T = T \quad \text{for} \quad a \in W; \tag{11}$$

$$w^{-1}(T) = \{e\}. \tag{12}$$

Then $f, M, \circ$ satisfy equation (1).

Proof. First we show that f is well defined. Take $a, b \in W$ and suppose that there exist $x, y \in T$ such that

$$w(a) + x = w(b) + y. \tag{13}$$

Since $(W, \circ|_{W^2})$ is a loop, there exists $c \in W$ with $a = b \circ c$. On account of (8) and (13) we get

$$\begin{aligned} y - x - M(b)w(c) &= w(a) - M(b)w(c) - w(b) \\ &= w(b \circ c) - M(b)w(c) - w(b) \in T. \end{aligned}$$

Thus $M(b)w(c) \in T$, because T is an additive group. By (11), we see that $w(c) \in T$, so (12) yields $c = e$. This means that $a = b$.

It remains to prove that $f, M, \circ$ satisfy equation (1). Hence fix $x, y \in X$. If $f(x) = 0$, then from (A2) and (5) we infer that $f(x) \circ f(y) = 0 = f(x) = f(x + M(f(x))y)$.

Now assume that $f(x) \neq 0$ and consider two cases:

1. $f(y) \neq 0$;
2. $f(y) = 0$.

Case 1. According to (10) there exist $a, b \in W$ with $x \in w(a) + T$ and $y \in w(b) + T$. From (8) and by the fact that T is an additive group, we obtain

$$w(a) + M(a)w(b) \in w(a \circ b) + T.$$

Consequently, in virtue of (10) and (11), we have

$$x + M(f(x))y = x + M(a)y \in w(a) + T + M(a)w(b) + M(a)T \in w(a \circ b) + T.$$

Thus

$$f(x + M(f(x))y) = a \circ b = f(x) \circ f(y).$$

Case 2. From (5) it follows that $f(x) \circ f(y) = 0$. To obtain a contradiction, suppose that $f(x+M(f(x))y) \neq 0$. Then, on account of (10), there exist $a, b \in W$ such that $x \in w(a) + T$ and $x + M(f(x))y \in w(b) + T$. Since $(W, \circ|_{W^2})$ is a loop, so $b = a \circ c$ for some $c \in W$. Therefore from (8) and (10) we derive

$$M(a)y = M(f(x))y \in w(b) - x + T = w(a \circ c) - w(a) + T = M(a)w(c) + T.$$

This and (11) yield $y \in w(c) + T$. Consequently $f(y) = c \neq 0$, which brings the contradiction. Finally $f(x + M(f(x))y) = 0 = f(x) \circ f(y)$. □

In the following two theorems we assume associativity of the operation $\circ$, instead of condition (5).

Theorem 3. *Assume that conditions* (A1)–(A3) *hold, the operation* $\circ$ *is associative,* $0 \notin f(X)$ *and* $M(f(X)) \setminus \{1\} \neq \emptyset$. *Then* $f, M, \circ$ *satisfy* (1) *iff there exists* $e \in K \setminus \{0\}$ *such that* $f \equiv e$ *and* $e \circ e = e$.

Proof. Assume that $f, M, \circ$ satisfy (1). In view of Theorem 1 it is easily seen that $f(X)$ is a subgroup, with neutral element $e := f(0)$, of the semigroup $(K, \circ)$. The inverse of an element $f(x)$ in this subgroup will be denoted by $f(x)^{-1}$.

First, we show that $M(f(x)) = M(e)$ for every $x \in X$. Suppose that there exists $x_0 \in X$ with $M(f(x_0)) \neq M(e)$. Since $0 \notin f(X)$, we get $M(e) \neq 0$. Define the function $M_1 : K \to K$ by

$$M_1(z) = \frac{M(z)}{M(e)} \quad \text{for} \quad z \in K.$$

It is clear that $M_1^{-1}(\{0\}) = \{0\}$ (i.e. M_1 satisfies (A2)). Further, on account of Lemma 1 (iv), for every $x, y \in X$, we obtain

$$f\big(x + M_1(f(x))y\big) = f\left(x + \frac{M(f(x))}{M(e)}y\right) = f(x) \circ f\left(\frac{y}{M(e)}\right) = f(x) \circ f(y).$$

This means that $f, M_1, \circ$ satisfy equation (1). Next, observe that $M_1(f(x_0)) \neq 1$. Let $y_0 := \frac{x_0}{1 - M_1(f(x_0))}$. Then

$$f(x_0) \circ f(y_0) = f(x_0 + M_1(f(x_0))y_0) = f(y_0),$$

and

$$f(x_0) = (f(x_0) \circ f(y_0)) \circ (f(y_0))^{-1} = f(y_0) \circ (f(y_0))^{-1} = e.$$

Hence $M(f(x_0)) = M(e)$, which brings a contradiction with $M(f(x_0)) \neq M(e)$.

