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Positive radial solutions for p-Laplacian systems

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Summary. The paper deals with the existence of positive radial solutions for the p-Laplacian system $\operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i) + f^i(u_1,\dots,u_n) = 0$, $|x| < 1$, $u_i(x) = 0$, on $|x| = 1$, $i = 1,\dots,n$, $p > 1$, $x \in \mathbb{R}^N$. Here f^i, $i = 1,\dots,n$, are continuous and nonnegative functions. Let $\mathbf{u} = (u_1,\dots,u_n)$, $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$, $f_0^i = \lim_{\|\mathbf{u}\|\to 0} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$, $f_\infty^i = \lim_{\|\mathbf{u}\|\to\infty} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$, $i = 1,\dots,n$, $\mathbf{f} = (f^1,\dots,f^n)$, $\mathbf{f}_0 = \sum_{i=1}^n f_0^i$ and $\mathbf{f}_\infty = \sum_{i=1}^n f_\infty^i$. We prove that $\mathbf{f}_0 = \infty$ and $\mathbf{f}_\infty = 0$ (sublinear), guarantee the existence of positive radial solutions for the problem. Our methods employ fixed point theorems in a cone.

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1. Introduction

In this paper we consider the existence of positive radial solutions for the p-Laplacian system

$$\begin{cases} \operatorname{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) + f^1(u_1,\dots,u_n) = 0 \ \text{ in } \ B \\ \cdots \\ \operatorname{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right) + f^n(u_1,\dots,u_n) = 0 \ \text{ in } \ B \\ u_i = 0 \ \text{ on } \ \partial B, \ \ i = 1,\dots,n \end{cases} \tag{1.1}$$

where $p > 1$, $B = \left\{x \in \mathbb{R}^N : |x| < 1,\ N \geq 2\right\}$.

When $p = 2$, (1.1) becomes

$$\begin{cases} \Delta u_1 + f^1(u_1,\dots,u_n) = 0 \ \text{ in } \ B \\ \cdots \\ \Delta u_n + f^n(u_1,\dots,u_n) = 0 \ \text{ in } \ B \\ u_i = 0 \ \text{ on } \ \partial B, \ \ i = 1,\dots,n. \end{cases} \tag{1.2}$$

When $n = 1$ and $p = 2$, (1.1) becomes

$$\begin{cases} \Delta u + f(u) = 0 \ \text{ in } \ B \\ u = 0 \ \text{ on } \ \partial B. \end{cases} \tag{1.3}$$

Notice that (1.3) has received extensive investigation in the past several decades. Lions in [5] discussed, under various combinations of superlinearity or sublinearity of f at infinity, $f(0) = 0$ and $f(0) > 0$, the existence and nonexistence of positive solutions of (1.3) in a general bounded regular domain in $\mathbb{R}^N$. The results of [5] are also interpreted in terms of bifurcation diagrams. If $0 < \beta < 1$, it is understood that $f(u) = (1 + \alpha u)^\beta$ (or u^β) is sublinear. Note that $f_0 = \lim_{u\to 0^+} \frac{f(u)}{u} = \infty$ and $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u} = 0$. It is clear that the notation in (1.5), $\mathbf{f}_0$ and $\mathbf{f}_\infty$, is a convenient extension of f_0 and f_∞ defined above for the scalar cases. We shall use $\mathbf{f}_0$ and $\mathbf{f}_\infty$ to characterize sublinearity and superlinearty of (1.1). Thus, we say (1.1) is sublinear if $\mathbf{f}_0 = \infty$ and $\mathbf{f}_\infty = 0$. In contrast, (1.1) is superlinear if $\mathbf{f}_0 = 0$ and $\mathbf{f}_\infty = \infty$.

Wang in [8] showed that the following parameterized problem,

$$\begin{cases} \operatorname{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) + \lambda f^1(u_1,\dots,u_n) = 0 \ \text{ in } \ B \\ \qquad\qquad\qquad\qquad\cdots \\ \operatorname{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right) + \lambda f^n(u_1,\dots,u_n) = 0 \ \text{ in } \ B \\ \qquad u_i = 0 \ \text{ on } \ \partial B, \ \ i = 1,\dots,n \end{cases} \tag{1.4}$$

has a positive solution when $\lambda > 0$ is sufficiently small under some assumptions. For the ODE case ($N = 1$), Wang in [7] proved that the existence, multiplicity and nonexistence of positive solutions of (1.1) can be determined by appropriate combinations of superlinearity and sublinearity of $\mathbf{f}(u)$ at zero and infinity.

