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A method of solving functional equations on convex subsets of linear spaces

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Summary. We introduce a method of solving a wide range of functional equations stemming from Mean Value Theorems. We generalize the results of Z. Daróczy and Gy. Maksa (Corollary 2; [2]) in the spirit of M. Sablik's lemma (Lemma 2.3; [11]). Next, we illustrate the method taking into account two functional equations. The first one is connected with Flett's Mean Value Theorem ([6]) and the second one is derived from Simpson's rule ([10]).

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1. Basic lemmas

Let us recall the definition of a locally polynomial function. Throughout the paper X and Y will denote linear spaces over a field $\mathbb{K} \subset \mathbb{R}$.

Definition 1.1. Let K be a non-empty subset of X. A map $f : K \to Y$ is called a locally polynomial function of degree at most n on K, if

$$\Delta_{y_{n+1},\dots,y_1} f(x) = 0$$

holds for every x, $y_i \in X$ such that $x + \sum_{i \in Z} y_i \in K$, $Z \subset \{1, \dots, n+1\}$.

In the sequel we use the notations

$$I := \{(\alpha, \beta) \in \mathbb{K} \times \mathbb{K} : |\alpha| + |\beta| \le 1\},$$
$$I^0 := \{(\alpha, \beta) \in I : \beta \ne 0\}.$$

We will work on absolutely convex (convex and balanced) sets with non-empty algebraic interior ([1]). Let us remind the definition of an algebraic interior.

Definition 1.2. Let $A \subset X$. The algebraic interior of the set A is the set

$$\operatorname{alg\,int} A = \Big\{ y \in A : \bigwedge_{x \in X} \bigvee_{\varepsilon > 0} \bigwedge_{\alpha \in (-\varepsilon, \varepsilon) \cap \mathbb{K}} \alpha x + y \in A \Big\}.$$

Now, let us present a lemma which follows easily from the well known properties of absolutely convex sets (cf. [5], [13]).

Lemma 1.3. *Let $\emptyset \neq K \subset X$ be absolutely convex and suppose that $J \subset I$ is finite. Further, let $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \in I^0$ for a fixed $n \in \mathbb{N}$. If*

$$r \geq \max \left\{ n+1, \ \max \left\{ |\alpha| + |\beta| + \sum_{i=1}^{n} \left| \frac{\alpha \beta_i - \alpha_i \beta}{\beta_i} \right| : (\alpha, \beta) \in J \right\} \right\}, \qquad (1)$$

then for every x, y, $u_1, \ldots, u_n \in \frac{1}{r} K$, every $(\alpha, \beta) \in J$, and $S \subset \{1, \ldots, n\}$ we have

(a) $x + \sum_{i \in S} u_i \in K$,

(b) $\alpha x + \beta y + \sum_{i \in S} \frac{\alpha \beta_i - \alpha_i \beta}{\beta_i} u_i \in K$.

The next lemma is a "convex" version of Lemma 2.1 in [11]. We state it without a proof.

Lemma 1.4. *Fix $N \in \mathbb{N}$. Suppose that K is a non-empty convex subset of X such that $0 \in \operatorname{alg\,int} K$, and G a group uniquely divisible by $N!$. Let $B_i \colon K \to G$, $i \in \{0, \ldots, N\}$ be functions homogeneous of the i-th order with respect to $\{2, \ldots, N+1\}$, i.e. satisfying for every $i \in \{0, \ldots, N\}$, every $k \in \{2, \ldots, N+1\}$ and every $z \in \frac{1}{N+1} K$*

$$B_i(kz) = k^i B_i(z).$$

If

$$B_N(z) + \cdots + B_0(z) = 0 \qquad (2)$$

for every $z \in K$, then $B_N = \cdots = B_0 = 0$.

Using the above lemma we can compare terms which are homogeneous of the same order in the functional equations we are interested in.

From now on we follow the notation used in [11]. If $r \in \mathbb{N}$, then $A^r(X;Y)$ denotes the group of all r-additive functions from X^r into Y. $A^1(X;Y)$ obviously denotes the group of all homomorphisms from X into Y. By $SA^r(X;Y)$ we denote the group of all symmetric functions from $A^r(X;Y)$. We also admit that $A^0(X;Y) = SA^0(X;Y)$ is the family of all constant mappings from X into Y. If $\Phi \in A^r(X;Y)$, then Φ^d will denote the diagonalization of Φ. For such a Φ we will write $\Phi(x^r)$, if all the variables are equal to x, and, if $\Phi \in SA^r(X;Y)$, then

$\Phi(x^s, y^{r-s})$ will stand for the value of Φ at any r-tuple in which s entries are equal to x, and the remaining ones to y.

The following lemma is used in the proof of our main result.

Lemma 1.5. *Fix n, $s \in \mathbb{N}$. Let $K \subset X$ be a non-empty and absolutely convex set and $\psi\colon K \to SA^s(X;Y)$. Suppose further that $(\alpha,\beta) \in I$, $(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n) \in I^0$ and a, $b \in \mathbb{K}$ are arbitrary. Let $\Psi : K \times K \to Y$ be defined by*

$$\Psi(x,y) = \psi(\alpha x + \beta y)((ax+by)^s).$$

Suppose that

$$\mathbb{N} \ni r \geq |\alpha| + |\beta| + \sum_{i=1}^{n} \left| \frac{\alpha\beta_i - \alpha_i\beta}{\beta_i} \right| + n + 1, \tag{3}$$

and let $u \in \frac{1}{r}K$ be fixed. Then there exist functions $\psi_j : \frac{1}{r}K \to SA^j(X;Y)$, $j \in \{0,\dots,s-1\}$, such that for every x, $y \in \frac{1}{r}K$

$$\begin{aligned}\Delta_{(u,-\frac{\alpha_n}{\beta_n}u),\dots,(u,-\frac{\alpha_1}{\beta_1}u)}\Psi(x,y) = {}& \Delta_{\frac{\alpha\beta_n-\alpha_n\beta}{\beta_n}u,\dots,\frac{\alpha\beta_1-\alpha_1\beta}{\beta_1}u}\psi(\alpha x+\beta y)((ax+by)^s) \\ &+ \sum_{j=0}^{s-1} \psi_j(\alpha x+\beta y)((ax+by)^j).\end{aligned} \tag{4}$$

Proof. We start with the observation that by Lemma 1.3 the expressions on the left and the first one on the right-hand side of (4) are well defined for all x, y, $u \in \frac{1}{r}K$. We now proceed by induction with respect to n. For $n = 1$ we get for every x, $y \in \frac{1}{r}K$

$$\begin{aligned}&\Delta_{(u,-\frac{\alpha_1}{\beta_1}u)}\Psi(x,y) \\ &= \psi\left(\alpha x + \beta y + \frac{\alpha\beta_1-\alpha_1\beta}{\beta_1}u\right)\left(\left(ax+by+\frac{a\beta_1-\alpha_1 b}{\beta_1}u\right)^s\right) \\ &\quad -\psi(\alpha x+\beta y)((ax+by)^s) \\ &= \sum_{j=0}^{s}\binom{s}{j}\psi\left(\alpha x+\beta y+\frac{\alpha\beta_1-\alpha_1\beta}{\beta_1}u\right)\left((ax+by)^j, \left(\frac{a\beta_1-\alpha_1 b}{\beta_1}u\right)^{s-j}\right) \\ &\quad -\psi(\alpha x+\beta y)((ax+by)^s) \\ &= \Delta_{\frac{\alpha\beta_1-\alpha_1\beta}{\beta_1}u}\psi(\alpha x+\beta y)((ax+by)^s) + \sum_{j=0}^{s-1}\psi_j(\alpha x+\beta y)((ax+by)^j),\end{aligned}$$

where $\psi_j : \frac{1}{r}K \to SA^j(X;Y)$, $j \in \{0,\dots,s-1\}$ is defined by

$$\psi_j(w)(z_1,\dots,z_j) = \binom{s}{j}\psi\left(w+\frac{\alpha\beta_1-\alpha_1\beta}{\beta_1}u\right)\left(z_1,\dots,z_j,\left(\frac{a\beta_1-\alpha_1 b}{\beta_1}u\right)^{s-j}\right).$$

Let us note that by (3) the function ψ_j is well defined. This completes the proof in the case $n=1$. Now we assume that the lemma holds true for a fixed $n \in \mathbb{N}$. Suppose that (α,β), $(\alpha_i,\beta_i)\in I^0$, $i\in\{1,\ldots,n+1\}$, and $r\ge|\alpha|+|\beta|+\sum_{i=1}^{n+1}|\frac{\alpha\beta_i-\alpha_i\beta}{\beta_i}|+n+2$ is a positive integer. Let $u\in\frac{1}{r}K$ be fixed. Suppose further that there exist functions $\psi_j:\frac{1}{r}K\to SA^j(X;Y)$, $j\in\{0,\ldots,s-1\}$ such that (4) holds. For simplicity we write v_i instead of $\frac{\alpha\beta_i-\alpha_i\beta}{\beta_i}u$, $i\in\{1,\ldots,n\}$, and t instead of $\frac{a\beta_{n+1}-\alpha_{n+1}b}{\beta_{n+1}}u$. From the induction hypothesis we get for every x, $y\in\frac{1}{r}K$

$$\begin{aligned}
&\Delta_{(u,-\frac{\alpha_{n+1}}{\beta_{n+1}}u),\ldots,(u,-\frac{\alpha_1}{\beta_1}u)}\Psi(x,y)\\
&=\Delta_{(u,-\frac{\alpha_{n+1}}{\beta_{n+1}}u)}(\Delta_{(u,-\frac{\alpha_n}{\beta_n}u),\ldots,(u,-\frac{\alpha_1}{\beta_1}u)})\Psi(x,y)\\
&=\Delta_{v_n,\ldots,v_1}\psi(\alpha x+\beta y+v_{n+1})((ax+by+t)^s)\\
&\quad-\Delta_{v_n,\ldots,v_1}\psi(\alpha x+\beta y)((ax+by)^s)\\
&\quad+\sum_{j=0}^{s-1}\psi_j(\alpha x+\beta y+v_{n+1})((ax+by+t)^j)-\sum_{j=0}^{s-1}\psi_j(\alpha x+\beta y)((ax+by)^j)\\
&=\Delta_{v_{n+1},\ldots,v_1}\psi(\alpha x+\beta y)((ax+by)^s)+\sum_{j=0}^{s-1}\tilde{\psi}_j(\alpha x+\beta y)((ax+by)^j),
\end{aligned}$$

where $\tilde{\psi}_j:\frac{1}{r}K\to SA^j(X;Y)$, $j\in\{0,\ldots,s-1\}$ is given by the formula

$$\begin{aligned}
&\tilde{\psi}_j(w)(z_1,\ldots,z_j)\\
&=\binom{s}{j}\Delta_{v_n,\ldots,v_1}\psi(w+v_{n+1})(z_1,\ldots,z_j,t^{s-j})\\
&\qquad+\Delta_{v_{n+1}}\psi_j(w)(z_1,\ldots,z_j)+\sum_{i=0}^{j-1}\binom{j}{i}\psi_k(w+v_{n+1})(z_1,\ldots,z_j,t^{k-j}).
\end{aligned}$$

Now the proof is complete.

It is easy to check that the following remark holds true (by Lemma 1.5).