Now write $d := M(e)$. Since $M(f(X)) \setminus \{1\} \neq \emptyset$, we have $d \neq 1$. Let $x \in X$, $y := \frac{x}{1-d}$. Since $M(f(x)) = d$, equation (1) implies

$$f(x) \circ f(y) = f(x + dy) = f\left(x + d\frac{x}{1-d}\right) = f(y).$$

Thus

$$f(x) = (f(x) \circ f(y)) \circ (f(y))^{-1} = f(y) \circ (f(y))^{-1} = e.$$

Consequently $f(x) = e$ for $x \in X$. Moreover, in view of (1), we see that $e \circ e = e$.

Since the converse is obvious, this completes the proof. □

Theorem 4. *Assume that conditions* (A1)–(A3) *hold,* $M(f(X)) \setminus \{1\} \neq \emptyset$*, the operation* $\circ$ *is associative and* $f \not\equiv const$*. Then* $f, M, \circ$ *satisfy equation* (1) *iff there exist an additive subgroup* T *of* X*, a subgroup* $W \subset K \setminus \{0\}$*, with neutral element* e*, of the semigroup* $(K, \circ)$ *and a function* $w : W \to X$ *such that*

$$M(a)T = T \quad \textit{for} \ \ a \in W; \tag{14}$$

$$w(a \circ b) - M(a)w(b) - w(a) \in T \quad \textit{for} \ \ a, b \in W; \tag{15}$$

$$w^{-1}(T) = \{e\}; \tag{16}$$

$$f(x) = \begin{cases} a, \ \textit{if } x \in w(a) + T \ \textit{and } a \in W; \\ 0 \qquad\qquad \textit{otherwise} \end{cases} \quad \textit{for} \ \ x \in X; \tag{17}$$

$$u \circ 0 = 0 \quad \textit{for} \ \ u \in W \cup \{0\}. \tag{18}$$

Proof. First assume that $f, M, \circ$ satisfy (1) and put $W := f(X) \setminus \{0\}$. On account of Lemma 4 condition (5) is valid and according to Theorem 3, we have $0 \in f(X)$. Theorem 1 and associativity of the operation $\circ$ imply that W is a subgroup, with neutral element $e := f(0)$, of the semigroup $(K, \circ)$. Moreover, there exist an additive subgroup T of X and a function $w : W \to X$ such that conditions (7)–(10) hold. Consequently, we obtain conditions (15)–(17). Clearly, equivalence (5) yields equation (18). Next, from (16) and (17) we derive $T = f^{-1}(\{e\})$, whence, in virtue of Lemma 3, we get (14).

For the converse, note that $(W, \circ|_{W^2})$ is a loop with neutral element e. Moreover (17) implies that $W = f(X) \setminus \{0\}$, hence by (18) we obtain (5). Finally, in view of Theorem 2, $f, M, \circ$ satisfy equation (1). □

In the following example we consider equation (1) for the operation $\circ : \mathbb{R}^2 \to \mathbb{R}$ defined by $u \circ v := \min\{u, v\}$.

Example 1. Let $f : X \to \mathbb{R}$, $M : \mathbb{R} \to \mathbb{R}$, $\circ : \mathbb{R}^2 \to \mathbb{R}$ be defined by $u \circ v := \min\{u, v\}$ and condition (A2) be valid.

Assume that $f, M, \circ$ satisfy equation (1) and $f \not\equiv 0$. Let $W := f(X) \setminus \{0\}$. Clearly the operation $\circ$ is commutative and associative, whence, on account of

Lemma 4, condition (5) holds. Next, in view of Theorem 1, the set W is a subgroup, with neutral element $f(0)$, of the semigroup $(\mathbb{R}, \circ)$. Moreover, there exist an additive subgroup T of X and a function $w : W \to X$ such that conditions (7)–(10) are valid. Setting $e := f(0)$ we get

$$a = a \circ e = \min\{a, e\} \quad \text{for} \quad a \in W,$$

and

$$e = a \circ a^{-1} = \min\{a, a^{-1}\} \quad \text{for} \quad a \in W.$$

This implies that $a \le e$ and $e \le a$ for $a \in W$, so $W = \{e\}$. By (9) and (10) we have $T = f^{-1}(\{e\})$, and according to Lemma 3, $M(e)T = T$. If $0 \in f(X)$, then from (5) we infer that $0 = e \circ 0 = \min\{e, 0\}$, thus as $e \in W \subset \mathbb{R} \setminus \{0\}$, we get $e > 0$.

