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Orthogonal stability of additive type equations

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Summary. Suppose that $(\mathcal{X}, \perp)$ is a symmetric orthogonality module and $\mathcal{Y}$ a Banach module over a unital Banach algebra $\mathcal{A}$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying

$$\|f(ax_1 + ax_2) + (-1)^{k+1} f(ax_1 - ax_2) - 2af(x_k)\| \le \epsilon,$$

for $k = 1$ or 2, for some $\epsilon \ge 0$, for all a in the unit sphere $\mathcal{A}_1$ of $\mathcal{A}$ and all $x_1, x_2 \in \mathcal{X}$ with $x_1 \perp x_2$. Assume that the mapping $t \mapsto f(tx)$ is continuous for each fixed $x \in \mathcal{X}$. Then there exists a unique $\mathcal{A}$-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying $T(ax) = aT(x), a \in \mathcal{A}, x \in \mathcal{X}$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{5}{2}\epsilon,$$

for all $x \in \mathcal{X}$.

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1. Introduction

There are a number of definitions of orthogonality in vector spaces, in addition to the usual one for inner product spaces. They have appeared in the literature during the past century. Many of these are mentioned in the article [8] by H. Drljević. In giving his axiomatic definition of orthogonality, J. Rätz (cf. [26]) modified the definition of S. Gudder and D. Strawther from [10] and arrived at the following.

Suppose that $\mathcal{X}$ is a real vector space (algebraic module) with $\dim \mathcal{X} \ge 2$ and $\perp$ is a binary relation on $\mathcal{X}$ with the following properties:

(O1) *totality of* $\perp$ *for zero*: $x \perp 0, 0 \perp x$ for all $x \in \mathcal{X}$;

(O2) *independence*: if $x, y \in \mathcal{X} - \{0\}, x \perp y$, then x, y are linearly independent;

(O3) *homogeneity*: if $x, y \in \mathcal{X}, x \perp y$, then $\alpha x \perp \beta y$ for all α, β in the real number field $\mathbb{R}$;

(O4) *the Thalesian property*: Let P be a 2-dimensional subspace of $\mathcal{X}$. If $x \in P$ and λ is in the set of nonnegative real numbers $\mathbb{R}_+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(\mathcal{X}, \perp)$ is called an *orthogonality space (module)*. By an *orthogonality normed space* (normed module) we mean an orthogonality space (module) having a normed space (normed module) structure.

J. Rätz pointed out that his definition of orthogonality space is more restrictive than that given by S. Gudder and D. Strawther, but he showed in [26] that his definition includes the following basic examples (see also [25]):

(i) The trivial orthogonality on a vector space $\mathcal{X}$ defined by (O1), and for non-zero elements $x, y \in \mathcal{X}$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(\mathcal{X}, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff–James orthogonality on a normed space $(\mathcal{X}, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$; cf. [13] (see also [7]).

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in \mathcal{X}$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff–James orthogonality is symmetric (see [1]).

Let $\mathcal{X}$ be a vector space (an orthogonality space) and $(\mathcal{Y}, +)$ be an abelian group. Then a mapping $f : \mathcal{X} \to \mathcal{Y}$ is called

(i) *(orthogonally) additive* if it satisfies the so-called (orthogonally) additive functional equation $f(x + y) = f(y) + f(x)$ for all $x, y \in \mathcal{X}$ (with $x \perp y$);

(ii) *(orthogonally) quadratic* if it satisfies the so-called (orthogonally) Jordan–von Neumann quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in \mathcal{X}$ (with $x \perp y$).

In 1940, S. M. Ulam posed in [27] the following problem: "Give conditions in order for a group homomorphism near an approximately homomorphism to exist." The Ulam problem was first solved in the context of Banach spaces by D. H. Hyers (see [11]) in 1941. In 1951, D. G. Bourgin treated the Ulam problem for additive mappings (cf. [4]). In 1978, Th. M. Rassias in [23] extended the theorem of Hyers by considering an unbounded Cauchy difference. Beginning around the year 1980 the subject of stability of functional equations has been investigated by a number of mathematicians. The reader is referred to [5, 6, 12, 15, 24] for a comprehensive account of the subject.

