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The Gołąb–Schinzel equation and its generalizations

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Summary. This paper gives a survey on the Gołąb–Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y)$$

and its generalizations.

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In connection with some problems in the theory of geometric objects J. Aczél has in [1] introduced the functional equation (for real functions)

$$f(x+y) = f(x)f\left(\frac{y}{f(x)}\right). \tag{1}$$

Is is easily seen that replacing in (1) y by $f(x)y$ we obtain

$$f(x + f(x)y) = f(x)f(y). \tag{GS}$$

For the first time equation (GS) has been studied extensively by S. Gołąb and A. Schinzel in [43], in the class of functions $f : \mathbb{R} \to \mathbb{R}$; S. Gołąb came across the equation while looking for subgroups of the centroaffine group of $\mathbb{R}^2$ (cf. e.g. [81], p. 12–13). After that the equation has been named the Gołąb–Schinzel equation. In other classes of functions (GS) has been studied later by K. Baron ([7]), N. Brillouët-Belluot ([11]), N. Brillouët and J. Dhombres ([14]), Z. Daróczy ([38]), H. Gebert ([40]), O. E. Gheorgiu and S. Gołąb ([42]), D. Ilse, I. Lehmann and W. Schulz ([44]), H. Lüneburg and P. Plaumann ([52]) (cf. Math. Reviews 0612184 (58#29542)), P. Javor ([45], [46]), P. Plaumann and S. Strambach ([71]), S. Wołodźko ([81], [82]), and J. Brzdęk ([16], [19], [21]–[23], [26], [27]).

Some further applications of (GS) have been given by

- P. Javor ([45], [46]): associative operations;
- J. Aczél and S. Gołąb ([4], cf. [3], p. 311–315): subsemigroups of the semigroup of the affine mappings of $\mathbb{R}$ ($t \to \alpha t + \beta$);
- P. Plaumann and S. Strambach ([71]): classification of quasialgebras;

- N. Brillouët and J. Dhombres ([14]): subsemigroups of the group of affine mappings of $\mathbb{R}^2$;
- J. Brzdęk ([19], see also [16]): subgroups of the group L_2^1;
- P. Kahlig and J. Matkowski ([49]): differential equations in meteorology and fluid mechanics;
- E. Aichinger and M. Farag ([6]): classification of near-rings.

Generalizations

In what follows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ stand, as usual, for the sets of positive integers, integers, reals and complex numbers, respectively. Below we give (in chronological order) successive examples of functional equations, which are generalizations of (GS) (by which we mean here that (GS) can be obtained from a given equation by some specification of terms occurring in it).

1° O. E. Gheorghiu ([41], see also [39], [53] and [50]):

$$f(x+g(x)y)=h(x)k(y). \tag{2}$$

2° E. Vincze ([80]):

$$f(x+g(x)y)=L(h(x),k(y)). \tag{3}$$

3° S. Midura ([56]) (for $n\in\mathbb{N}, n>1$):

$$f(f(y)^n x+f(x)y)=f(x)f(y). \tag{4}$$

4° S. Midura and P. Urban ([67], see also [76]–[79]) (for $n,k\in\mathbb{N}$):

$$f(f(y)^n x+f(x)^k y)=f(x)f(y). \tag{5}$$

5° W. Benz ([8], see also [9]–[12], [14], [18], [21]) (for $n,k\in\mathbb{N}\cup\{0\}, t\in\mathbb{R}$):

$$f(f(y)^n x+f(x)^k y)=tf(x)f(y). \tag{6}$$

6° J. Brzdęk ([24]) (for $n,k,t,A\in\mathbb{R}$, $f:\mathbb{R}\to(0,\infty)$):

$$f(f(y)^n x+f(x)^k y+Axy)=tf(x)f(y). \tag{7}$$

7° N. Brillouët-Belluot ([13]) (for $n,k\in\mathbb{N}$, $n\neq k$, $F:\mathbb{R}^2\to\mathbb{R}$):

$$f(f(y)^n x+f(x)^k y)=F(f(x),f(y)). \tag{8}$$

8° J. Chudziak ([33], [34]):