In this paper we shall show that if (1.1) is sublinear, or $\mathbf{f}_0 = \infty$ and $\mathbf{f}_\infty = 0$, then (1.1) has a positive solution.

We note that the ODE system (2.7) has a singularity at zero. It seems that the fixed point theorem of compression/expansion type does not work in the case. However we are able to use fixed point index to carry out our proof.

We now turn to general assumptions for this paper. Let $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_+^n = \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n}$. Also, for $\mathbf{u} = (u_1,\dots,u_n) \in \mathbb{R}_+^n$, let $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$. We make the assumption:

(H1) $f^i : \mathbb{R}_+^n \to \mathbb{R}_+$ is continuous, $i = 1,\dots,n$.

In order to state our results we introduce the notation

$$\mathbf{f}(\mathbf{u}) = (f^1(\mathbf{u}),\dots,f^n(\mathbf{u})) = (f^1(u_1,\dots,u_n),\dots,f^n(u_1,\dots,u_n)),$$

$$f_0^i = \lim_{\|\mathbf{u}\|\to 0} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}, \qquad f_\infty^i = \lim_{\|\mathbf{u}\|\to\infty} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}, \ \ i = 1,\dots,n,$$

where $\mathbf{u} = (u_1,\dots,u_n) \in \mathbb{R}_+^n$,

$$\mathbf{f}_0 = \sum_{i=1}^n f_0^i, \qquad \mathbf{f}_\infty = \sum_{i=1}^n f_\infty^i. \tag{1.5}$$

Our main result is

Theorem 1.1. *Assume* (H1) *holds. If* $\mathbf{f}_0 = \infty$ *and* $\mathbf{f}_\infty = 0$*, then* (1.1) *has a positive radial solution.*

Example. Consider the example

$$\begin{cases} \operatorname{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) + (u_1 + \cdots + u_n)^{p_1} = 0 \text{ in } B \\ \cdots \\ \operatorname{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right) + (u_1 + \cdots + u_n)^{p_n} = 0 \text{ in } B \\ u_i = 0 \text{ on } \partial B, \;\; i = 1, \dots, n \end{cases} \tag{1.6}$$

where $p > 1$, $0 < p_1, p_2, \dots, p_n < p-1$, $B = \left\{x \in \mathbb{R}^N : |x| < 1,\ N \geq 2\right\}$. It is easy to see that $\mathbf{f}_0 = \infty$ and $\mathbf{f}_\infty = 0$, then the example has a positive radial solution according to Theorem 1.1.

2. Preliminaries

Let $\varphi(t) = |t|^{p-2}t$, then, for $t > 0$, $\varphi(t) = t^{p-1}$ and $\varphi^{-1}(t) = t^{\frac{1}{p-1}}$. It is easy to see that $\varphi^{-1}(\sigma\varphi(t)) = \varphi^{-1}(\sigma)t$ for $t > 0$ and $\sigma > 0$.

A radial solution of (1.1) can be considered as a solution of the system

$$\begin{cases} \left(r^{N-1}\varphi(u_1'(r))\right)' + r^{N-1}f^1(\mathbf{u}) = 0, \;\; 0 < r < 1 \\ \cdots \\ \left(r^{N-1}\varphi(u_n'(r))\right)' + r^{N-1}f^n(\mathbf{u}) = 0, \;\; 0 < r < 1 \\ \mathbf{u}'(0) = \mathbf{u}(1) = 0, \;\; i = 1, \dots, n. \end{cases} \tag{2.7}$$

We shall treat classical solutions of (2.7), namely a vector-valued function $\mathbf{u} = (u_1(r), \dots, u_n(r))$ with $u_i \in C^1[0,1]$, and $\varphi(u_i') \in C^1(0,1), i = 1, \dots, n$, which satisfies (2.7). A solution $\mathbf{u}(r) = (u_1(r), \dots, u_n(r))$ is positive if $u_i(r) \geq 0$, $i = 1, \dots, n$, for all $r \in (0,1)$ and there is at least one nontrivial component of $\mathbf{u}$. In fact, it is easy to prove that such a nontrivial component of $\mathbf{u}$ is positive on $(0,1)$.