Remark 1.6. If $\alpha=\alpha_k$, and $\beta=\beta_k$ for a fixed $k\in\{1,\ldots,n\}$, then the expression on the right-hand side of (4) is a polynomial function of order at most $s-1$ with respect to the variable $ax+by$. Moreover, if $\alpha=1$, and $\beta=0$, then

$$\Delta_{(u,-\frac{\alpha_n}{\beta_n}u),\ldots,(u,-\frac{\alpha_1}{\beta_1}u)}\Psi(x,y)=\Delta_u^n\psi(x)((ax+by)^s)+\sum_{j=0}^{s-1}\psi_j(x)((ax+by)^j).$$

2. The main result

Now let us present our main result which is very helpful in solving a wide range of functional equations.

Theorem 2.1. *Let X, Y be two linear spaces over a field $\mathbb{K} \subset \mathbb{R}$ and let K be an absolutely convex set with $0 \in \operatorname{alg\,int} K$. Fix $N, M \in \mathbb{N} \cup \{0\}$ and $a, b \in \mathbb{K}$, $b \neq 0$. Assume that $I_0, \dots, I_M$ are finite subsets of I^0. If the functions $\varphi_i : K \to SA^i(X;Y)$, $i \in \{0, \dots, N\}$, and $\psi_{j,(\alpha,\beta)} : K \to SA^j(X;Y)$, $(\alpha, \beta) \in I_j$, $j \in \{0, \dots, M\}$ satisfy the equation*

$$\sum_{i=0}^{N} \varphi_i(x)((ax+by)^i) = \sum_{i=0}^{M} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha x + \beta y)((ax+by)^i) \tag{5}$$

for every x, $y \in K$, then there exists a positive integer p, such that φ_N is a locally polynomial function of the order at most equal to

$$\sum_{i=0}^{M} \operatorname{card}\Big(\bigcup_{k=i}^{M} I_k\Big) - 1$$

on $\frac{1}{p}K$.

Proof. Let us define the rank of the right-hand side of (5) which we denote by r_R. We set $r_R = 0$ if $I_j = \emptyset$ for $j \in \{0, \dots, M\}$, and $r_R = s \in \{1, \dots, M+1\}$ if $I_{s-1} \neq \emptyset = I_s = \dots = I_M$. To prove the theorem we proceed by induction with respect to the r_R.

1. For $r_R = 0$ equation (5) gets the form

$$\sum_{i=0}^{N} \varphi_i(x)((ax+by)^i) = 0,$$

or, by the properties of multiadditive and symmetric functions

$$\varphi_N(x)((by)^N) + \sum_{i=0}^{N-1} \sum_{r=i}^{N} \binom{r}{i} \varphi_r(x)((ax)^{r-i}, (by)^i) = 0.$$

If we fix x in the above equation we see that the left-hand side is the sum of monomials with respect to the variable by. Applying Lemma 1.4 we receive that $\varphi_N(x) = 0$, for all $x \in K$. It means that φ_N is a locally polynomial function of the degree 0 on the set K. (In this case we set $p = 1$.)

2. Now we assume that, if $r_R \leq s$ for a fixed $s \in \{0, \dots, M\}$, then φ_N is a locally polynomial function of order at most equal to

$$\sum_{i=0}^{s-1} \operatorname{card}\Big(\bigcup_{k=i}^{s-1} I_k\Big) - 1$$

on the set $\frac{1}{q}K$, $q \in \mathbb{N}$. We take into account (5) for $r_R = s+1$, i.e. $I_s \neq \emptyset = I_{s+1} = \dots = I_M$. Let $I_s = \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$, $n \in \mathbb{N}$. Suppose that $r \in \mathbb{N}$ satisfies (1), where $J = \bigcup_{j=0}^{s} I_j$. Fix $u \in \frac{1}{r}K$. Now we apply the difference operator

$$\Delta_{(u, -\frac{\alpha_n}{\beta_n}u), \dots, (u, -\frac{\alpha_1}{\beta_1}u)}$$

to the right and the left-hand side of (5). Using the linearity of this operator, Lemma 1.5, and Remark 1.6, and arranging terms according to their degree with respect to the expression $ax+by$ we get

$$\Delta_u^n\varphi_N(x)((ax+by)^N) + \sum_{i=0}^{N-1}\tilde{\varphi}_i(x)((ax+by)^i) = \sum_{k=0}^{s-1}\sum_{(\alpha,\beta)\in L_k}\tilde{\psi}_{k,(\alpha,\beta)}(\alpha x+\beta y)((ax+by)^k),$$

where $L_k=\bigcup_{j=k}^s I_j$, $k\in\{0,\dots,s-1\}$, $\tilde{\varphi}_i:\frac{1}{r}K\to SA^i(X;Y)$, $i\in\{0,\dots,N-1\}$, and $\tilde{\psi}_{k,(\alpha,\beta)}:\frac{1}{r}K\to SA^k(X;Y)$, $(\alpha,\beta)\in L_k$, $k\in\{0,\dots,s-1\}$, x, $y\in\frac{1}{r}K$. In the above equation we have $r_R\le s$. Then we apply the induction hypothesis to the set $\frac{1}{r}K$, and we get that there exists $q\in\mathbb{N}$ such that $\Delta_u^n\varphi_N$ is a locally polynomial function of order at most equal to

$$m=\sum_{k=0}^{s-1}\operatorname{card}L_k-1=\sum_{k=0}^{s-1}\operatorname{card}\Bigl(\bigcup_{j=k}^s I_j\Bigr)-1\ge 0$$

(let us note that $L_k\subset L_{k-1}$, $k\in\{1,\dots,s-1\}$) on $\frac{1}{qr}K$. Equivalently, the following equation

$$\Delta_v^{m+1}\Delta_u^n\varphi_N(x)=0 \tag{6}$$

is satisfied for every $(x,v)\in X\times X$ such that

$$x,x+v,\dots,x+(m+1)v\in\frac{1}{qr}K,$$

and every $u\in\frac{1}{r}K$. By the absolute convexity of the set K we have the following property:

If

$$x,x+v,\dots,x+(m+n+1)v\in\frac{1}{qr}K,$$

then

$$v\in\frac{1}{m+n+1}\left(\frac{1}{qr}K-x\right)\subset\frac{1}{qr}K\subset\frac{1}{r}K.$$

Setting $u=v$ in (6) we obtain

$$\Delta_v^{m+n+1}\varphi_N(x)=\Delta_v^{m+1}\Delta_v^n\varphi_N(x)=0.$$

It follows that φ_N is a locally polynomial function of order at most equal to

$$m+n=\sum_{k=0}^{s}\operatorname{card}\Bigl(\bigcup_{j=k}^s I_j\Bigr)-1$$

on $\frac{1}{p}K$, where $p=qr$. This completes the inductive proof.

Let us note that if $\bigcup_{i=0}^{M} I_i \subset \{(\alpha,\beta) \in I^0 : 0 \le \alpha \wedge \beta = 1-\alpha\}$ (i.e. $\alpha x + \beta y$ is a convex combination of x and y, $(\alpha,\beta) \in \bigcup_{i=0}^{M} I_i$) in (5), then we can weaken the assumption on the set K by not requiring that $0 \in K$. This is stated in the following corollary.

Corollary 2.2. *Let X, Y be linear spaces over a field $\mathbb{K} \subset \mathbb{R}$ and let N, $M \in \mathbb{N} \cup \{0\}$ be fixed. Suppose further that $J_0, \ldots, J_M$ are finite subsets of $\mathbb{K} \cap [0,1)$. If $\emptyset \neq K \subset X$ is a convex set such that $x_0 \in \operatorname{alg\,int} K$, and if the functions $\varphi_i : K \to SA^i(X;Y)$, $i \in \{0,\ldots,N\}$, and $\psi_{j,\alpha} : K \to SA^j(X;Y)$, $\alpha \in J_j$, $j \in \{0,\ldots,M\}$, satisfy the equation*

$$\sum_{i=0}^{N} \varphi_i(x)((ax+by)^i) = \sum_{i=0}^{M} \sum_{\alpha \in I_i} \psi_{i,\alpha}(\alpha x + (1-\alpha)y)((ax+by)^i) \tag{5'}$$

for every x, $y \in K$, then there exists a convex subset $K' \subset K$ such that $x_0 \in \operatorname{alg\,int} K'$, and φ_N is a locally polynomial function of degree at most

$$\sum_{i=0}^{M} \operatorname{card}\left(\bigcup_{k=i}^{M} J_k\right) - 1$$

on K'.

Proof. Suppose that $x_0 \in \operatorname{alg\,int} K$. Then the set $\tilde{K} = (K - x_0) \cap (-K + x_0)$ is absolutely convex. Let us define the functions $\tilde{\varphi}_i$, $\tilde{\psi}_{i,(\alpha,\beta)} : \tilde{K} \to SA^i(X;Y)$ by

$$\begin{aligned} \tilde{\varphi}_i(s) &= \varphi_i(s+x_0), && i \in \{0,\ldots,N\} \\ \tilde{\psi}_{i,\alpha}(s) &= \psi_{i,\alpha}(s+x_0), && \alpha \in J_i,\ i \in \{0,\ldots,M\}. \end{aligned}$$

These functions satisfy for every s, $t \in \tilde{K}$ the equation

$$\sum_{i=0}^{N} \tilde{\varphi}_i(s)((as+bt+(a+b)x_0)^i) = \sum_{i=0}^{M} \sum_{\alpha \in J_i} \tilde{\psi}_{i,\alpha}\big(\alpha s+(1-\alpha)t\big)\big((as+bt+(a+b)x_0)^i\big). \tag{7}$$

Now, we define new functions $\tilde{\tilde{\varphi}}_i$, $\tilde{\tilde{\psi}}_{i,\alpha} : \tilde{K} \to SA^i(X;Y)$ by

$$\begin{aligned} \tilde{\tilde{\varphi}}_i(s)(u_1,\ldots,u_i) &= \sum_{r=i}^{N} \binom{r}{i} \tilde{\varphi}_r(s)\big(u_1,\ldots,u_i,((a+b)x_0)^{r-i}\big), \quad i \in \{0,\ldots,N\}, \\ \tilde{\tilde{\psi}}_{i,\alpha}(s)(u_1,\ldots,u_i) &= \sum_{r=i}^{M} \binom{r}{i} \tilde{\psi}_{r,\alpha}(s)\big(u_1,\ldots,u_i,((a+b)x_0)^{r-i}\big), \end{aligned}$$

$\alpha \in J_i$, $i \in \{0,\ldots,M\}$. We can rewrite equation (7) in the form

$$\sum_{i=0}^{N} \tilde{\tilde{\varphi}}_i(s)((as+bt)^i) = \sum_{i=0}^{M} \sum_{\alpha \in J_i} \tilde{\tilde{\psi}}_{i,\alpha}(\alpha s + (1-\alpha)t)((as+bt)^i). \tag{8}$$

It is easily seen that $0 \in \operatorname{alg\,int} \tilde{K}$. Then, applying Lemma 2.1, we get that $\tilde{\tilde{\varphi}}_N = \tilde{\varphi}_N$ is a locally polynomial function of order at most

$$m := \sum_{i=0}^{M} \operatorname{card}\left(\bigcup_{k=i}^{M} J_k\right) - 1$$

on a convex subset $\tilde{K}' \subset \tilde{K}$ such that $0 \in \operatorname{alg\,int} \tilde{K}'$. Put $K' = x_0 + \tilde{K}'$. We infer that $K' \subset K$ is convex and $x_0 \in \operatorname{alg\,int} K'$. Moreover, for every $x \in K$ we get

$$\varphi_N(x) = \tilde{\varphi}_N(x - x_0) = \tilde{\tilde{\varphi}}_N(x - x_0).$$

But $x - x_0, x - x_0 + u, \ldots, x - x_0 + (m+1)u \in K' - x_0 = \tilde{K}'$, and $\tilde{\tilde{\varphi}}_N$ is a locally polynomial function of order not greater than m on $\tilde{K}'$. Thus, for every x, $u \in X$ such that $x, x+u, \ldots, x+(m+1)u \in K'$ we obtain

$$\Delta_u^{m+1} \varphi_N(x) = \Delta_u^{m+1} \tilde{\tilde{\varphi}}_N(x - x_0) = 0.$$

This means that φ_N is a locally polynomial function of order at most equal to m on the set K'.