$$f(\varphi(f(y))x+\psi(f(x))y)=f(x)f(y). \tag{9}$$

9° J. Brzdęk ([30], [31]):

$$f(x+M(f(x))y)=H(f(x),y), \tag{10}$$

$$f(x+M(f(x))y)=H(x,y). \tag{11}$$

We also should mention here the systems of functional equations introduced by S. Midura in connection with the problem of finding algebraic substructures in the Lie groups L_s^1. The system from which one can obtain some particular cases of equation (5) has been studied in [56], [58]–[64] (see also [54], [66], [68], [17]).

Equations derived from the Gołąb–Schinzel equation

Single variable functional equations derived from (GS) have been considered by J. Matkowski in [55] (for $A, B, a, b, \alpha, \beta \in \mathbb{R}$):

$$f(Ax+\alpha) = af(x), \quad f(Bx+\beta) = bf(x),$$

and J. Knop, T. Kostrzewski, M. Lupa and M. Wróbel in [51]:

$$f(x+f(x)x) = f(x)^2. \tag{12}$$

Motivated by a problem of P. Kahlig (cf. [49]), J. Aczél and J. Schwaiger ([5]) and L. Reich ([73]) have considered the two conditional equations

$$\text{if } \ x, y \geq 0, \ \text{then } \ f(x+f(x)y) = f(x)f(y),$$

$$\text{if } x, y, x+f(x)y \geq 0, \ \text{then } f(x+f(x)y) = f(x)f(y)$$

(for $f : \mathbb{R} \to \mathbb{R}$). Some further results connected with those equations one may find in [49], [69], [70], [74], [75], [32] and [29].

Connections with some other equations

Let $n \in \mathbb{N} \cup \{0\}$ and X be a linear space over a field $\mathbb{K}$. S. Midura ([56], [57], cf. also [15], [20], [23]) has introduced the functional equation

$$f\left(\frac{f(y)x - f(x)y}{f(y)^{n+1}}\right) = \frac{f(x)}{f(y)}, \tag{13}$$

where $f : X \to \mathbb{K} \setminus \{0\}$ is the unknown function.

Assume $f : X \to \mathbb{K} \setminus \{0\}$ satisfies (13). With $x = y = 0$ we get $f(0) = 1$ and next with $x = 0$

$$f\left(\frac{-y}{f(y)^{n+1}}\right) = \frac{1}{f(y)}.$$

Finally replacing y with $\frac{-y}{f(y)^{n+1}}$ we obtain that f satisfies

$$f(f(y)^n x + f(x)y) = f(x)f(y),$$

i.e. equation (4).

Moreover in [23] it is proved that for $n > 0$ there is a strict connection between solutions $f, g : X \to \mathbb{K}$ of the two functional equations

$$\text{if } f(x)f(y) \neq 0, \ \text{then } f\left(\frac{f(y)x - f(x)y}{f(y)^{n+1}}\right) = \frac{f(x)}{f(y)},$$

and

$$g(x + g(x)^{n-1}y) = g(x)g(y).$$

Let $\mathbb{R}^+ := (0, \infty)$, $f : \mathbb{R}^+ \to \mathbb{R}$ and $F(x, y) = x + f(x)y$ for $x, y \in \mathbb{R}^+$. Suppose F satisfies the conditional associativity equation

$$\text{if } \; F(x, y), F(y, z) > 0, \; \text{then } \; F(x, F(y, z)) = F(F(x, y), z). \tag{14}$$

Take $x, y \in \mathbb{R}^+$ with $x + f(x)y = F(x, y) > 0$. Then $F(y, z) = y + f(y)z > 0$ for some $z > 0$ and consequently

$$\begin{aligned} x + f(x)(y + f(y)z) &= F(x, F(y, z)) \\ = F(F(x, y), z) &= x + f(x)y + f(x + f(x)y)z, \end{aligned}$$

which implies $f(x + f(x)y) = f(x)f(y)$. Hence f satisfies the equation

$$\text{if } \; x + f(x)y > 0, \; \text{then } \; f(x + f(x)y) = f(x)f(y).$$

The converse is true as well.