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.1 (([2, 4])). *Let E be a Banach space and K a cone in E. For $R > 0$, define $K_R = \left\{u \in K : \|x\| < R\right\}$. Assume that $T : \bar{K}_R \to K$ is completely continuous and $\partial K_R = \{u \in K : \|x\| = R\}$.*

(i) *If there exists a $x_0 \in K \setminus \{0\}$ such that*

$$x - Tx \neq tx_0, \text{ for all } x \in \partial K_R \text{ and } t \geq 0,$$

then

$$i(T, K_R, K) = 0.$$

(ii) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_R$ and $Tx \neq x$ for $x \in \partial K_R$, then*

$$i(T, K_R, K) = 1.$$

In order to apply Lemma 2.1 to (2.7), let X be the Banach space

$$\underbrace{C[0,1]\times\cdots\times C[0,1]}_{n}$$

and, for $\mathbf{u}=(u_1,\dots,u_n)\in X$,

$$\|\mathbf{u}\|=\sum_{i=1}^{n}\sup_{t\in[0,1]}|u_i(t)|.$$

For $\mathbf{u}\in X$ or $\mathbb{R}_+^n$, $\|\mathbf{u}\|$ denotes the norm of $\mathbf{u}$ in X or $\mathbb{R}_+^n$, respectively. Of course, a constant function is an element of $C[0,1]$.

Let K be a cone in X defined by

$$K=\big\{(u_1,\dots,u_n)\in X: u_i(t)\ge 0,\ t\in[0,1],\ i=1,\dots,n\big\}.$$

Also, for R a positive number, define Ω_R by

$$\Omega_R=\big\{\mathbf{u}\in K:\|\mathbf{u}\|<R\big\}.$$

Note that $\partial\Omega_R=\big\{\mathbf{u}\in K:\|\mathbf{u}\|=R\big\}$.

Let $\mathbf{T}:K\to X$ be a map with components $(T^1,\dots,T^n)$. We define T^i, $i=1,\dots,n$, by

$$T^i\mathbf{u}(r)=\int_r^1\varphi^{-1}\bigg(\frac{1}{s^{N-1}}\int_0^s\tau^{N-1}f^i(\mathbf{u}(\tau))d\tau\bigg)ds,\ r\in[0,1]. \tag{2.8}$$

It is straightforward to verify that (2.7) is equivalent to the fixed point equation

$$\mathbf{Tu}=\mathbf{u}\quad\text{in}\quad K.$$

Lemma 2.2. *Assume* (H1) *holds. Then* $\mathbf{T}(K)\subset K$ *and* $\mathbf{T}:K\to K$ *is compact and continuous.*

Proof. It is clear that $\mathbf{T}(K)\subset K$. We now show that $\mathbf{T}$ is compact. Let $(\mathbf{u}_m)_{m\in\mathbb{N}}$ be a bounded sequence in K and let $R>0$ be such that $\|\mathbf{u}_m\|\le R$ for all $m\in\mathbb{N}$. Hence, by the definition of $\mathbf{T}$, we have, $i=1,\dots,n$,

$$(T^i\mathbf{u}_m)'(r)=\begin{cases}-\varphi^{-1}\Big(\frac{1}{r^{N-1}}\int_0^r\tau^{N-1}f^i(\mathbf{u}_m(\tau))d\tau\Big), & 0<r<1,\\ 0, & r=0.\end{cases}$$

Then it is easy to see that both $(\mathbf{Tu}_m)_{m\in\mathbb{N}}$ and $((\mathbf{Tu}_m)')_{m\in\mathbb{N}}$ are uniformly bounded sequences (so $(\mathbf{Tu}_m)_{m\in\mathbb{N}}$ is equicontinuous on [0,1]). It follows from the Arzela–Ascoli theorem that there exists a $\mathbf{v}\in K$ and a subsequence of $\mathbf{Tu}_m$ converging to $\mathbf{v}$ in X.