Now we have got that the map φ_N is a locally polynomial function only on a subset of K. However, using the results of Roman Ger from [4], we are able to show that actually φ_N is the restriction of a polynomial function defined on the whole space. Taking into account well-known results on representation of polynomial functions (cf. for example Székelyhidi [12], Theorem 9.1), we get the following

Corollary 2.3. *Under the assumptions of Corollary* 2.2 *there exists a convex subset* $K' \subset K$ *such that* $x_0 \in \operatorname{alg\,int} K'$, *and functions* $A_i \in SA^i(X; SA^N(X;Y))$, $i \in \{0, \ldots, m\}$, *such that for every* $x \in K'$

$$\varphi_N(x) = A_0 + A_1(x) + A_2^d(x) + \cdots + A_m^d(x),$$

where

$$m := \sum_{i=0}^{M} \operatorname{card}\left(\bigcup_{k=i}^{M} I_k\right) - 1.$$

The functions A_i, $i \in \{0, \ldots, m\}$ *are defined uniquely.*

3. Applications

Now let us present a method of solving some functional equations stemming from Mean Value Theorems.

1. We rewrite the equation in order to get a form similar to (5) and from Theorem 2.1 we obtain that one of the unknown mappings is a locally polynomial function.

2. Then applying an analogous procedure as above enriched possibly with substituting new variables or by elementary transformations we get that the remaining unknown mappings are also locally polynomial functions.
3. After that from Corollary 2.3 we get the representation of the unknown functions on a convex subset K' of the domain of the equation. Then, comparing terms which are homogeneous of the same order on the left- and right-hand side of the equation, we determine these polynomial functions.
4. The next step is to verify that the forementioned polynomial extensions of the unknown functions on the whole space satisfy the equation for every argument from the linear space.
5. On account of the linear character of the equation it remains to show that if a solution (usually a vector of functions) vanishes on a convex subset of the domain of the equation with nonempty algebraic interior, then it vanishes everywhere in the domain.

3.1. Flett's functional equation

In [10] T. Riedel and P. K. Sahoo proved the following extension of Flett's Mean Value Theorem

Theorem 3.1. *If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, then there exists a point $c \in (a, b)$ such that*

$$f(c) - f(a) = (c - a)f'(c) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(c - a)^2.$$

T. Riedel and M. Sablik dealt in [9] with the functional equation

$$f(c) - f(a) = (c - a)h(c) - \frac{1}{2}\frac{h(b) - h(a)}{b - a}(c - a)^2 \tag{9}$$

motivated by Theorem 3.1. They proved that (9) characterizes cubic polynomials for $c = \frac{a+3b}{4}$.

In [6] we gave the following extension of Flett's Mean Value Theorem.

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be an n-times differentiable function. Then there exists a $t = t(a, b) \in (0, 1)$ such that*

$$\begin{aligned} f(a) = {} & \sum_{k=0}^{n} \frac{t^k f^{(k)}(a + t(b - a))}{k!}(a - b)^k \\ & + \frac{t^{n+1}}{(n + 1)!}(f^{(n)}(a) - f^{(n)}(b))(a - b)^n. \end{aligned} \tag{10}$$

Motivated by the work of T. Riedel and M. Sablik we deal with the functional equation

$$f(x) = \sum_{k=0}^{n} g_k(x + t(y - x))\big((t(x - y))^k\big) + (\Phi(x) - \Phi(y))\big((t(x - y))^n\big), \quad (11)$$

which reduces to (10), if $g_0, \dots, g_n, \Phi : [a, b] \to \mathbb{R}$ are defined by

$$g_i(x) = \frac{f^{(i)}(x)}{i!}, \qquad i \in \{0, \dots, n\}, \quad (12)$$

$$\Phi(x) = \frac{tf^{(n)}(x)}{(n+1)!} \quad (13)$$

for all $x, y \in [a, b]$and for a fixed $t \in (0, 1)$.

This equation was solved in [7] on the real line and in [8] on groups. Let us note that Flett's Mean Value Theorem concerns the behaviour of functions on a real interval. So it is interesting to consider functional equations stemming from Flett's MVT also in a restricted domain. In the real case it is an interval. Our aim is to solve (11) in a more abstract case that is in a linear space in which the interval is replaced by a convex set. First, let us note that equation (11) makes sense in such domains. We have to explain what the multiplications on the right-hand side mean. We can think about g_k and Φ in terms of linear mappings which act on the k-tuple or the n-tuple $((t(x - y))^k)$ or $((t(x - y))^n)$. Then we replace these linear mappings by homomorphisms. Now let us present the solution of equation (11) for functions defined in a convex set with non-empty algebraic interior. For the simplicity we assume that 0 is in its algebraic interior. We can do it because the solution in the general case can be obtained by superposition of a solution in the special case with a translation. But the solutions are locally polynomial functions and translations do not change this property and their degree. In the sequel, K denotes a nonempty convex subset of X with $0 \in \operatorname{alg\,int} K$. In particular, the functions in (11) are defined as $f : K \to Y$, $g_i : K \to SA^i(X, Y)$, $i \in \{0, \dots, n\}$, $\Phi : K \to SA^n(X, Y)$. Because of our way of solving (11) we take into account a subset of K depending on the expression $u := 1 - \frac{1}{t}$ such that every substitution is well defined. If $|u| \geq 1$ then we take $x, y \in \frac{1}{8|u|}K$, and in the opposite case $x, y \in \frac{|u|}{8}K$. Without loss of generality, we can assume that $x, y \in \frac{1}{8|u|}K$. The case $x, y \in \frac{|u|}{8}K$ is analogous. We show now that the substitutions defined below are correct. For $x, y \in \frac{1}{8|u|}K$ we have $c := x - y \in \frac{1}{4|u|}K$, $z := tc \in \frac{1}{4|u|}K$, $y := \frac{a}{x} - z \in \frac{1}{2|u|}K$ and $y + z, y + uz \in K$.

Now we can present the solution of the functional equation (11). Equation (11) can be written equivalently in the form

$$\Phi(x)\big((t(x-y))^n\big) - f(x) = \sum_{k=0}^{n} -g_k((1-t)x + ty)\big((t(x-y))^k\big) + \Phi(y)\big((t(x-y))^n\big).$$

Thus taking $N = M = n$, $a = t$, $b = -t$, $\varphi_N = \Phi$, $\varphi_0 = -f$, $\varphi_i = 0$, $i \in \{1, ..., N-1\}$,

$I_i = \{(1-t,t)\}$, $i \in \{0,\dots,M-1\}$, $I_M = \{(1-t,t);(0,1)\}$, $\psi_{i,(1-t,t)} = -g_i$, $i \in \{0,\dots,M\}$, $\psi_{M,(0,1)} = \Phi$ in Theorem 2.1 we get that Φ is a locally polynomial function on $\frac{1}{8|u|}K' = \frac{1}{8|u|}\frac{1}{p}K$ for a $p \in \mathbb{N}$. Applying Corollary 2.3 we obtain that there exist functions $B_i \in SA^i(X;SA^n(X;Y))$, $i \in \{0,\dots,s\}$, such that

$$\Phi(z) = B_0 + \cdots + B_s^d(z), \tag{14}$$

for every $z \in \frac{1}{8|u|}K'$, and any $s \in \mathbb{N}$. Suppose that x, $y \in \frac{1}{8|u|}K'$ and set $c := x-y$. Then equation (11) has the form

$$f(x) = \sum_{k=0}^{n} g_k(x-tc)\big((tc)^k\big) + (\Phi(x) - \Phi(x-c))\big((tc)^n\big). \tag{15}$$

If we put $z = tc$, $y = x - z$ in the above equation, we obtain

$$f(y+z) = \sum_{k=0}^{n} g_k(y)(z^k) + (\Phi(y+z) - \Phi(y+uz))(z^n). \tag{16}$$

This equation is well defined for every $y \in \frac{1}{2|u|}K'$, $z \in \frac{1}{4|u|}K'$. Setting $y = 0$ in (16) we get

$$f(z) = \sum_{k=0}^{n} g_k(0)(z^k) + (\Phi(z) - \Phi(uz))(z^n).$$

Taking (14) into account we have

$$f(z) = \sum_{k=0}^{n} a_k(z^k) + (A_1(z) + \cdots + A_s^d(z))(z^n), \tag{17}$$

where $a_k := g_k(0)$, $k \in \{0,\dots,n\}$, $A_i \in SA^i(X;SA^n(X;Y))$ are proportional to B_i, $i \in \{1,\dots,s\}$, and $z \in \frac{1}{4|u|}K'$. Hence f is a locally polynomial function on $\frac{1}{4|u|}K'$. Let us put $z \in \frac{1}{4|u|}K'$ and insert (14) and (17) in (16). This yields

$$\begin{aligned} &\sum_{l=0}^{n}\sum_{i=0}^{l}\binom{l}{i}a_l(y^{l-i},z^i) + \sum_{l=1}^{s}\sum_{j=0}^{l}\sum_{i=0}^{n}\binom{l}{j}\binom{n}{i}A_l(y^{l-j},z^j)(y^{n-i},z^i) \\ &\quad = \sum_{k=0}^{n} g_k(y)(z^k) + \left[\sum_{l=0}^{s}\sum_{j=0}^{l}\binom{l}{j}(1-u^j)B_l(y^{l-j},z^j)\right](z^n). \end{aligned} \tag{18}$$

We can now proceed analogously to the case of the real line in [7] or groups in [8], i.e. comparing terms of the same order of the left and the right-hand side of (18). Consequently, we get that $s \le 2$, and

$$A_s^d(z)(z^n) = \frac{s(s-1)\cdot\dots\cdot(s-k+1)}{(n+s)(n+s-1)\cdot\dots\cdot(n+s-k+1)}(1-u^{s-k})B_s^d(z)(z^n). \tag{19}$$

Thus the functions f and Φ are locally polynomial of order not greater than $n+2$ and 2 on the set $\frac{1}{8|u|}K'$. Then from (14) and (19) we receive

$$\Phi(z) = B_0 + B_1(z) + B_2^d(z),$$

and

$$f(z) = \sum_{k=0}^{n} a_k(z^k) + (1-u)B_1(z)(z^n) + (1-u^2)B_2^d(z)(z^n)$$

for all $z \in \frac{1}{8|u|}K'$. Inserting the formulas for the functions f and Φ in (18) we have that if $t = \frac{n+2}{2(n+1)}$ then for every $z \in \frac{1}{8|u|}K'$

$$B_2(y,z)(z^n) = B_2^d(z)(y, z^{n-1}), \tag{20}$$

and if $t \neq \frac{n+2}{2(n+1)}$ then $B_2 = 0$. Now we proceed analogously to the case of solving equation (11) in groups ([8]). We need the definition of the derivative of a polynomial function.