In [16] S.-M. Jung and P. K. Sahoo proved the stability of the quadratic equation of Pexider type (see also [14]). As a corollary one can conclude the stability of the so-called additive type equations $f(x + y) + f(x - y) = 2f(x)$ and $f(x + y) - f(x - y) = 2f(y)$. R. Ger, J. Sikorska in [9] and the first author in [18, 19] studied the orthogonal stability of (Pexiderized) Cauchy and quadratic functional equations.

Our main aim in this paper is to consider the orthogonal stability of additive type equations in Banach modules in the spirit of Hyers–Ulam stability.

2. Orthogonal stability in Banach modules

Applying some ideas from [9, 16, 21] and [19], we deal with the conditional stability problem for

$$f(ax+ay)+f(ax-ay)=2af(x) \qquad x\perp y$$

and

$$f(ax+ay)-f(ax-ay)=2af(y) \qquad x\perp y$$

where $a\in\mathcal{A}$, $x,y\in\mathcal{X}$ and $\perp$ is a symmetric orthogonality in the sense of J. Rätz. We will use a sequence of Hyers' type [11] which is a useful tool in the theory of stability of equations. In the first two propositions we will describe the solutions of additive type equations.

Throughout this section, $\mathcal{A}$ is a unital real Banach algebra with unit 1 and unit sphere $\mathcal{A}_1$, and $(\mathcal{X},\perp)$ denotes an orthogonality normed real left $\mathcal{A}$-module with the property $1x=x$ and $(\mathcal{Y},\|.\|)$ is a real Banach left $\mathcal{A}$-module. By definition, a real left $\mathcal{A}$-module is among other things a real vector space ([2], p. 49, Definition 11). The reader is referred to [2] for more details on the theory of normed modules.

Proposition 2.1. *If $\varphi:\mathcal{X}\to\mathcal{Y}$ fulfills $\varphi(x+y)+\varphi(x-y)=2\varphi(x)$ for all $x,y\in\mathcal{X}$ with $x\perp y$ and if $\perp$ is symmetric, then $\varphi(x)-\varphi(0)$ is orthogonally additive.*

Proof. Setting $x=0$, we get $-\varphi(y)=\varphi(-y)-2\varphi(0), y\in\mathcal{X}$. Let $x\perp y$. Then $y\perp x$ and so $\varphi(y-x)=-\varphi(y+x)+2\varphi(y)$. Hence $\varphi(x+y)=-\varphi(x-y)+2\varphi(x)=(\varphi(y-x)-2\varphi(0))+2\varphi(x)=(-\varphi(y+x)+2\varphi(y))-2\varphi(0)+2\varphi(x)$. Thus $\varphi(x+y)-\varphi(0)=(\varphi(x)-\varphi(0))+(\varphi(y)-\varphi(0))$, so that $\varphi(x)-\varphi(0)$ is orthogonally additive. □

Proposition 2.2. *If $\varphi:\mathcal{X}\to\mathcal{Y}$ fulfills $\varphi(x+y)-\varphi(x-y)=2\varphi(y)$ for all $x,y\in\mathcal{X}$ with $x\perp y$ and if $\perp$ is symmetric, then φ is orthogonally additive.*

Proof. Setting $x=0$, we obtain $\varphi(y)-\varphi(-y)=2\varphi(y)$ or $\varphi(-y)=-\varphi(y), y\in\mathcal{X}$. Suppose that $x\perp y$. By the assumption one has

$$\varphi(x+y)-\varphi(x-y)=2\varphi(y). \tag{2.1}$$

Since $\perp$ is symmetric, $y\perp x$ and so

$$2\varphi(x)=\varphi(x+y)-\varphi(y-x)=\varphi(x+y)+\varphi(x-y). \tag{2.2}$$

It follows from (2.1) and (2.2) that $\varphi(x)+\varphi(y)=\varphi(x+y)$ for all $x,y\in\mathcal{X}$ with $x\perp y$. □

Now we establish the orthogonal stability of the equation $f(x+y)+f(x-y)=2f(x)$.