Finally, if $f, M, \circ$ satisfy (1), then one of the following two assertions holds.

a) $f \equiv e$ for some $e \in \mathbb{R}$.

b) There exists $e > 0$ and an additive subgroup T of X such that $M(e)T = T$ and

$$f(x) = \begin{cases} e, & \text{if } x \in T; \\ 0, & \text{if } x \in X \setminus T \end{cases} \qquad \text{for} \quad x \in X.$$

Conversely, it is easily seen that if one of the conditions a)–b) is valid, then $f, M, \circ$ satisfy equation (1).

Similarly as in Example 1, we could consider equation (1) for $\circ : \mathbb{R}^2 \to \mathbb{R}$ defined by one of the formulas $u \circ v := \max\{u, v\}$, $u \circ v := \min\{|u|, |v|\}$, $u \circ v := \max\{|u|, |v|\}$.

If conditions (A1)–(A3) hold and the function $X \setminus f^{-1}(\{0\}) \ni x \to f(x) \in K$ is injective, then, under some additional assumptions, we get a useful description of the general solution of equation (1). It has been applied in [16].

First we introduce the following definition.

Definition 2. We say that a function $f : X \to K$ is trivial iff $f(X \setminus \{0\}) = \{0\}$.

Theorem 5. *Assume that conditions* (A1)–(A3) *and* (5) *hold, f is not trivial, $M(f(X)) \setminus \{0, 1\} \neq \emptyset$ and the function $X \setminus f^{-1}(\{0\}) \ni x \to f(x) \in K$ is injective. Then $f, M, \circ$ satisfy equation* (1) *iff there exist a multiplicative subgroup $D \neq \{1\}$ of $K \setminus \{0\}$, an injective function $h : D \cup \{0\} \to K$, and $x_0 \in X \setminus \{0\}$ such that*

$$h(0) = 0; \tag{19}$$

$$M(y) = h^{-1}(y) \quad \text{for} \quad y \in h(D \cup \{0\}); \tag{20}$$

$$f(x) = \begin{cases} h(d), & \text{if } x = (d-1)x_0 \text{ and } d \in D; \\ 0 & \text{otherwise} \end{cases} \quad \text{for} \quad x \in X; \tag{21}$$

$$a \circ b = h(h^{-1}(a)h^{-1}(b)) \quad \text{for} \quad a, b \in h(D \cup \{0\}). \tag{22}$$

Proof. First assume that $f, M, \circ$ satisfy (1) and put $W := f(X) \setminus \{0\}$. Since $M(f(X)) \setminus \{0,1\} \neq \emptyset$, we see that $f \not\equiv 0$. In view of Theorem 1, $(W, \circ|_{W^2})$ is a loop with neutral element $f(0) \neq 0$. Moreover, there exist an additive subgroup T of X and a function $w : W \to X$ such that conditions (7)–(10) are valid. The function $X \setminus f^{-1}(\{0\}) \ni x \to f(x)$ is injective, so, taking into account (10), we get $T = \{0\}$ and

$$f(x) = \begin{cases} a, \text{ if } x = w(a) \text{ and } a \in W; \\ 0 \qquad\qquad \text{otherwise} \end{cases} \qquad \text{for } \ x \in X. \tag{23}$$

From (8) we infer that

$$w(a \circ b) - M(a)w(b) - w(a) = 0 \quad \text{for } \ a, b \in W \tag{24}$$

and

$$w(b \circ a) - M(b)w(a) - w(b) = 0 \quad \text{for } \ a, b \in W. \tag{25}$$

Subtracting (25) from (24) we have

$$M(b)w(a) + w(b) - M(a)w(b) - w(a) = 0 \quad \text{for } \ a, b \in W,$$

so

$$w(a)(M(b) - 1) = w(b)(M(a) - 1) \quad \text{for } \ a, b \in W. \tag{26}$$

Suppose that $M(a) = 1$ and $a \neq f(0)$ for some $a \in W$. Then, by (23), $w(a) \neq 0$, which in view of (26), implies $M(b) = 1$ for every $b \in W$. This contradicts our assumption that $M(f(X)) \setminus \{0,1\} \neq \emptyset$. Consequently

$$M(a) \neq 1 \quad \text{for } \ a \in W, a \neq f(0).$$

Thus, according to (26), we obtain

$$\frac{w(a)}{M(a) - 1} = \frac{w(b)}{M(b) - 1} =: x_0 \quad \text{for } \ a, b \in W \setminus \{f(0)\}.$$

Such an element x_0 is well defined, because f is not trivial and so there exists $b_0 \in W \setminus \{f(0)\}$. Moreover, by (23), we get $w(b_0) \neq w(f(0)) = 0$. Thus we see that $w \not\equiv 0$ and $x_0 \neq 0$.