It is easy to check that an analogous connection holds in the unconditional case as well (see [45] and [46]).

Connections between (GS) and some other functional equations have been studied in [47] and [50].

Continuous solutions of the Gołąb–Schinzel equation

S. Gołąb and A. Schinzel [43] proved that every continuous solution $f : \mathbb{R} \to \mathbb{R}$ of (GS) must be of one of the three forms

$$f \equiv 0 \;\text{ or }\; f(x) = cx + 1 \;\text{ or }\; f(x) = \max\{cx + 1, 0\}$$

with some $c \in \mathbb{R}$. Next, from a result of P. Plaumann and S. Strambach ([71]) we obtain that for every continuous solution $g : \mathbb{C} \to \mathbb{C}$ of (GS) there are $c \in \mathbb{C}$, a continuous solution $f : \mathbb{R} \to \mathbb{R}$ of (GS) and an $\mathbb{R}$-linear function $L : \mathbb{C} \to \mathbb{R}$ such that

$$g(x) = cx + 1 \qquad \text{or} \qquad g = f \circ L.$$

Other results concerning the continuous solutions of (GS) one may find in [7], [14], [38], [40], [42], [46], [81], [82]. In particular, we have the following

Theorem 1. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let X be a topological linear space over $\mathbb{K}$. Every continuous solution $g : X \to \mathbb{K}$ of* (GS) *is of the form*

$$g = f \circ L$$

with some continuous solution $f : \mathbb{K} \to \mathbb{K}$ of (GS) *and a continuous linear functional $L : X \to \mathbb{K}$.*

Solutions of (GS) under the assumptions of continuity at a point, quasicontinuity, continuity on rays have been considered in [16] and [19].

Every solution $g : (0, \infty) \to \mathbb{R}$ of the conditional equation

$$\text{if } x + g(x)y > 0, \text{ then } g(x + g(x)y) = g(x)g(y), \tag{15}$$

continuous at a point $x \in (0, \infty)$ such that $g(x) \neq 0$, is a restriction of a continuous solution $f : \mathbb{R} \to \mathbb{R}$ of (GS) (see [70]; cf. also [69] and [74]).

Measurable solutions of the Gołąb–Schinzel equation

The first examples of non-measurable solutions $f : \mathbb{R} \to \mathbb{R}$ of (GS) are due to W. Sierpiński and S. Marcus (see [43], p. 118 and 123); another one is given in [2] (p. 134–5). C. G. Popa ([72]) has proved that every Lebesgue measurable solution $f : \mathbb{R} \to \mathbb{R}$ of (GS) is continuous or equal zero almost everywhere. That result has been generalized in [26] and [27] in the following way.

Theorem 2. *Let $n \in \mathbb{N}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, X be a separable Fréchet space over $\mathbb{K}$ and $f : X \to \mathbb{K}$ be a Christensen (respectively Baire) measurable solution of*

$$f(x + f(x)^n y) = f(x)f(y). \tag{16}$$

Then f is continuous or $X \setminus f^{-1}(\{0\})$ is a Christensen zero set (of first category, resp.).

Actually the result in [27], concerning the Baire measurability, is stronger; namely it is proved that solutions $f : X \to \mathbb{K}$ of (16), with $|f(A)| \subset (0, a)$ (i.e. $0 < |f(x)| < a$ for $x \in A$) for some $a > 0$ and some $A \subset \mathbb{R}$ of the second category with the Baire property, are bounded or continuous. In connection with this there arise the following two problems.

Problem. Is it true that an analogous statement holds for solutions of (16), bounded on a Christensen measurable non-zero set?

Problem. Let $\mathcal{I}$ be a σ-ideal of subsets of a linear topological space X over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Find conditions, as weak as possible, for X and $\mathcal{I}$ such that every solution $f : X \to \mathbb{K}$ of (16), with $|f(U \setminus D)| \subset (0, a)$ for some $a > 0$, $D \in \mathcal{I}$, and an open nonempty set $U \subset X$, must be bounded or continuous.