Proposition 2.3. *Suppose that* $\perp$ *is symmetric on* $\mathcal{X}$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping satisfying*

$$\big\|f(ax+ay)+f(ax-ay)-2af(x)\big\| \le \epsilon, \tag{2.3}$$

for some $\epsilon \ge 0$, *for all* $a \in \mathcal{A}_1$ *and all* $x, y \in \mathcal{X}$ *with* $x \perp y$. *Assume that* f *is odd. Then there exists a unique additive mapping* $T : \mathcal{X} \to \mathcal{Y}$ *satisfying* $T(ax) = aT(x)$, $a \in \mathcal{A}_1$, $x \in \mathcal{X}$ *such that*

$$\big\|f(x) - T(x)\big\| \le 2\epsilon,$$

for all $x \in \mathcal{X}$. *Moreover, if the mapping* $t \mapsto f(tx)$ *is continuous for each fixed* $x \in \mathcal{X}$, *then* T *is* $\mathcal{A}$*-linear.*

Proof. Fix $x \in \mathcal{X}$ and $a \in \mathcal{A}_1$. By (O4), there exists $y_0 \in \mathcal{X}$ such that $x \perp y_0$ and $x + y_0 \perp x - y_0$. Since $\perp$ is symmetric one has $x - y_0 \perp x + y_0$, too. Using inequality (2.3) and the oddness of f we get

$$\big\|f(x+y_0)+f(x-y_0)-2f(x)\big\| \le \epsilon,$$

$$\big\|f(2ax)+f(2ay_0)-2af(x+y_0)\big\| \le \epsilon,$$

$$\big\|f(2ax)-f(2ay_0)-2af(x-y_0)\big\| \le \epsilon.$$

Thus

$$\begin{aligned}\big\|f(2ax)-2af(x)\big\| \le{}& \big\|af(x+y_0)+af(x-y_0)-2af(x)\big\| \\ &+\frac{1}{2}\big\|f(2ax)+f(2ay_0)-2af(x+y_0)\big\| \\ &+\frac{1}{2}\big\|f(2ax)-f(2ay_0)-2af(x-y_0)\big\| \\ \le{}& 2\epsilon,\end{aligned}$$

whence

$$\big\|f(2ax)-2af(x)\big\| \le 2\epsilon. \tag{2.4}$$

Using (2.4) with $a = 1$ and induction on n one can verify that

$$\big\|2^{-n}f(2^n x) - f(x)\big\| \le 2\epsilon \sum_{k=1}^{n}\Big(\frac{1}{2}\Big)^k,$$

for all n, and

$$\big\|2^{-n}f(2^n x) - 2^{-m}f(2^m x)\big\| \le 2\epsilon \sum_{k=m+1}^{n}\Big(\frac{1}{2}\Big)^k$$

for all $m < n$. Thus $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence in the Banach module $\mathcal{Y}$. Hence $\lim\limits_{n\to\infty} 2^{-n}f(2^n x)$ exists and the mapping $\varphi(x) := \lim\limits_{n\to\infty} 2^{-n}f(2^n x)$ from $\mathcal{X}$

into $\mathcal{Y}$ satisfies

$$\big\|f(x)-\varphi(x)\big\| \le 2\epsilon,$$

for all $x \in \mathcal{X}$. Let $x, y \in \mathcal{X}$ with $x \perp y$. Applying inequality (2.3) and (O3) we obtain

$$\big\|2^{-n}f(2^n(x+y)) + 2^{-n}f(2^n(x-y)) - 2^{-n+1}f(2^nx)\big\| \le 2^{-n}\epsilon.$$