We have proved that there exists $x_0 \in X \setminus \{0\}$ with

$$w(a) = (M(a) - 1)x_0 \quad \text{for } \ a \in W \setminus \{f(0)\}. \tag{27}$$

Now, setting $a = f(0)$ and $b = b_0$ in (26), we have $M(f(0)) = 1$. Consequently $w(f(0)) = 0 = (M(f(0)) - 1)x_0$. This, (27) and (23) imply that

$$f(x) = \begin{cases} a, \text{ if } x = (M(a) - 1)x_0 \ \text{ and } \ a \in W; \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases} \qquad \text{for } \ x \in X. \tag{28}$$

From this and (A2) we derive that the function $M_0 := M|_{f(X)}$ is injective.

Now we show that

$$a \circ b = M_0^{-1}(M_0(a)M_0(b)) \quad \text{for } \ a, b \in f(X). \tag{29}$$

Take $a, b \in f(X)$. By equation (1) we get $a \circ b \in f(X)$. If $ab = 0$, then from (5), we obtain $M_0(a \circ b) = 0 = M_0(a)M_0(b)$, and further $a \circ b = M_0^{-1}(M_0(a)M_0(b))$. It remains to consider the case where $ab \neq 0$. Then there exist $x, y \in X$ with $f(x) = a$ and $f(y) = b$. By (28) we infer that

$$\begin{aligned} x + M_0(f(x))y &= x + M_0(a)y \\ &= (M_0(a) - 1)x_0 + M_0(a)(M_0(b) - 1)x_0 = (M_0(a)M_0(b) - 1)x_0. \end{aligned}$$

On the other hand

$$f(x + M_0(f(x))y) = f(x) \circ f(y) = a \circ b \neq 0.$$

Consequently, taking into account (28), we get

$$(M_0(a)M_0(b) - 1)x_0 = (M_0(a \circ b) - 1)x_0.$$

Thus $M_0(a)M_0(b) = M_0(a \circ b)$, so $a \circ b = M_0^{-1}(M_0(a)M_0(b))$.

Next we prove that $D := M_0(W)$ is a multiplicative subgroup of $K \setminus \{0\}$. By the definition of W and (A2) we obtain $D \subset K \setminus \{0\}$. From (29) and by the fact that $(W, \circ|_{W^2})$ is a loop one can easily see that D is a multiplicative subgroup of $K \setminus \{0\}$. Clearly $D \neq \{1\}$.

By (29) the operation $\circ$ is associative on the set $f(X)$. We define the operation $* : K^2 \to K$ by

$$a * b = \begin{cases} a \circ b, & \text{if } \ a, b \in f(X); \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } \ a, b \in K.$$

It is easily seen that the operation $*$ is commutative and associative. Moreover $f, M, *$ satisfy equation (1).

Suppose that $0 \notin f(X)$. Then $0 \notin M(f(X))$, whence by the assumption $M(f(X)) \setminus \{0, 1\} \neq \emptyset$, we get $M(f(X)) \setminus \{1\} \neq \emptyset$. Consequently, in view of Theorem 3, $f \equiv c$ for some $c \in K \setminus \{0\}$. This contradicts the assumption that the function $X \setminus f^{-1}(\{0\}) \ni x \to f(x) \in K$ is injective. We have shown that $0 \in f(X)$.

Now, from the definition of the set D and the condition (A2), we obtain $M_0(f(X)) = D \cup \{0\}$. Let $h : D \cup \{0\} \to K$ be given by

$$h(t) = M_0^{-1}(t) \quad \text{for } \ t \in D \cup \{0\}.$$

Then $f(X) = h(D \cup \{0\})$ and $W = h(D)$, so on account of (28) and (29) we get (21) and (22). Conditions (19) and (20) we derive from the definition of the function h.

For the converse, let us set $W := h(D)$, $T := \{0\}$ and define the function $w : W \to X$ by

$$w(a) = (M(a) - 1)x_0 \quad \text{for } \ a \in W.$$

In view of (22), one can check that $(W, \circ|_{W^2})$ is a loop with a neutral element $h(1)$. Further, by (19), we have $W \subset K \setminus \{0\}$. Clearly, conditions (10) and (11) are valid.

We show that (12) holds. Let $w(a) \in T = \{0\}$ for some $a \in W = h(D)$. Then $(M(a) - 1)x_0 = 0$, so $M(a) = 1$. Hence, according to (20), $a = h(1)$. Moreover

$$w(h(1)) = (M(h(1)) - 1)x_0 = (1 - 1)x_0 = 0 \in T,$$

which ends the proof of (12). One can easily check that condition (8) is valid. Finally, in virtue of Theorem 2, $f, M, \circ$ satisfy equation (1). □

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Manuscript received: May 21, 2007 and, in final form, September 10, 2007.