Solutions of a generalization of the Gołąb–Schinzel equation

We consider now equation (6).

Let $n, k \in \mathbb{N} \cup \{0\}, k > 0$, $t \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ be a solution of (6) and $y \in \mathbb{R}$, $f(y) \neq 0$. Suppose f has the Darboux property and is in the Baire class I. Then the mapping

$$\varphi_y : \mathbb{R} \ni x \to f(y)^n x + f(x)^k y \in \mathbb{R}$$

has the Darboux property (see e.g. [10], Lemma 2) and it is easy to show (see e.g. [18]) that, for $t \neq 0$, it is injective (for $t = 0$ see [10]). So φ_y is continuous, whence f is continuous.

The continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of equation (6) are listed below (see [10], [11], [14], [77], [78], [18], [21]).

1° $f \equiv 0$.

2° $f \equiv \frac{1}{t}$ – only for $t \neq 0$.

3° $f(x) = (cx)^{\frac{1}{k}}$ with some $c \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ – only for $t = 2$ and odd $k = n$.

4° $f(x) = (\max\{cx, 0\})^{\frac{1}{k}}$ with some $c \in \mathbb{R}_0$ – only for $t = 2$ and $k = n$.

5° $f(x) = t(\max\{cx + 1, 0\})^{\frac{1}{k}}$ with some $c \in \mathbb{R}_0$ – only for $t^k = 1$, $n = 0$.

6° $f(x) = (cx + 1)^{\frac{1}{k}}$ with some $c \in \mathbb{R}_0$ – only for $t = 1$, $n = 0$ and odd k.

We have also the following

Theorem 3. *Let $n, k \in \mathbb{N} \cup \{0\}$, $k > 0$, $t \in \mathbb{R}$, X be a real linear space, $f : X \to \mathbb{R}$, and*

$$f_x : \mathbb{R} \ni t \to f(tx) \in \mathbb{R}$$

be continuous for every $x \in X$. Then f satisfies equation (6) *if and only if there exist a linear $L : X \to \mathbb{R}$ and a continuous solution $f_0 : \mathbb{R} \to \mathbb{R}$ of* (6) *such that $f = f_0 \circ L$.*

As we have already shown, it is actually enough to assume in the above theorem that, for every $x \in X$, f_x has the Darboux property and is in the Baire class I.

Continuous solutions $f : [0, \infty) \to \mathbb{R}$ of (6) have been studied by P. Urban in [78], [79]. Numerous examples of discontinuous solutions $f : \mathbb{R} \to \mathbb{R}$ of (6) have been given by W. Benz ([8], see also [9]).

In the complex case we only have the subsequent results (see [21]).

Theorem 4. *Let X be a complex linear topological space, $k \in \mathbb{N}$, $k > 1$, and $t \in \mathbb{C}$.*

1° *If $t^k \neq 1$, then there are no non-constant continuous solutions $f : X \to \mathbb{C}$ of*

$$f(x + f(x)^k y) = tf(x)f(y). \tag{17}$$

2° *If $t^k = 1$ and $f : X \to \mathbb{C}$ is a continuous solution of* (17)*, then $tf(X) \subset \mathbb{R}$ and tf is a solution of* (6) *with $n = 0$ and $t = 1$.*

We have as well the following result (see Proposition 1 in [21] and [22]).

Theorem 5. *Let $t \in \mathbb{R}$, $k \in \mathbb{N}$, and X be a real linear topological space. Each solution $f : X \to \mathbb{R}$ of* (17)*, that has the Darboux property, is continuous.*

Problem. There arises a natural question whether the last statement is true for equation (6) with $n > 0$? Is that statement true as well, for equation (17) (or even for (6)), in the case where X is a complex linear topological space, $t \in \mathbb{C}$ and $f : X \to \mathbb{C}$?

Solutions of (17) under some algebraic conditions are described in [21]. A conditional version of (5) has been studied in [25].