It remains to show the continuity of $\mathbf{T}$. Let us take a sequence $(\mathbf{u}_m)_{m\in\mathbb{N}}$ in K converging to $\mathbf{u}\in K$ in X and fix i, $i=1,\dots,n$. Note that φ^{-1} and $f^i(\mathbf{u})$ are continuous. It is not hard to see that the Dominated Convergence Theorem guarantees that

$$\lim_{m\to\infty}T^i\mathbf{u}_m(r)=T^i\mathbf{u}(r) \tag{2.9}$$

for each $r \in [0,1]$. Moreover, the compactness of T^i implies that $T^i\mathbf{u}_m(r)$ converges uniformly to $T^i\mathbf{u}(r)$ on $[0,1]$. Suppose this is false. Then there exists $\varepsilon_0 > 0$ and a subsequence $(\mathbf{u}_{m_j})_{j\in\mathbb{N}}$ of $(\mathbf{u}_m)_{m\in\mathbb{N}}$ such that

$$\sup_{r\in[0,1]} |T^i\mathbf{u}_{m_j}(r) - T^i\mathbf{u}(r)| \geq \varepsilon_0,\ j \in \mathbb{N}. \tag{2.10}$$

Now, it follows from the compactness of T^i that there exists a subsequence of $(\mathbf{u}_{m_j})_{j\in\mathbb{N}}$ (without loss of generality assume the subsequence is $(\mathbf{u}_{m_j})_{j\in\mathbb{N}}$) such that $(T^i\mathbf{u}_{m_j})_{j\in\mathbb{N}}$ converges uniformly to $y_0 \in C[0,1]$. Thus, from (2.10), we easily see that

$$\sup_{r\in[0,1]} |y_0(r) - T^i\mathbf{u}(r)| \geq \varepsilon_0. \tag{2.11}$$

On the other hand, from the pointwise convergence (2.9) we obtain

$$y_0(r) = T^i\mathbf{u}(r),\ r \in [0,1].$$

This is a contradiction to (2.11). Therefore $\mathbf{T}$ is continuous. □

For each $i = 1,\ldots,n$, define a new function $\hat{f}^i(t) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\hat{f}^i(t) = \max\big\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}^n_+ \text{ and } \|\mathbf{u}\| \leq \mathrm{t}\big\}.$$

Note that $\hat{f}^i_0 = \lim_{t\to 0^+} \frac{\hat{f}^i(t)}{\varphi(t)}$ and $\hat{f}^i_\infty = \lim_{t\to\infty} \frac{\hat{f}^i(t)}{\varphi(t)}$.

Lemma 2.3 ([7])**.** *Assume* (H1) *hold. Then* $\hat{f}^i_0 = f^i_0$ *and* $\hat{f}^i_\infty = f^i_\infty$, $i = 1,\ldots,n$.

Proof. It is easy to see that $\hat{f}^i_0 = f^i_0$. For the second part, we consider the two cases, (a) $f^i(\mathbf{u})$ is bounded and (b) $f^i(\mathbf{u})$ is unbounded. For case (a), it follows, from $\lim_{t\to\infty}\varphi(t) = \infty$, that $\hat{f}^i_\infty = 0 = f^i_\infty$. For case (b), for any $\delta > 0$, let $M = \hat{f}^i(\delta)$ and

$$N_\delta = \inf\big\{\|\mathbf{u}\| : \mathbf{u} \in \mathbb{R}^n_+,\ \|\mathbf{u}\| \geq \delta, f^i(\mathbf{u}) \geq M\big\} \geq \delta,$$

then

$$\max\big\{f^i(\mathbf{u}) : \|\mathbf{u}\| \leq N_\delta,\ \mathbf{u} \in \mathbb{R}^n_+\big\} = M = \max\big\{f^i(\mathbf{u}) : \|\mathbf{u}\| = N_\delta,\ \mathbf{u} \in \mathbb{R}^n_+\big\}.$$

Thus, for any $\delta > 0$, there exists a $N_\delta \geq \delta$ such that

$$\hat{f}^i(t) = \max\big\{f^i(\mathbf{u}) : N_\delta \leq \|\mathbf{u}\| \leq t,\ \mathbf{u} \in \mathbb{R}^n_+\big\} \text{ for } t > N_\delta.$$