Definition 3.3. Let n, $s \in \mathbb{N}$. Suppose further that $A_s \in SA^s(X;Y)$. Then the mapping $D_x A_s^d \in SA^1(X;Y)$ is called the derivative of the function A_s at the point x provided that

$$D_x A_s^d(z) = sA_s(x^{s-1}, z)$$

for every x, $z \in X$. Further, if $A_s \in SA^s(X; SA^n(X;Y))$ and the function $B : X \to Y$ is represented in the form

$$B(x) = A_s^d(x)(x^n),$$

then for every $x \in X$ we define the derivative $D_x B \in SA^1(X;Y)$ as

$$D_x B(z) = sA_s(x^{s-1}, z)(x^n) + nA_s^d(x)(x^{n-1}, z).$$

Definition 3.4. Let the function $f : X \to Y$ be of the form $f(z) = A_0(z^n) + A_1(z)(z^n) + \cdots + A_s^d(z)(z^n)$ where $A_i \in SA^s(X; SA^n(X;Y))$. Fix $x \in X$. The function $D_x f : X \to SA^1(X;Y)$ is said to be the derivative of the polynomial function f at the point x if and only if for every $z \in X$

$$D_x f(z) = D_x(B_1(z)) + \cdots + D_x(B_s^d(z))$$

where $B_i^d(z) = A_i^d(z)(z^n)$.

The second derivative we define as $D_x^2 f = D_x(D_x f)$. Analogously $D_x^j f = D_x(D_x^{j-1} f)$, $j \leq n+s$. We admit that $D_x^{n+s+1} f = 0$ and $D_x^0 f = f$.

From investigations analogous to those in [8] we have the following

Lemma 3.5. *Let $\emptyset \neq K \subset X$ be a convex set such that $0 \in \operatorname{alg\,int} K$. If the functions $f : K \to Y$, $g_i : K \to SA^i(X,Y)$, $i \in \{0,\dots,n\}$, $\Phi : K \to SA^n(X,Y)$ satisfy the equation*

$$f(x) = \sum_{k=0}^{n} g_k((1-t)x + ty)((t(x-y))^k) + (\Phi(x) - \Phi(y))((t(x-y))^n), \qquad (11)$$

for every x, $y \in K$, and for a fixed $t \in \mathbb{Q} \cap (0,1)$, then there exist a convex set $\hat{K} \subset K$ such that $0 \in \operatorname{alg\,int} \hat{K}$, and functions $a_k \in SA^k(X;Y)$, $k \in \{0,\dots,n\}$, $B_i \in SA^i(X; SA^n(X;Y))$, $i \in \{0,1,2\}$, such that the formula

$$B_2(x,y)(y^n) = B_2^d(y)(x, y^{n-1}) \qquad (20)$$

holds for every x, $y \in \hat{K}$ and for every $x \in \hat{K}$:

$$f(x) = \begin{cases} \sum_{k=0}^{n} a_k(x^k) + \frac{1}{t} B_1(x)(x^n) + \frac{1}{t}\left(2 - \frac{1}{t}\right) B_2^d(x)(x^n), & \text{if } t = \frac{n+2}{2(n+1)}, \\ \sum_{k=0}^{n} a_k(x^k) + \frac{1}{t} B_1(x)(x^n), & \text{if } t \neq \frac{n+2}{2(n+1)}, \end{cases}$$

$$\Phi(x) = \begin{cases} B_0 + B_1(x) + B_2^d(x), & \text{if } t = \frac{n+2}{2(n+1)}, \\ B_0 + B_1(x), & \text{if } t \neq \frac{n+2}{2(n+1)}, \end{cases}$$

$$g_i(x) = \frac{D_x^i f}{i!}, \qquad i \in \{0,\dots,n\}.$$

Conversely, if the functions $a_k \in SA^k(X;Y)$, $k \in \{0,\dots,n\}$, $B_i \in SA^i(X; SA^n(X;Y))$, $i \in \{0,1,2\}$ satisfy eq. (20), then the functions $P_f : X \to Y$, $P_{g_i} : X \to SA^i(X,Y)$, $i \in \{0,\dots,n\}$, $P_\Phi : X \to SA^n(X,Y)$ defined by

$$P_f(x) = \begin{cases} \sum_{k=0}^{n} a_k(x^k) + \frac{1}{t} B_1(x)(x^n) + \frac{1}{t}\left(2 - \frac{1}{t}\right) B_2^d(x)(x^n), & \text{if } t = \frac{n+2}{2(n+1)}, \\ \sum_{k=0}^{n} a_k(x^k) + \frac{1}{t} B_1(x)(x^n), & \text{if } t \neq \frac{n+2}{2(n+1)}, \end{cases}$$

$$P_\Phi(x) = \begin{cases} B_0 + B_1(x) + B_2^d(x), & \text{if } t = \frac{n+2}{2(n+1)}, \\ B_0 + B_1(x), & \text{if } t \neq \frac{n+2}{2(n+1)}, \end{cases}$$

$$P_{g_i}(x) = \frac{D_x^i f}{i!} \qquad i \in \{0,\dots,n\}.$$

satisfy equation (11) for every $x \in X$.

Now we show that local solutions of equation (11) can be uniquely extended on the whole set K except maybe for its boundary points.

Let p_K denote the Minkowski functional of K (cf. [1]). We will use the fact that, if $p_K(x) < 1$, then $x \in K$. The set $\{x \in X : p_K(x) < 1\}$ we denote by K^0; by the edge of K we understand the set $\{x \in X : p_K(x) = 1\}$. A point $x \in K$ we call an extreme point if $x = \frac{u+v}{2}$ for any $u,\ v \in K$ implies that $u = v = x$.

Suppose that $\hat{K} \subset K$ is a convex set such that $0 \in \operatorname{alg\,int} \hat{K}$, and let $P_f : X \to Y$, $P_{g_i} : X \to SA^i(X,Y)$, $i \in \{0,\dots,n\}$, $P_\Phi : X \to SA^n(X,Y)$ be polynomial functions such that $P_f|_{\hat{K}} = f|_{\hat{K}}$, $P_{g_i}|_{\hat{K}} = g_i|_{\hat{K}}$, $P_\Phi|_{\hat{K}} = \Phi|_{\hat{K}}$. Suppose further that for every $x,\ y \in X$

$$P_f(x) = \sum_{k=0}^{n} P_{g_k}((1-t)x+ty)((t(x-y))^k) + (P_\Phi(x) - P_\Phi(y))((t(x-y))^n).$$

We take $\hat{K} \subset K$ as in Lemma 3.5.

Let us define

$$\hat{f} = f - P_f|_K, \ \ \hat{g}_i = g_i - P_{g_i}|_K, \ \ \hat{\Phi} = \Phi - P_\Phi|_K.$$

Then $(\hat{f},\ \hat{g}_0,\dots,\ \hat{g}_n, \hat{\Phi})$ is a solution of (11) for every $x,\ y \in K$ and

$$\hat{f}|_{\hat{K}} = 0 \in Y^{\hat{K}},\ \hat{g}_i|_{\hat{K}} = 0 \in SA^i(X,Y)^{\hat{K}},\ i \in \{0,\dots,n\},\ \hat{\Phi}|_{\hat{K}} = 0 \in SA^n(X,Y)^{\hat{K}}.$$

Now we show that the function $\hat{f}$ vanishes on K. Precisely,

Lemma 3.6. *If the functions $(\hat{f},\ \hat{g}_0,\dots,\ \hat{g}_n, \hat{\Phi})$ satisfy the equation*

$$f(x) = \sum_{k=0}^{n} g_k((1-t)x+ty)((t(x-y))^k) + (\Phi(x) - \Phi(y))((t(x-y))^n), \tag{11}$$

for every $x,\ y \in K$, and $\hat{f}|_{\hat{K}} = 0 \in Y^{\hat{K}}$, $\hat{g}_i|_{\hat{K}} = 0 \in SA^i(X,Y)^{\hat{K}}$, $i \in \{0,\dots,n\}$, $\hat{\Phi}|_{\hat{K}} = 0 \in SA^n(X,Y)^{\hat{K}}$, then $\hat{f} = \hat{g}_0 = 0 \in Y^K$, $\hat{\Phi}(x)(x^n) = 0$, and $\hat{g}_i(x)(x^n) = 0$, $i \in \{1,\dots,n\}$, $x \in K^0$.

Proof. It is obvious that $\hat{f} = \hat{g}_0$ (putting $y = x$ w (11)). Let us notice that it is sufficient to analyze the behaviour of the functions $\hat{f},\ \hat{g}_0,\dots,\ \hat{g}_n,\ \hat{\Phi}$ on the intersection of every straight line $l_z = \mathbb{K}z$, $z \neq 0$ with the set K. Fix $z \in X \setminus \{0\}$. Suppose that $K_z = \{q \in \mathbb{K} : qz \in K\}$. Let us define new functions $\hat{f}_z : K_z \to Y$, $\hat{g}_{i,z} : K_z \to SA^i(l_z, Y)$, $i \in \{0,\dots,n\}$, $\hat{\Phi}_z : K_z \to SA^n(l_z, Y)$ by

$$\hat{f}_z(q) = \hat{f}(qz), \ \ \hat{g}_{i,z}(q)(r^i) = \hat{g}_i(qz)((rz)^i), \ \ \hat{\Phi}_z(q)(r^n) = \hat{\Phi}(qz)((rz)^n).$$

We see at once that the functions $\hat{f}_z,\ \hat{g}_{0,z},\dots,\ \hat{g}_{n,z},\ \hat{\Phi}_z$ satisfy the equation

$$\begin{aligned}\hat{f}_z(q) = &\sum_{k=0}^{n} (t(q-r))^k \hat{g}_{k,z}((1-t)q+tr)(z^k)\\ &+ (t(q-r))^n(\hat{\Phi}_z(q) - \hat{\Phi}_z(r))(z^n)\end{aligned} \tag{21}$$

for every $q,\ r \in K_z$, and $\hat{f}_z|_{\hat{K}_z} = 0 \in Y^{\hat{K}_z}$, $\hat{g}_{i,z}|_{\hat{K}_z} = 0 \in SA^i(X,Y)^{\hat{K}_z}$, $i \in \{0,\ldots,n\}$, $\hat{\Phi}_z|_{\hat{K}_z} = 0 \in SA^n(X,Y)^{\hat{K}_z}$, where $\hat{K}_z = \{q \in \mathbb{K} : qz \in \hat{K}\}$. Without loss of generality we can assume that $\hat{K}_z$ is a maximal interval of numbers from $\mathbb{K}$ such that $0 \in \operatorname{alg\,int} \hat{K}_z$, and the functions $\hat{f}_z, \hat{g}_{0,z}, \ldots, \hat{g}_{n,z}, \hat{\Phi}_z$ vanish on $\hat{K}_z$.