The general solution of the Gołąb–Schinzel equation

Let X be a linear space over a field $\mathbb{K}$. A description of the general solution of (GS) in the class of functions $f : X \to \mathbb{K}$ has been given for the first time by S. Wołodźko (see [81]) and then, in a modified form, by P. Javor (see [45]; cf. also [3], p. 315–318). Here we present a little more general result from [21].

Theorem 6. *Let X be a linear space over a field $\mathbb{K}$, $k \in \mathbb{N}$ and $t \in \mathbb{K}, t \neq 0$. $f : X \to \mathbb{K}$, $f \not\equiv 0$, is a solution of equation (17) if and only if there are subgroups A of $(X, +)$ and W of $(\mathbb{K} \setminus \{0\}, \cdot)$ and a mapping $w : W \to X$ such that*

$$\textit{if} \quad w(\alpha) \in A, \ \textit{then} \quad \alpha = 1;$$

$$\alpha^k A = A, \qquad \textit{for } \alpha \in W \cup \{t\};$$

$$(t^k - 1)w(\alpha) \in A, \qquad \textit{for } \alpha \in W;$$

$$w(\alpha\beta) - \alpha^k w(\beta) - w(\alpha) \in A, \quad \textit{for } \alpha, \beta \in W;$$

$$f(x) = \begin{cases} t^{-1}\alpha, & \textit{if } x \in w(\alpha) + A \textit{ for some } \alpha \in W; \\ 0, & \textit{otherwise.} \end{cases}$$

Suppose f has the form described in the theorem. If $A = \{0\}$, then $w(\alpha) = (\alpha^k - 1)z$ with some $z \in X$. Furthermore, if $\beta^k - 1 \in W$ for some $\beta \in W$ or if W is cyclic and $\alpha^k \neq 1$ for some $\alpha \in W$, then there is $z \in X$ with $w(\alpha) - (\alpha^k - 1)z \in A$ (see [21]). So in all these cases we may write

$$f(x) = \begin{cases} t^{-1}\alpha, & \text{if } x \in (\alpha^k - 1)z + A \text{ for some } \alpha \in W; \\ 0, & \text{otherwise.} \end{cases}$$

There are solutions of (17) which are not of this form (see [2], p. 134–5, and [16])!

In connection with a problem in [28] it is proved in [32] that every solution $g : (0, \infty) \to \mathbb{R}$ of (15), with $x_0 + g(x_0)y_0 > 0$ for some $x_0, y_0 > 0$, is a restriction of a solution $f : \mathbb{R} \to \mathbb{R}$ of (GS). The following seems to be of interest.

Problem. Let $n \in \mathbb{N}$, X be a real linear space and $\mathcal{C} \subset X$ be a cone (i.e. $x + y$, $\alpha x \in \mathcal{C}$ for $x, y \in \mathcal{C}$, $\alpha \in [0, \infty)$). Under what conditions, as weak as possible, is a solution $g : \mathcal{C} \to \mathbb{R}$ of the conditional equation

$$\text{if } x + g(x)^n y \in \mathcal{C}, \text{ then } g(x + g(x)^n y) = g(x)g(y),$$

a restriction of a solution $f : X \to \mathbb{R}$ of (16)?

Final remarks

During his survey talk on stability of functional equations (see Aequationes Math. *61* (2001), p. 284), at the 38th International Symposium on Functional Equations (Noszvaj, Hungary, 2000), R. Ger presented a list of open problems among others concerning the stability of (GS) and of all the other equations considered in this paper. Some stability results for (GS) have been obtained recently by J. Chudziak and J. Tabor (see [35]–[37]). It seems that the problem of stability of (GS) and its generalizations supplies a very large field for research.

It follows from [31] that there are similarities between the classes of continuous solutions (e.g. $f : \mathbb{R} \to \mathbb{R}$) of equations

$$f(x + M(f(x))y) = H(f(x), f(y)) \tag{18}$$

and (17) (or even (GS)). So the following problem seems to be of interest.

Problem. To what extent can the results proved for (17) (or (16)) be carried over to the case of (18)?

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