Letting n tend to infinity we deduce that $\varphi(x+y)+\varphi(x-y)-2\varphi(x)=0$. Moreover, $\varphi(0)=\lim\limits_{n\to\infty} 2^{-n}f(2^n.0)=0$. Using Proposition 2.1 we conclude that φ is an orthogonally additive mapping. Given $a \in \mathcal{A}_1$ and $x \in \mathcal{X}$, replace x in (2.4) by 2^nx, where $n \in \mathbb{N}$. Then

$$\Big\|\frac{1}{2^{n+1}}f(2^{n+1}ax) - \frac{1}{2^n}af(2^nx)\Big\| \le \frac{1}{2^n}\epsilon.$$

Letting n tend to infinity we conclude that $\varphi(ax)=a\varphi(x)$.

Since f is odd, so is φ, whence from Corollary 7 of [26], φ, denoted there by T, is additive. Thus

$$\big\|T(x)-f(x)\big\| \le 2\epsilon.$$

If $T' : \mathcal{X} \to \mathcal{Y}$ is another additive mapping satisfying $\big\|T'(x)-f(x)\big\| \le 2\epsilon$, then $\big\|T(x)-T'(x)\big\| \le \frac{1}{n}\big(\big\|T(nx)-f(nx)\big\| + \big\|T'(nx)-f(nx)\big\|\big) \le \frac{4\epsilon}{n}$. Letting n tend to infinity we infer that $T=T'$ which proves the uniqueness assertion.

Now assume that for each fixed $x \in \mathcal{X}$ the mapping $t \mapsto f(tx)$ is continuous. By the same argument as in the proof of the theorem of [23], we can deduce that T is $\mathbb{R}$-linear.

Now for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$ we have

$$T(ax) = T\bigg(\|a\|\frac{a}{\|a\|}x\bigg) = \|a\|T\bigg(\frac{a}{\|a\|}x\bigg) = \|a\|\frac{a}{\|a\|}T(x) = aT(x). \qquad \square$$

Proposition 2.4. *Suppose that* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping satisfying* (2.3) *for some* $\epsilon \ge 0$, *for* $a = 1$ *and all* $x, y \in \mathcal{X}$ *with* $x \perp y$. *Assume that* f *is even and* $f(0)=0$. *Then* $\|f(x)\| \le \frac{\epsilon}{2}$ *for all* $x \in \mathcal{X}$.

Proof. Setting $x=0$ in (2.3) we get $\big\|f(y)+f(-y)-2f(0)\big\| \le \epsilon$ and so $\|f(y)\| \le \frac{\epsilon}{2}$ for all $y \in \mathcal{X}$. $\square$

Theorem 2.5. *Suppose that* $\perp$ *is symmetric on* $\mathcal{X}$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping satisfying* (2.3) *for some* $\epsilon \ge 0$, *for all* $a \in \mathcal{A}_1$ *and all* $x, y \in \mathcal{X}$ *with* $x \perp y$. *Then there exists a unique additive mapping* $T : \mathcal{X} \to \mathcal{Y}$ *satisfying* $T(ax) = aT(x)$, $a \in \mathcal{A}_1$, $x \in \mathcal{X}$ *such that*

$$\big\|f(x)-f(0)-T(x)\big\| \le \frac{5}{2}\epsilon,$$

for all $x \in \mathcal{X}$. *Moreover, if the mapping* $t \mapsto f(tx)$ *is continuous for each fixed* $x \in \mathcal{X}$, *then* T *is* $\mathcal{A}$*-linear.*

Proof. Define $F(x) = f(x) - f(0)$ and denote the even and odd parts of F by F^e, F^o, respectively. Clearly $F^e(0) = F^o(0) = F(0) = 0$.