Now, suppose that $f^i_\infty < \infty$. In other words, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$f^i_\infty - \varepsilon < \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)} < f^i_\infty + \varepsilon,\ \text{for } \mathbf{u} \in \mathbb{R}^n_+,\ \|\mathbf{u}\| > \delta. \tag{2.12}$$

Thus, for $t > N_\delta$, there exist $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^n_+$ such that $\|\mathbf{u}_1\| = t$, $t \geq \|\mathbf{u}_2\| \geq N_\delta$ and $f^i(\mathbf{u}_2) = \hat{f}^i(t)$. Therefore,

$$\frac{f^i(\mathbf{u}_1)}{\varphi(\|\mathbf{u}_1\|)} \leq \frac{\hat{f}^i(t)}{\varphi(t)} = \frac{f^i(\mathbf{u}_2)}{\varphi(t)} \leq \frac{f^i(\mathbf{u}_2)}{\varphi(\|\mathbf{u}_2\|)}. \tag{2.13}$$

Now (2.12) and (2.13) yield that

$$f_\infty^i - \varepsilon < \frac{\hat{f}^i(t)}{\varphi(t)} < f_\infty^i + \varepsilon \text{ for } t > N_\delta. \tag{2.14}$$

Hence $\hat{f}_\infty^i = f_\infty^i$. Similarly, we can show $\hat{f}_\infty^i = f_\infty^i$ if $f_\infty^i = \infty$.

Lemma 2.4. *Assume* (H1) *hold and let* $r > 0$. *If there exits an* $\varepsilon > 0$ *such that*

$$\hat{f}^i(r) \le \varphi(\varepsilon)\varphi(r), \quad i = 1, \ldots, n,$$

then

$$\|\mathbf{Tu}\| \le n\varepsilon\|\mathbf{u}\| \text{ for } \mathbf{u} \in \partial\Omega_r.$$

Proof. From the definition of T, for $\mathbf{u} \in \partial\Omega_r$, we have

$$\begin{aligned}
\|\mathbf{Tu}\| &= \sum_{i=1}^{n} \sup_{t\in[0,1]} |T^i\mathbf{u}(t)| \\
&= \sum_{i=1}^{n} \int_0^1 \varphi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \tau^{N-1} f^i(\mathbf{u}(\tau))d\tau\right] ds \\
&\le \sum_{i=1}^{n} \int_0^1 \varphi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \tau^{N-1} d\tau \hat{f}^i(r)\right] ds \\
&\le n\varphi^{-1}[\varphi(\varepsilon)\varphi(r)] \\
&= n\varphi^{-1}[\varphi(\varepsilon r)] \\
&= n\varepsilon\|\mathbf{u}\|.
\end{aligned}$$

□

3. Proof of Theorem 1.1

Proof. Since $\mathbf{f}_0 = \infty$, there exists a component f^i such that $f_0^i = \infty$. Therefore, there is an $r_1 > 0$ such that

$$f^i(\mathbf{u}) \ge \varphi(\eta)\varphi(\|\mathbf{u}\|) \tag{3.15}$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \le r_1$, where $\eta > 0$ is chosen so that

$$\frac{\eta}{2}\varphi^{-1}\left(\frac{1}{N4^N}\right) \ge 1. \tag{3.16}$$

If $\mathbf{u} - \mathbf{Tu} = 0$ for some $\mathbf{u} \in \partial U_{r_1}$, we already have the desired solution of (1.1). Therefore we assume that

$$\mathbf{u} - \mathbf{Tu} \ne 0 \text{ for all } \mathbf{u} \in \partial U_{r_1}, \tag{3.17}$$

We now claim that

$$\mathbf{u} - \mathbf{Tu} \ne t\mathbf{v}, \text{ for all } \mathbf{u} \in \partial\Omega_{r_1} \text{ and } t \ge 0, \tag{3.18}$$