On the contrary, suppose that $s_1 = \sup \hat{K}_z < s_2 = \sup K_z$. Let ε be a positive number such that $s_1 + \varepsilon < s_2$ and $\varepsilon < s_1 \cdot \frac{t}{1-t}$. Then $(1-t)q < s_1$ for every $q \in [s_1, s_1+\varepsilon) \cap \mathbb{K} =: J$ and for every $r \in \hat{K}_z \cap (-\infty, s_1 - \frac{1-t}{t}\varepsilon)$, and $q \in J$ we have

$$(1-t)q + tr < s_1.$$

We can rewrite equation (21) in the form

$$\hat{f}_z(q) - t(q-r))^n \hat{\Phi}_z(q)(z^n) = \sum_{k=0}^{n} (t(q-r))^k \hat{g}_{k,z}((1-t)q+tr)(z^k) - (t(q-r))^n \hat{\Phi}_z(r)(z^n).$$

By our assumptions, the right-hand side of the above equation vanishes for every $q \in J$, and $r \in \hat{K}_z \cap (-\infty, s_1 - \frac{1-t}{t}\varepsilon)$. Thus we get

$$\hat{f}_z(q) - (t(q-r))^n \hat{\Phi}_z(q)(z^n) = 0. \tag{22}$$

Taking $r = 0$ in (22) we obtain

$$\hat{f}_z(q) - (tq)^n \hat{\Phi}_z(q)(z^n) = 0.$$

If we subtract side by side the above equations, we receive $\hat{\Phi}_z(q)(z^n) = 0$ for every $q \in J$. Hence $\hat{\Phi}_z(q) = 0 \in SA^n(l_z; Y)$, $q \in J$. From (22) it follows that $\hat{f}_z(q) = 0$, $q \in J$, whence $\hat{\Phi}_z|_{\hat{K}_z \cup J} = 0$ and $\hat{f}_z|_{\hat{K}_z \cup J} = 0$.

Fix $p \in J$. Then we have $p < s_1 + \varepsilon$. It is easy to check that

$$\frac{p - t(s_1+\varepsilon)}{1-t} < p.$$

Let $q \in (\hat{K}_z \cup J) \cap \left(\frac{p-t(s_1+\varepsilon)}{1-t}, p\right) =: I_p$. We have, therefore,

$$r := \frac{1}{t}p - \frac{1-t}{t}q < \frac{1}{t}p - \frac{1-t}{t}\frac{p-t(s_1+\varepsilon)}{1-t} = s_1 + \varepsilon$$

and $r > p$. Thus we get $r \in J$, and

$$p = tr + (1-t)q.$$

Let us notice that, if q goes across the set I_p, then $t(q-r) = q - p$ goes across $I_p - p$ (it is an interval in $\mathbb{K}$). Substituting $q \in I_p$, and $r = \frac{1}{t}p - \frac{1-t}{t}q$ in (21), and applying $\hat{\Phi}_z|_{\hat{K}_z \cup J} = 0$, $\hat{f}_z|_{\hat{K}_z \cup J} = 0$, we get

$$\sum_{k=0}^{n} (q-p)^k \hat{g}_{k,z}(p)(z^k) = 0.$$

Taking into account our earlier investigations, we showed that the polynomial

$$s \longmapsto w(s) = \sum_{k=0}^{n} s^k \hat{g}_{k,z}(p)(z^k) = 0$$

vanishes on $I_p - p$. It follows that its coefficients are $\hat{g}_{k,z}(p)(z^k) = 0$, $k \in \{0, \dots, n\}$. Thus $\hat{g}_{k,z}(p) = 0 \in SA^k(l_z; Y)$. Since $p \in J$ is arbitrary we state that $\hat{g}_{k,z}|_J = 0$, $k \in \{0, \dots, n\}$. This implies $\hat{g}_{k,z}|_{\hat{K}_z \cup J} = 0$, $k \in \{0, \dots, n\}$, and with the equalities $\hat{\Phi}_z|_{\hat{K}_z \cup J} = 0$, and $\hat{f}_z|_{\hat{K}_z \cup J} = 0$ this contradicts the definition of s_1. Therefore $\sup \hat{K}_z = \sup K_z$. Applying similar considerations, we get $\inf \hat{K}_z = \inf K_z$.

We showed that for every $z \neq 0$ the functions $\hat{f}_z$, $\hat{\Phi}_z$, and $\hat{g}_{k,z}$, $k \in \{0, \dots, n\}$ vanish on $[(\inf \hat{K}_z, \sup \hat{K}_z) \cap \mathbb{K}]z$. Fix $z \neq 0$. Suppose that $q = \sup K_z \in K_z$, i.e. $qz \in K$. Thus, for every $r \in (\inf \hat{K}_z, \sup \hat{K}_z) \cap \mathbb{K}$ we get $(1-t)q + tr \in (\inf \hat{K}_z, \sup \hat{K}_z) \cap \mathbb{K}$. According to the first part of the proof and (21), we have that (22) holds for every $r \in (\inf \hat{K}_z, \sup \hat{K}_z) \cap \mathbb{K}$. Therefore (as above) we obtain $\hat{\Phi}_z(q) = 0 \in SA^n(l_z, Y)$, and $\hat{f}_z(q) = 0 \in Y$. We proceed analogously in the case $q = \inf K_z \in K_z$, so we get that $\hat{\Phi}_z(q)$ and $\hat{f}_z(q)$ vanish. Since z is arbitrary we conclude that $\hat{f}_z = 0 \in Y^K$ and $\hat{\Phi}_z(z)(z^n) = 0$, $z \in K$.

To prove the last part it is sufficient to see that if $z \in K^0$ then $z \in K$, and $\sup K_z = \sup\{q \in \mathbb{K} : qz \in K\} > 1 > \inf K_z$, hence $\hat{g}_{i,z}(1) = 0$, thus $\hat{g}_i(z)(z^i) = 0$, $i \in \{0, \dots, n\}$.

Let us notice that, taking $X = \mathbb{R}$ and multiplication by $(x-y)^k$, $k \in \{0, \dots, n\}$, on the right-hand side of (11), and dividing $\Phi(z)(z^n)$ by z^n, and $g_i(z)(z^i)$ by z^i, $i \in \{0, \dots, n\}$, the above lemma implies that g_i, $i \in \{0, \dots, n\}$, vanish on K^0 (in particular, g_i, $i \in \{0, \dots, n\}$, vanish everywhere possibly except at the boundary points of K). Similar results can be obtained in an arbitrary space X but the proof is not so immediate.

Applying Lemma 3.6 we get that (21) is reduced to

$$0 = \sum_{k=0}^{n} \hat{g}_k((1-t)x + ty)((t(x-y))^k) + (\hat{\Phi}(x) - \hat{\Phi}(y))((t(x-y))^n) \tag{23}$$

for every x, $y \in K$. Suppose again that $\hat{K}$ is a convex subset of K such that $0 \in \operatorname{alg\,int} \hat{K}$ and $\hat{g}_i|_{\hat{K}} = 0 \in SA^i(X; Y)^{\hat{K}}$, $i \in \{0, \dots, n\}$, $\hat{\Phi}|_{\hat{K}} = 0 \in SA^n(X; Y)^{\hat{K}}$. Let $K_1 = \{x \in K : p_{\hat{K}}(x) < 1 + t\}$. It follows that for every $y \in t\hat{K} \subset \hat{K}$ we get

$$p_{\hat{K}}((1-t)x + ty) \leq (1-t)p_{\hat{K}}(x) + tp_{\hat{K}}(y) < (1-t)(1+t) + t^2 = 1,$$

hence $(1-t)x + ty \in \hat{K}$, $x \in K_1$, $y \in t\hat{K}$. Consequently (see (23))

$$\hat{\Phi}(x)((x-y)^n) = 0$$

for all $x \in K_1$ and $y \in t\hat{K}$. Fix $x \in K_1$. From this we conclude that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \hat{\Phi}(x)(x^{n-k}, y^k) = 0$$

for every $y \in t\hat{K}$. The set $t\hat{K}$ is $\mathbb{Q}$-convex, and $0 \in \operatorname{alg\,int} t\hat{K}$. Applying Theorem 2.1 (for $N = n$), we get $\hat{\Phi}(x) = 0$. In other words, this means that for every $x, y \in K_1$ equation (23) is reduced to

$$0 = \sum_{k=0}^{n} \hat{g}_k((1-t)x + ty)((t(x-y))^k). \tag{24}$$

Let us define a mapping $L : K_1 \times K_1 \to X \times X$ by

$$L(x, y) = ((1-t)x + ty, t(x-y)) = (x - t(x-y), t(x-y)) =: (u, v).$$

It is a one-to-one function, and $u = u(x, y) \in K_1$ for all $x, y \in K_1$. Fix an arbitrary $u \in K_1$ such that $p_{K_1}(u) < 1$. We take into account the set

$$U := [L(K_1 \times K_1)]_u = \{v \in X : (u, v) \in L(K_1 \times K_1)\}.$$

Obviously we have $0 \in U$ $((u, 0) = L(u, u))$. Suppose further that $v_1, v_2 \in U$, and $\lambda \in [0, 1] \cap \mathbb{K}$. Thus $(u, v_i) = L(x_i, y_i)$ for any $x_i, y_i \in K_1$, $i \in \{1, 2\}$. Consequently,

$$\begin{aligned}
&(u, \lambda v_1 + (1-\lambda)v_2)\\
&\qquad = \lambda(u, v_1) + (1-\lambda)(u, v_2)\\
&\qquad = \lambda L(x_1, y_1) + (1-\lambda)L(x_2, y_2)\\
&\qquad = \lambda(x_1 - t(x_1 - y_1), t(x_1 - y_1)) + (1-\lambda)(x_2 - t(x_2 - y_2), t(x_2 - y_2))\\
&\qquad = (\lambda x_1 + (1-\lambda)x_2 - t(\lambda x_1 + (1-\lambda)x_2 - \lambda y_1 - (1-\lambda)y_2),\\
&\quad t(\lambda x_1 + (1-\lambda)x_2 - \lambda y_1 - (1-\lambda)y_2))\\
&\qquad = L(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2),
\end{aligned}$$

hence $\lambda v_1 + (1-\lambda)v_2 \in U$. This gives convexity of U. Let $0 \neq z \in X$. For every $\lambda \in \mathbb{K}$ we get

$$p_{K_1}(u + \lambda z) \le p_{K_1}(u) + |\lambda| p_{K_1}(z),$$

whence $p_{K_1}(u) < 1$ implies that for any $\varepsilon > 0$, if $|\lambda| < \varepsilon$ then $p_{K_1}(u + \lambda z) < 1$. Thus $u + \lambda z \in K_1$, $|\lambda| < \varepsilon$. Taking ε small enough we get

$$p_{K_1}\left(u + \left(1 - \frac{1}{t}\right)\lambda z\right) < 1$$

for $|\lambda| < \varepsilon$. Put $x = u + \lambda z$, and $y = u + (1 - \frac{1}{t})|\lambda| z$. Therefore $x, y \in K_1$ provided that $|\lambda| < \varepsilon$, and

$$\begin{aligned}
L(x, y) \;=\; &(u + \lambda z - t(u + \lambda z - u - \left(1 - \frac{1}{t}\right)\lambda z), t(u + \lambda z\\
&-u - \left(1 - \frac{1}{t}\right)\lambda z)) = (u, \lambda z).
\end{aligned}$$

It follows that $\lambda z \in U$, $|\lambda| < \varepsilon$. This gives $0 \in \operatorname{alg\,int} U$. Equation (24) implies that

$$0 = \sum_{k=0}^{n} \hat{g}_k(u)(v^n)$$

for $v \in U$. Applying Theorem 2.1 we have $\hat{g}_k(u) = 0$, $k \in \{0, \ldots, n\}$. Since u is arbitrary we obtain $\hat{g}_k|_{K_1^0} = 0$.