Setting $x = y = 0$ in (2.3) and subtracting the argument of the norm of the resulting inequality from that of inequality (2.3) we get

$$\big\|F(ax + ay) + F(ax - ay) - 2aF(x)\big\| \leq 2\epsilon. \tag{2.5}$$

If $x \perp y$ then, by (O3), $-x \perp -y$. Hence we can replace x by $-x$ and y by $-y$ in (2.5) to obtain

$$\big\|F(-ax - ay) + F(-ax + ay) - 2aF(-x)\big\| \leq 2\epsilon. \tag{2.6}$$

By virtue of the triangle inequality and (2.5) and (2.6) we obtain

$$\begin{aligned} \big\|F^e(ax + ay) + F^e(ax - ay) - 2aF^e(x)\big\| &\leq 2\epsilon, \\ \big\|F^o(ax + ay) + F^o(ax - ay) - 2aF^o(x)\big\| &\leq 2\epsilon, \end{aligned}$$

for all $a \in \mathcal{A}_1$ and $x, y \in \mathcal{X}$.

In light of Proposition 2.3 there exists an additive mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying $T(ax) = aT(x), a \in \mathcal{A}_1, x \in \mathcal{X}$ such that $\big\|F^o(x) - T(x)\big\| \leq 2\epsilon$. By Proposition 2.4, $\big\|F^e(x)\big\| \leq \frac{\epsilon}{2}$. Hence

$$\big\|f(x) - f(0) - T(x)\big\| \leq \big\|F^e(x)\big\| + \big\|F^o(x) - T(x)\big\| \leq \frac{\epsilon}{2} + 2\epsilon = \frac{5}{2}\epsilon,$$

for all $x \in \mathcal{X}$. The uniqueness and $\mathcal{A}$-linearity of T are obtained from Proposition 2.3. □

Corollary 2.6. *Suppose that $(\mathcal{X}, \perp)$ is an orthogonality complex normed space, $(\mathcal{Y}, \|.\|)$ is a complex Banach space, $\perp$ is symmetric on $\mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying*

$$\big\|f(\lambda x + \lambda y) + f(\lambda x - \lambda y) - 2\lambda f(x)\big\| \leq \epsilon$$

for some $\epsilon \geq 0$, for all $\lambda \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and all $x, y \in \mathcal{X}$ with $x \perp y$. If the mapping $t \mapsto f(tx)$ is continuous for each fixed $x \in \mathcal{X}$, then there exists a unique $\mathbb{C}$-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - f(0) - T(x)\big\| \leq \frac{5}{2}\epsilon$$

for all $x \in \mathcal{X}$.

Proof. Consider $\mathcal{A}$ to be $\mathbb{C}$ in Theorem 2.5. □

The following corollary gives the stability of orthogonally additive type equation $f(x + y) + f(x - y) = 2f(x), \quad x \perp y$.

Corollary 2.7. *Suppose that* $(\mathcal{X}, \perp)$ *is an orthogonality complex normed space,* $(\mathcal{Y}, \|.\|)$ *is a complex Banach space,* $\perp$ *is symmetric on* $\mathcal{X}$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping satisfying*

$$\big\|f(x+y) + f(x-y) - 2f(x)\big\| \le \epsilon$$

for some $\epsilon \ge 0$ *and for all* $x, y \in \mathcal{X}$ *with* $x \perp y$*. Then there exists a unique additive mapping* $T : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\big\|f(x) - f(0) - T(x)\big\| \le \frac{5}{2}\epsilon$$

for all $x \in \mathcal{X}$*.*

Proof. Use the same reasoning as in the proof of Theorem 2.5 with $a = 1$. □

Now we are going to establish the orthogonal stability of $f(ax+ay) - f(ax-ay) = 2af(y)$.