where $\mathbf{v} = (\theta(r), \ldots, \theta(r))$, and $\theta \in C[0,1]$ such that $0 \le \theta(r) \le 1$ on $[0,1]$, $\theta(r) \equiv 1$ on $[0, \frac{1}{4}]$ and $\theta(r) \equiv 0$ on $[\frac{1}{2}, 1]$. Thus, $\mathbf{v} \in K \setminus \{0\}$. If there exists $\mathbf{u}^* = (u_1^*, \ldots, u_n^*) \in \partial\Omega_{r_1}$ and $t_0 \ge 0$ such that $\mathbf{u}^* - \mathbf{T}\mathbf{u}^* = t_0\mathbf{v}$, we shall show this leads to a contradiction. Since (3.17) is true, we have $t_0 > 0$. Since $\mathbf{T}(K) \subset K$, we obtain that $u_i^*(r) \ge t_0\theta(r)$ for all $r \in [0,1]$. Let

$$t^* = \sup\bigl\{t : u_i^*(r) \ge t\theta(r) \ \text{ for all } r \in [0,1]\bigr\}.$$

It follows that $t_0 \le t^* < \infty$ and $u_i^*(r) \ge t^*\theta(r)$ for all $r \in [0,1]$. Now, for $r \in [0, \frac{1}{2}]$, we have

$$\begin{aligned} u_i^*(r) &= \mathbf{T}^i\mathbf{u}^*(r) + t_0\theta(r) \\ &= \int_r^1 \varphi^{-1}\left(\frac{1}{s^{N-1}}\int_0^s \tau^{N-1} f^i(\mathbf{u}^*(\tau))d\tau\right) ds + t_0\theta(r). \end{aligned}$$

Note that $\sum_{j=1}^n u_j^*(r) \le r_1$ for $r \in [0,1]$. Also (3.15) implies that, for $r \in [0, \frac{1}{2}]$,

$$\begin{aligned} u_i^*(r) &\ge \int_{\frac{1}{2}}^1 \varphi^{-1}\left(\frac{1}{s^{N-1}}\int_0^s \tau^{N-1}\varphi(\eta)\varphi\left(\sum_{j=1}^n u_j^*(\tau)\right)d\tau\right)ds + t_0\theta(r) \\ &\ge \int_{\frac{1}{2}}^1 \varphi^{-1}(\int_0^s \tau^{N-1}\varphi(\eta)\varphi(u_i^*(\tau))d\tau)ds + t_0\theta(r) \\ &\ge \frac{1}{2}\varphi^{-1}\left(\int_0^{\frac{1}{4}} \tau^{N-1}\varphi(\eta)\varphi(t^*\theta(\tau))d\tau\right) + t_0\theta(r) \\ &= \frac{1}{2}\varphi^{-1}\left(\int_0^{\frac{1}{4}} \tau^{N-1}d\tau\varphi(\eta)\varphi(t^*)\right) + t_0\theta(r) \\ &= \frac{1}{2}\varphi^{-1}\left(\frac{1}{N4^N}\varphi(\eta t^*)\right) + t_0\theta(r). \end{aligned}$$

Now, in view of the fact that $\varphi^{-1}(\sigma\varphi(t)) = \varphi^{-1}(\sigma)t$, we have, for $r \in [0, \frac{1}{2}]$,

$$\begin{aligned} u_i^*(r) &\ge t^*\frac{\eta}{2}\varphi^{-1}\left(\frac{1}{N4^N}\right) + t_0\theta(r). \\ &\ge t^* + t_0\theta(r) \\ &\ge (t^* + t_0)\theta(r), \end{aligned}$$

and hence since $\theta(r) = 0$ on $\left[\frac{1}{2}, 1\right]$ we have

$$u_i^*(r) \ge (t^* + t_0)\theta(r), \ r \in [0,1],$$

which is a contradiction to the definition of t^*. Thus, in view of Lemma 2.1,

$$i(\mathbf{T}, \Omega_{r_1}, K) = 0.$$

We now determine Ω_{r_2}. Notice that $\mathbf{f}_\infty = 0$ implies that $f_\infty^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.3 that $\hat{f}_\infty^i = 0$, $i = 1, \ldots, n$. Therefore there is an $r_2 > 2r_1$ such that

$$\hat{f}^i(r_2) \le \varphi(\varepsilon)\varphi(r_2), \quad i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$n\varepsilon < 1.$$

Thus, we have by Lemma 2.4 that

$$\|\mathbf{Tu}\| \leq n\varepsilon\|\mathbf{u}\| < \|\mathbf{u}\| \quad for \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i(\mathbf{T}, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1.1).

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