Summarizing we proved the following:

If $0 \in \operatorname{alg\,int} \hat{K}$, $\hat{K} \subset K$, $\hat{\Phi}|_{\hat{K}} = 0 \in SA^n(X;Y)^{\hat{K}}$, and $\hat{g}_i|_{\hat{K}} = 0 \in SA^i(X;Y)^{\hat{K}}$, $i \in \{0, \ldots, n\}$, then $\hat{\Phi}$ vanishes on $K_1 = K \cap \{x \in X : p_{\hat{K}}(x) < 1 + t\}$, and $\hat{g}_i$ vanish on K_1^0. We can continue in this fashion and taking into account that $\hat{K}$ is absorbing to obtain $\hat{\Phi} = 0 \in SA^n(X;Y)^K$ and $\hat{g}_i(x) = 0 \in SA^n(X;Y)^{K_0}$.

We have proved

Theorem 3.7. *Fix $t \in (0, 1) \cap \mathbb{Q}$. Let $K \subset X$ be a convex set such that $0 \in \operatorname{alg\,int} K$. If the functions $f : K \to Y$, $g_i : K \to SA^i(X, Y)$, $i \in \{0, \ldots, n\}$, $\Phi : K \to SA^n(X, Y)$ satisfy for every x, $y \in K$ the equation*

$$f(x) = \sum_{k=0}^{n} g_k((1-t)x + ty)((t(x-y))^k) + (\Phi(x) - \Phi(y))((t(x-y))^n), \tag{11}$$

then there exist functions $B_i \in SA^i(X; SA^n(X;Y))$, $i \in \{0, 1, 2\}$, such that for every x, $y \in X$

$$B_2(x, y)(y^n) = B_2^d(y)(x, y^{n-1}), \tag{20}$$

and for every $x \in K$

$$\begin{aligned} g_0(x) &= f(x) \\ &= \begin{cases} \displaystyle\sum_{k=0}^{n} a_k(x^k) + \frac{1}{t}B_1(x)(x^n) + \frac{1}{t}\left(2 - \frac{1}{t}\right) B_2^d(x)(x^n), & \text{if } t = \frac{n+2}{2(n+1)}, \\ \displaystyle\sum_{k=0}^{n} a_k(x^k) + \frac{1}{t}B_1(x)(x^n), & \text{if } t \neq \frac{n+2}{2(n+1)}, \end{cases} \end{aligned}$$

$$\Phi(x) = \begin{cases} B_0 + B_1(x) + B_2^d(x), & \text{if } t = \frac{n+2}{2(n+1)}, \\ B_0 + B_1(x), & \text{if } t \neq \frac{n+2}{2(n+1)}, \end{cases}$$

and for every $x \in K^0$

$$g_i(x) = \frac{D_x^i f}{i!}, \qquad i \in \{1, \ldots, n\}.$$

Remark 3.8. The above theorem is stated as a necessary condition. It can be reversed provided that every point on the edge C of K is an extreme point of K, because it is easily seen that the functions g_i, $i \geq 1$, can then be defined arbitrarily at every point of C. Indeed, if $u \in C$ is an extreme point of K, then $u = (1-t)x + ty$ for any x, $y \in K$ if and only if $x = y = u$. Thus the right-hand side of (11) is reduced to $g_0(u) = f(u)$.

3.2. Simpson's functional equation

Now we consider the equation derived from Simpson's rule. Simpson's rule is an elementary numerical method for evaluating definite integrals. The functional equation derived from this rule is

$$f(x) - g(y) = \left[h\left(\frac{x+y}{2}\right) + \Phi(x) + \Psi(y)\right](x-y). \tag{25}$$

J. Ger in [3] deals with equation (25) on the real interval. In [8] we solved a generalized version of (25) for functions defined in an abelian group. Let $f,\ g : K \to Y,\ h,\ \Phi,\ \Psi : K \to SA^1(X,Y)$. Equation (25) can be written equivalently in the form

$$f(x) - \Phi(x)(x-y) = g(y) + h\left(\frac{x+y}{2}\right)(x-y) + \Psi(y)(x-y). \tag{26}$$

The above equation is the special case of equation (5) from Theorem 2.1 for $N = M = 1$, $a = 1$, $b = -1$, $\varphi_0 = f$, $\varphi_1 = -\Phi$, $I_0 = \{(0,1)\}$, $I_1 = \{(\frac{1}{2},\frac{1}{2});(0,1)\}$, $\psi_{0,(0,1)} = g$, $\psi_{1,(\frac{1}{2},\frac{1}{2})} = h$ and $\psi_{1,(0,1)} = \Psi$. Applying Theorem 2.1 we get that Φ is a locally polynomial function of order at most equal to

$$\sum_{i=0}^{1} \operatorname{card}\left(\bigcup_{k=i}^{1} I_k\right) - 1 = \operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 = 3$$

on $\frac{1}{p_1}K$ for a $p_1 \in \mathbb{N}$. Thus we have

$$\Delta_u^4 \Phi(x) = 0$$

for every $(x,u) \in X \times X$ such that $x, x+u, \ldots, x+4u \in \frac{1}{p_1}K$. From Corollary 2.3 it follows that

$$\Phi(x) = A_0 + A_1(x) + A_2^d(x) + A_3^d(x), \tag{27}$$

where $A_i \in SA^i(X; SA^1(X,Y))$, $i \in \{0,\ldots,3\}$, $x \in \frac{1}{p_1}K$.

We rephrase (26) as

$$g(y) + \Psi(y)(x-y) = f(x) - h\left(\frac{x+y}{2}\right)(x-y) - \Phi(x)(x-y).$$

Interchanging x and y we get

$$g(x) + \Psi(x)(y-x) = f(y) - h\left(\frac{x+y}{2}\right)(y-x) - \Phi(y)(y-x).$$

Repeated application of Lemma 2.1 yields that Ψ is a locally polynomial function of the order not greater than 3 on $\frac{1}{p_2}K$ for a $p_2 \in \mathbb{N}$. Analogously, from Corollary 2.3 we obtain

$$\Psi(x) = B_0 + B_1(x) + B_2^d(x) + B_3^d(x), \tag{28}$$

where $B_i \in SA^i(X; SA^1(X,Y))$, $i \in \{0,\ldots,3\}$, $x \in \frac{1}{p_2}K$.

Let us note that the transformation $L : K \times K \to X \times X$ defined by $L(x, y) = (x + y, x - y)$ has the property

$$K \times K \subset L(K \times K)$$

(indeed, if $(u, v) \in K \times K$ then it is sufficient to put $x = \frac{u+v}{2} \in K$, and $y = \frac{u-v}{2} \in K$, then $(u, v) = L(x, y)$). Thus, substituting the new variables $u := x + y$ and $v := x - y$ in (26) it follows that

$$\tilde{h}(u)(v) = f\left(\frac{u+v}{2}\right) - g\left(\frac{u-v}{2}\right) - \Phi\left(\frac{u+v}{2}\right)(v) - \Psi\left(\frac{u-v}{2}\right)(v),$$

holds for every u, $v \in K$, where $\tilde{h} : K \to SA^1(X; Y)$ is defined by $\tilde{h}(u) = h(\frac{u}{2})$. We now apply Lemma 2.1 and Corollary 2.3 again, to obtain that

$$\tilde{h}(x) = \tilde{C}_0 + \tilde{C}_1(x) + \tilde{C}_2^d(x) + \tilde{C}_3^d(x),$$

where $\tilde{C}_i \in SA^i(X, SA^1(X, Y))$, $i \in \{0, \ldots, 3\}$, $x \in \frac{1}{p_3}K$, $p_3 \in \mathbb{N}$. Therefore h is also a locally polynomial function and can be written in the form

$$h(x) = C_0 + C_1(x) + C_2^d(x) + C_3^d(x), \tag{29}$$

$x \in \frac{1}{2p_3}K$, $C_i^d(x) = \tilde{C}_i^d(2x)$, $C_i \in SA^i(X, SA^1(X, Y))$, $i \in \{0, \ldots, 3\}$.

Let $p = \max\{p_1, p_2, 2p_3\}$. Hence equalities (27), (28), and (29) hold for every $x \in K' := \frac{1}{p}K$. Put $y = 0$ in (26)

$$f(x) = [\tilde{h}(x) + \Phi(x) + \Psi(0)](x) + g(0).$$

Taking into account (27), (28), and (29) the above equation can be rephrased as

$$\begin{aligned} f(x) = A_0(x) + B_0(x) + \tilde{C}_0(x) + A_1(x)(x) + \tilde{C}_1(x)(x) \\ + A_2^d(x)(x) + \tilde{C}_2^d(x)(x) + A_3^d(x)(x) + \tilde{C}_3^d(x)(x) + \alpha, \end{aligned} \tag{30}$$

where $\alpha := g(0)$. If we set $x = y$ in (26) we get that $f = g$. Putting $x = 0$ in (26) we obtain

$$g(y) = [\tilde{h}(y) + \Phi(0) + \Psi(y)](y) + \alpha.$$

Now we substitute (27), (28), and (29) in the above equation, replace y by x and get

$$\begin{aligned} g(x) \;=\; & A_0(x) + B_0(x) + \tilde{C}_0(x) + B_1(x)(x) + \tilde{C}_1(x)(x) \\ & + B_2^d(x)(x) + \tilde{C}_2^d(x)(x) + B_3^d(x)(x) + \tilde{C}_3^d(x)(x) + \alpha. \end{aligned}$$

The equality $f = g$ implies that

$$A_i^d(x)(x) = B_i^d(x)(x), \tag{31}$$

$i \in \{1, 2, 3\}$, $x \in K'$. Putting the formulas for f, g, $\tilde{h}$, Φ, Ψ in (26) and taking into account (31) we obtain

$$\begin{aligned}&- A_1(x)(y) + B_1(y)(x) - \tilde{C}_1(x)(y) + \tilde{C}_1(y)(x) - A_2^d(x)(y) + B_2^d(y)(x)\\&- \tilde{C}_2^d(x)(y) + 2\tilde{C}_2(x,y)(x) - 2\tilde{C}_2(x,y)(y) + \tilde{C}_2^d(y)(x)\\&- A_3^d(x)(y) + B_3^d(y)(x) - \tilde{C}_3^d(x)(y) + 3\tilde{C}_3(x^2,y)(x)\\&- 3\tilde{C}_3(x^2,y)(y) + 3\tilde{C}_3(x,y^2)(x) - 3\tilde{C}_3(x,y^2)(y) + \tilde{C}_3^d(y)(x) = 0.\end{aligned} \tag{32}$$

The left-hand side of this equation is the sum of functions homogeneous of i-th order with respect to the variables x and y, $i \in \{1,2,3\}$. Considering terms with third order homogeneity with respect to x we have

$$-A_3^d(x)(y) - \tilde{C}_3^d(x)(y) + 3\tilde{C}_3(x^2,y)(x) = 0. \tag{33}$$

Interchanging x and y we obtain

$$-A_3^d(y)(x) - \tilde{C}_3^d(y)(x) + 3\tilde{C}_3(x,y^2)(y) = 0. \tag{34}$$

Considering functions homogeneous of third degree with respect to y on both sides of (32), we find

$$B_3^d(y)(x) - 3\tilde{C}_3(x,y^2)(y) + \tilde{C}_3^d(y)(x) = 0. \tag{35}$$

Adding (34) to (35) we get

$$A_3^d(y)(x) = B_3^d(y)(x), \tag{36}$$

$x, y \in K'$. Hence by absorption of K', and by the properties of multiadditive and symmetric functions we get

$$A_3 = B_3.$$

Now we consider functions homogeneous of the second degree with respect to x in (32), and get

$$-A_2^d(x)(y) - \tilde{C}_2^d(x)(y) + 2\tilde{C}_2(x,y)(x) - 3\tilde{C}_3(x^2,y)(y) + 3\tilde{C}_3(x,y^2)(x) = 0. \tag{37}$$

Analogously we obtain

$$A_2^d(y)(x) = B_2^d(y)(x), \tag{38}$$

$x, y \in K'$, and the repeated reasoning give us $A_2 = B_2$. We look at terms which are homogeneous of i-th degree with respect to y in (37), $i \in \{1,2\}$. For $i = 2$ we get

$$-3\tilde{C}_3(x^2,y)(y) + 3\tilde{C}_3(x,y^2)(x) = 0, \tag{39}$$

and for $i = 1$

$$-A_2^d(x)(y) - \tilde{C}_2^d(x)(y) + 2\tilde{C}_2(x,y)(x) = 0. \tag{40}$$

At the end, we consider homogeneous functions with respect to x in (32) which yields

$$
\begin{aligned}
&-A_1(x)(y) + B_1(y)(x) - \tilde{C}_1(x)(y) + \tilde{C}_1(y)(x)\\
&+A_2^d(y)(x) + \tilde{C}_2^d(y)(x) - 2\tilde{C}_2(x,y)(y)\\
&+A_3^d(y)(x) + \tilde{C}_3^d(y)(x) - 3\tilde{C}_3(x,y^2)(y) = 0,
\end{aligned}
$$

hence we get the equality of homogeneous terms with respect to y

$$-A_1(x)(y) + B_1(y)(x) - \tilde{C}_1(x)(y) + \tilde{C}_1(y)(x) = 0. \tag{41}$$

Replacing $\tilde{C}_i^d(x)$ by $C_i^d(\frac{x}{2})$, $i \in \{0,\dots,3\}$, our result is the following. (The proof of the second part is immediate; it is sufficient to show that P_f, P_g, P_h, P_Φ and P_Ψ satisfy (25).)