Proposition 2.8. *Suppose that* $\perp$ *is symmetric on* $\mathcal{X}$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping satisfying*

$$\big\|f(ax+ay) - f(ax-ay) - 2af(y)\big\| \le \epsilon$$

for some $\epsilon \ge 0$*, for all* $a \in \mathcal{A}_1$ *and all* $x, y \in \mathcal{X}$ *with* $x \perp y$*. Assume that* f *is odd. Then there exists a unique additive mapping* $T : \mathcal{X} \to \mathcal{Y}$ *satisfying* $T(ax) = aT(x), a \in \mathcal{A}_1, x \in \mathcal{X}$ *such that*

$$\big\|f(x) - T(x)\big\| \le 2\epsilon$$

for all $x \in \mathcal{X}$*.*

Proof. Let $x \perp y$. Since $\perp$ is symmetric, $y \perp x$ and so

$$\big\|f(ay+ax) - f(ay-ax) - 2af(x)\big\| \le \epsilon.$$

Due to the fact that f is odd we conclude that

$$\big\|f(ax+ay) + f(ax-ay) - 2af(x)\big\| \le \epsilon.$$

Therefore we can apply Proposition 2.3 to get the required mapping. □

Proposition 2.9. *Suppose that* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping satisfying*

$$\big\|f(x+y) - f(x-y) - 2f(y)\big\| \le \epsilon \tag{2.7}$$

for some $\epsilon \ge 0$ *and all* $x, y \in \mathcal{X}$ *with* $x \perp y$*. Assume that* f *is even and* $f(0) = 0$*. Then*

$$\big\|f(x)\big\| \le \frac{1}{2}\epsilon$$

for all $x \in \mathcal{X}$*.*

Proof. Setting $x = 0$ in (2.7) we get $\|f(y) - f(-y) - 2f(y)\| \leq \epsilon$ and so $\|f(y)\| \leq \frac{1}{2}\epsilon$ for all $y \in \mathcal{X}$. □

Theorem 2.10. *Suppose that $\perp$ is symmetric on $\mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying*

$$\|f(ax + ay) - f(ax - ay) - 2af(y)\| \leq \epsilon$$

for some $\epsilon \geq 0$, for all $a \in \mathcal{A}_1$ and all $x, y \in \mathcal{X}$ with $x \perp y$. Then there exists a unique additive mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying $T(ax) = aT(x), a \in \mathcal{A}_1, x \in \mathcal{X}$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{5}{2}\epsilon$$

for all $x \in \mathcal{X}$. Moreover, if the mapping $t \mapsto f(tx)$ is continuous for each fixed $x \in \mathcal{X}$, then T is $\mathcal{A}$-linear.

Proof. One can use the same reasoning as the proof of Theroem 2.5. □

Remark 2.11. One can state and prove the results analogue to Corollaries 2.6 and 2.7 in a similar manner.

3. Orthogonal stability in Banach modules over Banach $*$-algebras

Applying some ideas from [3] and [22], we deal with the orthogonal stability problem for the additive type equations in Banach modules over Banach $*$-algebras.

Let $\mathcal{A}$ be a unital Banach $*$-algebra with unit 1, unit sphere $\mathcal{A}_1$, the unital group $U(\mathcal{A})$, and the positive cone $\mathcal{A}_+$. Let $(\mathcal{X}, \perp)$ denote an orthogonality normed left $\mathcal{A}$-module with the property $1x = x$ and $(\mathcal{Y}, \|.\|)$ be a Banach left $\mathcal{A}$-module.

Theorem 3.1. *Suppose that $\perp$ is symmetric on $\mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying*

$$\|f(ax_1 + ax_2) + (-1)^{k+1} f(ax_1 - ax_2) - 2af(x_k)\| \leq \epsilon$$

for $k = 1$ or 2, for some $\epsilon \geq 0$, for all $a \in (\mathcal{A}_1 \cap \mathcal{A}_+) \cup \{\mathbf{i}\}$ and all $x_1, x_2 \in \mathcal{X}$ with $x_1 \perp x_2$. Assume that the mapping $t \mapsto f(tx)$ is continuous for each fixed $x \in \mathcal{X}$. Then there exists a unique $\mathcal{A}$-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying $T(ax) = aT(x), a \in \mathcal{A}, x \in \mathcal{X}$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{5}{2}\epsilon$$

for all $x \in \mathcal{X}$.