Lemma 3.9. *Let $\emptyset \neq K \subset X$ be convex and symmetric by $0 \in \operatorname{alg\,int} K$. If the functions $f,\ g : K \to Y$, $h,\ \Phi,\ \Psi : K \to SA^1(X,Y)$ satisfy the equation*

$$f(x) - g(y) = \left[h\left(\frac{x+y}{2}\right) + \Phi(x) + \Psi(y)\right](x-y) \tag{25}$$

for every $x,\ y \in K$, then there exist a set $\hat{K} \subset K$ such that $0 \in \operatorname{alg\,int} \hat{K}$ and functions A_k, $C_k \in SA^k(X; SA^1(X;Y))$, $k \in \{0,\dots,3\}$, $B_i \in SA^i(X; SA^1(X;Y))$, $i \in \{0,1\}$, such that the conditions

$$A_i^d(x)(y) + 2^{-i}C_i^d(x)(y) = i2^{-i}C_i(x^{i-1},y)(x), \qquad i \in \{2,3\} \tag{w 1}$$

$$C_3(x^2,y)(y) = C_3(x,y^2)(x), \tag{w 2}$$

$$B_1(y)(x) - A_1(x)(y) = 2^{-1}C_1(x)(y) - 2^{-1}C_1(y)(x) \tag{w 3}$$

hold for every $x,\ y \in X$ and a constant $\alpha \in Y$ such that for every $x \in \hat{K}$

$$
\begin{aligned}
\Phi(x) &= A_0 + A_1(x) + A_2^d(x) + A_3^d(x),\\
\Psi(x) &= B_0 + B_1(x) + A_2^d(x) + A_3^d(x),\\
h(x) &= C_0 + C_1(x) + C_2^d(x) + C_3^d(x),\\
f(x) &= g(x) = A_0(x) + B_0(x) + C_0(x) + A_1(x)(x) + 2^{-1}C_1(x)(x)\\
&\quad + A_2^d(x)(x) + 2^{-2}C_2^d(x)(x) + A_3^d(x)(x) + 2^{-3}C_3^d(x)(x) + \alpha.
\end{aligned}
$$

Conversely, if the functions $A_k, C_k \in SA^k(X; SA^1(X;Y))$, $k \in \{0,\dots,3\}$, $B_i \in SA^i(X; SA^1(X;Y))$, $i \in \{0,1\}$, satisfy the conditions (w 1)–(w 3)*, and $\alpha \in Y$ is a constant, then the functions P_f, $P_g : X \to Y$, P_h, P_Φ, $P_\Psi : X \to SA^1(X,Y)$ defined by*

$$
\begin{aligned}
P_\Phi(x) &= A_0 + A_1(x) + A_2^d(x) + A_3^d(x),\\
P_\Psi(x) &= B_0 + B_1(x) + A_2^d(x) + A_3^d(x),\\
P_h(x) &= C_0 + C_1(x) + C_2^d(x) + C_3^d(x),
\end{aligned}
$$

$$P_f(x) = P_g(x) = A_0(x) + B_0(x) + C_0(x) + A_1(x)(x) + 2^{-1}C_1(x)(x) \\ +A_2^d(x)(x) + 2^{-2}C_2^d(x)(x) + A_3^d(x)(x) + 2^{-3}C_3^d(x)(x) + \alpha,$$

satisfy the equation

$$P_f(x) - P_g(y) = \left[P_h\left(\frac{x+y}{2}\right) + P_\Phi(x) + P_\Psi(y)\right](x-y) \tag{25}$$

for every $x,\ y \in X$.

Now we show that the assumption of symmetry of K with 0 can be omitted. Precisely, we want to prove that, if K is a convex set with $0 \in \operatorname{alg\,int} K$, and equation (25) holds for every $x,\ y \in K$, then the solutions have the properties defined in Lemma 3.9. Let $\hat{K} \subset K$ be a convex set such that $0 \in \operatorname{alg\,int} \hat{K}$, and the polynomial functions $P_f,\ P_g : X \to Y,\ P_h,\ P_\Phi,\ P_\Psi : X \to SA^1(X,Y)$ satisfy $P_f|_{\hat{K}} = f|_{\hat{K}},\ P_g|_{\hat{K}} = g|_{\hat{K}},\ P_h|_{\hat{K}} = h|_{\hat{K}},\ P_\Phi|_{\hat{K}} = \Phi|_{\hat{K}},\ P_\Psi|_{\hat{K}} = \Psi|_{\hat{K}}$. Suppose further that

$$P_f(x) - P_g(y) = \left[P_h\left(\frac{x+y}{2}\right) + P_\Phi(x) + P_\Psi(y)\right](x-y) \tag{25}$$

holds for every $x,\ y \in X$. We take $\hat{K} \subset K$ as in Lemma 3.9.

Let

$$\hat{f} = f - P_f|_K,\ \ \hat{g} = g - P_g|_K,\ \ \hat{h} = h - P_h|_K,\ \ \hat{\Phi} = \Phi - P_\Phi|_K\ \ \hat{\Psi} = \Psi - P_\Psi|_K.$$

Then $(\hat{f},\ \hat{g},\ \hat{h},\ \hat{\Phi},\ \hat{\Psi})$ is a solution of (25) for every $x,\ y \in K$, and

$$\hat{f}|_{\hat{K}} = \hat{g}|_{\hat{K}} = 0 \in Y^{\hat{K}},\ \ \hat{h}|_{\hat{K}} = \hat{\Phi}|_{\hat{K}} = \hat{\Psi}|_{\hat{K}} = 0 \in SA^1(X,Y)^{\hat{K}}.$$

We are going to show that $\hat{f}$, $\hat{g}$, $\hat{\Phi}$, $\hat{\Psi}$ vanish on K and $\hat{h}$ vanishes, except maybe at the boundary points of K.

Lemma 3.10. *Let K be a convex subset of X such that $0 \in \operatorname{alg\,int} K$ and $\hat{K}$ be a convex subset of K such that $0 \in \operatorname{alg\,int} \hat{K}$. If $(\hat{f},\ \hat{g},\ \hat{h},\ \hat{\Phi},\ \hat{\Psi})$ is a solution of*

$$\hat{f}(x) - \hat{g}(y) = \left[\hat{h}\left(\frac{x+y}{2}\right) + \hat{\Phi}(x) + \hat{\Psi}(y)\right](x-y)$$

for every $x,\ y \in K$ and

$$\hat{f}|_{\hat{K}} = \hat{g}|_{\hat{K}} = 0 \in Y^{\hat{K}},\ \ \hat{h}|_{\hat{K}} = \hat{\Phi}|_{\hat{K}} = \hat{\Psi}|_{\hat{K}} = 0 \in SA^1(X,Y)^{\hat{K}},$$

then $\hat{f} = \hat{g} = 0 \in Y^K$, $\hat{\Phi} = \hat{\Psi} = 0 \in SA^1(X,Y)^K$ and $\hat{h}(x) = 0$ for every $x \in K^0$.

Proof. We proceed analogously to the proof of Lemma 3.6. First, we analyze the behaviour of the functions $\hat{f}$, $\hat{g}$, $\hat{h}$, $\hat{\Phi}$, and $\hat{\Psi}$ on the straight lines $l_z = \mathbb{K}z$, $z \neq 0$, intersected with K. Fix $z \in X \setminus \{0\}$. Let $K_z = \{q \in \mathbb{K} : qz \in K\}$. We define new functions $\hat{f}_z,\ \hat{g}_z : K_z \to Y$, $\hat{h}_z,\ \hat{\Phi}_z,\ \hat{\Psi}_z : K_z \to SA^1(l_z, Y)$ by

$$\hat{f}_z(q) = \hat{f}(qz),\ \ \hat{h}_z(q)(rz) = \hat{h}(qz)(rz),\ \ \hat{\Phi}_z(q)(rz) = \hat{\Phi}(qz)(rz),$$

$$\hat{\Psi}_z(q)(rz) = \hat{\Psi}(qz)(rz).$$

It is easy to see that $(\hat{f}_z,\ \hat{g}_z,\ \hat{h}_z,\ \hat{\Phi}_z,\ \hat{\Psi}_z)$ satisfies the equation

$$\hat{f}_z(q) - \hat{g}_z(r) = (q-r)\left[\hat{h}_z\left(\frac{q+r}{2}\right) + \hat{\Phi}_z(q) + \hat{\Psi}_z(r)\right](z) \tag{42}$$

for every $q,\ r \in K_z$, and $\hat{f}_z|_{\hat{K}_z} = \hat{g}_z|_{\hat{K}_z} = 0 \in Y^{\hat{K}_z}$, $\hat{h}_z|_{\hat{K}_z} = \hat{\Phi}_z|_{\hat{K}_z} = \hat{\Psi}_z|_{\hat{K}_z} = 0 \in SA^1(X,Y)^{\hat{K}_z}$, where $\hat{K}_z = \{q \in \mathbb{K} : qz \in \hat{K}\}$. Without loss of generality we can assume that $\hat{K}_z$ is a maximal interval of elements of $\mathbb{K}$ such that $0 \in \operatorname{alg\,int} \hat{K}_z$, and the functions $\hat{f}_z$, $\hat{g}_z$, $\hat{h}_z$, $\hat{\Phi}_z$, $\hat{\Psi}_z$ vanish on $\hat{K}_z$.