Proof. By the same reasoning as the proof of Theorem 2.5 and Theorem 2.10, there is a unique $\mathbb{R}$-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying $T(ax) = aT(x)$, $a \in \mathcal{A}_+ \cup \{\mathbf{i}\}$, $x \in \mathcal{X}$ such that $\|f(x) - f(0) - T(x)\| \leq \frac{5}{2}\epsilon$.

Each element of $\mathcal{A}$ can be represented as $a = (a_1 - a_2) + \mathbf{i}(a_3 - a_4)$ where $a_j \in \mathcal{A}_+$, $1 \le j \le 4$ (see [20]). Hence $T(ax) = aT(x)$, $a \in \mathcal{A}$, $x \in \mathcal{X}$. □

The following lemma is very useful when one deals with the unitaries of a C^*-algebra; cf. Theorem 1 of [17]:

Lemma 3.2. *Let a be an element of a C^*-algebra $\mathcal{A}$ and $\|a\| < 1 - (2/m)$ for some integer $m > 2$. Then there exist m elements $u_1, \cdots, u_m \in U(\mathcal{A})$ such that $a = (u_1 + \cdots + u_m)/m$.*

Now we are ready to end our work.

Theorem 3.3. *Suppose that $\mathcal{A}$ is a C^*-algebra, $\perp$ is symmetric on $\mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying*

$$\left\| f(ax_1 + ax_2) + (-1)^{k+1} f(ax_1 - ax_2) - 2af(x_k) \right\| \le \epsilon$$

for $k = 1$ or 2, for some $\epsilon \ge 0$, for all $a \in U(\mathcal{A})$ and all $x_1, x_2 \in \mathcal{X}$ with $x_1 \perp x_2$. Assume that the mapping $t \mapsto f(tx)$ is continuous for each fixed $x \in \mathcal{X}$. Then there exists a unique $\mathcal{A}$-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying $T(ax) = aT(x)$, $a \in \mathcal{A}$, $x \in \mathcal{X}$ such that

$$\left\| f(x) - f(0) - T(x) \right\| \le \frac{5}{2}\epsilon$$

for all $x \in \mathcal{X}$.

Proof. By the same reasoning as the proof of Theorem 2.5 and Theorem 2.10, there is a unique $\mathbb{R}$-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying $T(ax) = aT(x)$, $a \in U(\mathcal{A})$, $x \in \mathcal{X}$ such that $\left\| f(x) - f(0) - T(x) \right\| \le \frac{5}{2}\epsilon$ for all $x \in \mathcal{X}$.

Assume that $a \in \mathcal{A} (a \neq 0)$ and N is an integer greater than $4\|a\|$. Then

$$\frac{\|a\|}{N} < \frac{\|a\|}{4\|a\|} < 1/3 = 1 - \frac{2}{3}.$$

By Lemma 3.2, there exist three unitaries u_1, u_2, u_3 such that $3\frac{a}{N} = u_1 + u_2 + u_3$. By the additivity of T we get $T\left(\frac{1}{3}x\right) = \frac{1}{3}T(x)$ for all $x \in \mathcal{A}$. Therefore,

$$\begin{aligned} T(ax) &= T\Big(\frac{N}{3} \cdot 3 \cdot \frac{a}{N}x\Big) = NT\Big(\frac{1}{3} \cdot 3 \cdot \frac{a}{N}x\Big) = \frac{N}{3}T\Big(3 \cdot \frac{a}{N}x\Big) \\ &= \frac{N}{3}T(u_1x + u_2x + u_3x) = \frac{N}{3}\big(T(u_1x) + T(u_2x) + T(u_3x)\big) \\ &= \frac{N}{3}(u_1 + u_2 + u_3)T(x) = \frac{N}{3} \cdot 3 \cdot \frac{a}{N} = aT(x) \end{aligned}$$

for all $x \in \mathcal{A}$. In addition, $T(0x) = 0T(x)$ for all $x \in \mathcal{A}$. Hence T is $\mathcal{A}$-linear. □

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