On the contrary, we assume that $s_1 = \sup \hat{K}_z < s_2 = \sup K_z$. Let $\varepsilon \in (0, s_1)$ be chosen such that $s_1 + \varepsilon < s_2$. Thus we get $\frac{q}{2} < s_1$, $q \in [s_1, s_1+\varepsilon) \cap \mathbb{Q} =: J$. We consider a $r \in \hat{K}_z$ such that

$$r < \frac{s_1}{2}, \quad \frac{q+r}{2} < s_1 \tag{43}$$

for all $q \in J$. Let us note that (43) is satisfied by $r \in (0,\delta) \cap \mathbb{Q}$ for any $\delta > 0$. We can rewrite (42) in the form

$$(r-q)\hat{\Phi}_z(q)(z) + \hat{f}_z(q) = (q-r)\left[\hat{h}_z\left(\frac{q+r}{2}\right) + \hat{\Psi}_z(r)\right](z) + \hat{g}_z(r)$$

for every $q \in J$, and $r \in \hat{K}_z$ such that (43) hold. By our assumptions we get that the right-hand side of the above equation vanishes for every $q \in J$. It follows that

$$r\hat{\Phi}_z(q)(z) - q\hat{\Phi}_z(q)(z) + \hat{f}_z(q) = 0. \tag{44}$$

Taking $r = 0$ in (42) we obtain

$$-q\hat{\Phi}_z(q)(z) + \hat{f}_z(q) = 0.$$

Subtracting the above equations side by side we have $\hat{\Phi}_z(q)(z) = 0$, $q \in J$, hence $\hat{\Phi}_z(q) = 0 \in SA^1(l_z;Y)$, $q \in J$. (44) implies that $\hat{f}_z(q) = 0$, $q \in J$, whence $\hat{\Phi}_z|_{\hat{K}_z \cup J} = 0$, and $\hat{f}_z|_{\hat{K}_z \cup J} = 0$.

Repeated analysis of the equation (derived from (42))

$$(r-q)\hat{\Psi}_z(r)(z) - \hat{g}_z(q) = (q-r)\left[\hat{h}_z\left(\frac{q+r}{2}\right) + \hat{\Phi}_z(q)\right](z) - \hat{f}_z(q)$$

gives us $\hat{\Psi}_z|_{\hat{K}_z \cup J} = 0$, and $\hat{g}_z|_{\hat{K}_z \cup J} = 0$. Fix a $p \in J$, and let $0 < q < p < r < s_1+\varepsilon$ be rational numbers such that $p = \frac{q+r}{2}$. By the above reasoning we have (since q, $r \in \hat{K}_z \cup J$)

$$(q-r)\hat{h}_z(p)(z) = 0,$$

hence $\hat{h}_z(p)(z) = 0$. Since p is arbitrary it follows that $\hat{h}_z(p) = 0 \in SA^1(l_z;Y)$, $p \in J$ and consequently $\hat{h}_z|_{\hat{K}_z \cup J} = 0$, $k \in \{0,\dots,n\}$. This contradicts the formula for s_1, so we get $s_1 = s_2$. An analogous procedure can be applied in the case $\inf \hat{K}_z = \inf K_z$.

We can now proceed similarly to the proof of Lemma 3.6 by showing that if $q = \sup K_z \in K_z$ ($q = \inf K_z \in K_z$), then $\hat{f}_z(q) = \hat{g}_z(q) = \hat{\Phi}_z(q) = \hat{\Psi}_z(q) = 0$, whence $\hat{f}(z) = 0 = \hat{g}(z)$, $z \in K$.

An argument analogous to that one following Lemma 3.6 gives us $\hat{\Phi}(z) = \hat{\Psi}(z) = 0 \in SA^1(X;Y)$, and $h(z) = 0 \in SA^1(X;Y)$ for every $z \in K^0$.

We can now summarize our results in the following

Theorem 3.11. *Let $K \subset X$ be a convex and absorbing set. If the functions f, g, h, Φ, $\Psi : K \to SA^1(X,Y)$ satisfy for every x, $y \in K$ the equation*

$$f(x) - g(y) = \left[h\left(\frac{x+y}{2}\right) + \Phi(x) + \Psi(y)\right](x-y), \tag{25}$$

then there exist functions A_i, $C_i \in SA^i(X;SA^1(X;Y))$, $i \in \{0,\ldots,3\}$, $B_k \in SA^k(X;SA^1(X;Y))$, $k \in \{0,1\}$, such that the conditions

$$A_i^d(x)(y) + 2^{-i}C_i^d(x)(y) = i2^{-i}C_i(x^{i-1},y)(x), \qquad i \in \{2,3\} \tag{w 1}$$

$$C_3(x^2,y)(y) = C_3(x,y^2)(x), \tag{w 2}$$

$$B_1(y)(x) - A_1(x)(y) = 2^{-1}C_1(x)(y) - 2^{-1}C_1(y)(x) \tag{w 3}$$

hold for every x, $y \in X$, and a constant $\alpha \in Y$ such that for every $x \in K$

$$\begin{aligned}
\Phi(x) &= A_0 + A_1(x) + A_2^d(x) + A_3^d(x),\\
\Psi(x) &= B_0 + B_1(x) + A_2^d(x) + A_3^d(x),\\
f(x) = g(x) &= A_0(x) + B_0(x) + C_0(x) + A_1(x)(x) + 2^{-1}C_1(x)(x)\\
&\quad + A_2^d(x)(x) + 2^{-2}C_2^d(x)(x) + A_3^d(x)(x) + 2^{-3}C_3^d(x)(x) + \alpha,
\end{aligned}$$

and if $x \in K^0$ then

$$h(x) = C_0 + C_1(x) + C_2^d(x) + C_3^d(x).$$

Remark 3.12. Analogously to Theorem 3.7 this result can be conversed under the additional assumption that the edge of K consists only of the extreme points. Therefore, if h is defined arbitrarily on its extreme points, and the functions f, g, h, Φ, Ψ are defined by the formulas from Theorem 3.11, and the conditions (w 1)–(w 3) are satisfied, then it is easy to compute that (f, g, h, Φ, Ψ) satisfies (25).

It can be seen that, taking $X = \mathbb{R}$, $\mathbb{K} = \mathbb{Q}$, and $K = I$ as a nonempty interval we have the same solutions as in [3] (obviously we replace a homomorphism by multiplication). Analogously to the first part of this section, let us assume that $0 \in \operatorname{alg\,int} I$, whence $0 \in \operatorname{int} I$. This can be done since without the loss of generality a solution in the general case can be obtained by the composition of the solution in the special case with the translation by a point $x_0 \in \operatorname{int} I$. In this case the edge of I consists of the endpoints of the interval I and these points are extreme whenever they belong to I. Moreover $I^0 = \operatorname{int} I$. It is obvious that the solutions have the forms as in Lemma 3.9, and they satisfy the conditions (w 1)–(w 3), where instead of $D(x)(y)$ we have $D(x)y$ for all $D \in \{A_i^d, C_i^d, B_k^d\}$, $i \in \{0, \ldots, 3\}$, $k \in \{0, 1\}$, x, $y \in X$.

Taking $x = y$ in (w 3) we obtain

$$A_1 = B_1. \tag{45}$$

Hence and from (w 3) again we get for x, $y \neq 0$

$$\frac{A_1(y)}{y} + \frac{2^{-1}C_1(y)}{y} = \frac{A_1(x)}{x} + \frac{2^{-1}C_1(x)}{x},$$

whence there exists a constant $c \in \mathbb{R}$ such that

$$A_1(x) = cx - 2^{-1}C_1(x), \tag{46}$$

$x \in I$. Setting $y = x$ in (w 1) we obtain

$$A_i^d(x) = (i-1)2^{-i}C_i^d(x), \tag{47}$$

$x \in I$, $i \in \{2, 3\}$. Now we substitute this equality in (w 1) again. So we have for all x, $y \neq 0$

$$\frac{C_i(x^{i-1}, y)}{x^{i-1}y} = \frac{C_i^d(x)}{x^i} \tag{48}$$

and interchanging x and y we get

$$\frac{C_i(x, y^{i-1})}{xy^{i-1}} = \frac{C_i^d(y)}{y^i}, \tag{49}$$

x, $y \neq 0$. The comparison of the above equations and the application of the symmetry of C_2 give us for $i = 2$

$$\frac{C_2^d(x)}{x^2} = \frac{C_2^d(y)}{y^2},$$

therefore there exists a real constant b such that

$$C_2^d(x) := 4bx^2,$$

and from (47) we get

$$A_2^d(x) = bx^2.$$

Following (w 2) for x, $y \neq 0$ we get

$$\frac{C_3(x, y^2)}{xy^2} = \frac{C_3(x^2, y)}{x^2y},$$

and taking into account (48), and (49) for $i = 3$ we have

$$\frac{C_3^d(x)}{x^3} = \frac{C_3^d(y)}{y^3}.$$

Consequently there exists an $a \in \mathbb{R}$ such that

$$C_3^d(x) := 8ax^3$$

for all $x \in I$. From (47) it follows that

$$A_3^d(x) = 2ax^3.$$

We can now write the functions f, g, Φ, Ψ in the following for every $x \in I$ as

$$\begin{aligned} f(x) = g(x) &= 3ax^4 + 2bx^3 + cx^2 + (A_0 + B_0 + C_0)x + \alpha, \\ \Phi(x) &= 2ax^3 + bx^2 + cx - \frac{1}{2}C_1(x) + A_0, \\ \Psi(x) &= 2ax^3 + bx^2 + cx - \frac{1}{2}C_1(x) + B_0, \end{aligned}$$

and for every $x \in \operatorname{int} I$

$$h(x) = 8ax^3 + 4bx^2 + C_1(x) + C_0.$$

Interchanging x and y in (25) we get

$$f(y) - g(x) = \left[h\left(\frac{x+y}{2}\right) + \Phi(y) + \Psi(x)\right](y - x). \tag{50}$$

Adding side by side equations (25), and (50), and applying the equality $f = g$ we have

$$\Psi(x) - \Phi(x) = \Psi(y) - \Phi(y),$$

x, $y \in I$, hence

$$\Psi(x) = \Phi(x) - 2\gamma \tag{51}$$

for a constant $\gamma \in \mathbb{R}$. Let us define constants $d := A_0 + B_0 + C_0$ and $\beta := \frac{A_0+B_0}{2}$ and an additive function $A : \mathbb{R} \to \mathbb{R}$ by $A := \frac{1}{2}C_1$. Then applying (51) we finally obtain for all $x \in I$

$$\begin{aligned} f(x) = g(x) &= 3ax^4 + 2bx^3 + cx^2 + dx + \alpha, \\ \Phi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta + \gamma, \\ \Psi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta - \gamma \end{aligned}$$

and for all $x \in \operatorname{int} I$

$$h(x) = 8ax^3 + 4bx^2 + 2A(x) + d - 2\beta$$

for constants a, b, c, α, β, $\gamma \in \mathbb{R}$, and an additive function $A : \mathbb{R} \to \mathbb{R}$. The above results are stated in the following theorem (cf. [3; Theorem 1]):

Theorem 3.13. *Let $I \subset \mathbb{R}$ be a nonempty interval. The functions f, g, h, Φ, $\Psi : I \to \mathbb{R}$ satisfy the equation for every x, $y \in I$*

$$f(x) - g(y) = \left[h\left(\frac{x+y}{2}\right) + \Phi(x) + \Psi(y)\right](x-y), \tag{25}$$

if and only if

$$\begin{aligned}
f(x) = g(x) &= 3ax^4 + 2bx^3 + cx^2 + dx + \alpha, && x \in I,\\
h(x) &= 8ax^3 + 4bx^2 + 2A(x) + d - 2\beta, && x \in \operatorname{int} I,\\
\Phi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta + \gamma, && x \in I,\\
\Psi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta - \gamma, && x \in I,
\end{aligned}$$

where a, b, c, α, β, $\gamma \in \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$ is an additive function. If one of the endpoints of I belongs to I, then h can be defined arbitrarily in it.

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