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Neumann boundary value problems with singularities in a phase variable

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Summary. The singular Neumann boundary value problem $(g(x'))' = f(t,x,x')$, $x'(0) = x'(T) = 0$ is considered. Here $f(t,x,y)$ satisfies local Carathéodory conditions on $[0,T] \times \mathbb{R} \times (0,\infty)$ and f may be singular at the value 0 of the phase variable y. Conditions guaranteeing the existence of a solution to the above problem with a positive derivative on $(0,T)$ are given. The proofs are based on regularization and sequential techniques and use the topological transversality method.

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1. Introduction

Let T be a positive number, $J = [0,T]$. Consider the Neumann boundary value problem (BVP)

$$(g(x'(t)))' = f(t, x(t), x'(t)), \tag{1.1}$$

$$x'(0) = 0, \quad x'(T) = 0, \tag{1.2}$$

where $g \in C^0([0,\infty))$, $g(0) = 0$ and f satisfies local Carathéodory conditions on $J \times \mathbb{R} \times (0,\infty)$ ($f \in Car(J \times \mathbb{R} \times (0,\infty))$) and f may be singular at the value 0 of its second phase variable in the following sense: $\lim_{y\to 0^+} |f(t,x,y)| = \infty$ for a.e. $t \in J$ and each $x \in \mathbb{R}$, $x \neq \alpha(t)$, where α appears in assumption (H_2).

A function $x \in C^1(J)$ is said to be a *solution of the BVP* (1.1), (1.2) if $g(x') \in AC(J)$ (absolutely continuous functions on J), x satisfies the Neumann boundary conditions (1.2) and (1.1) holds a.e. on J.

In this paper we are interested in finding conditions guaranteeing the existence of a solution x of the BVP (1.1), (1.2) such that $x'(t) > 0$ for $t \in (0,T)$. We

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note that our problem is at resonance since each constant function on J is a solution of the associated homogeneous problem $(g(x'))' = 0$, (1.2) and, in addition, solutions of the BVP (1.1), (1.2) have 'maximal' smoothness (that is $x \in C^1(J)$ and $(g(x'))' \in AC(J)$) although f may be singular at the value 0 of its second phase variable. Also we remark here that the singular Neumann boundary value problem with f singular at the value 0 of its second phase variable has not been considered in the literature.

We note that the regular BVP (1.1), (1.2) with $g(u) \equiv u$ is usually considered by combining the method of lower and upper functions (see, e.g. [5], [6], [8], [9], [11], [12] and references therein) with the Mawhin continuation theorem ([11]) or the topological transversality method ([5], [6], [8]) or Schauder degree theory ([12]) or special procedures ([9]). The nonlinearity f in (1.1) is continuous ([5], [8], [12]) or satisfies local Carathéodory conditions ([6], [9], [11]). Existence results for the Neumann problem $-x''(t) = f_1(t, x(t)) + p(t)$, (1.2) with $f_1 \in C^0([0,1] \times \mathbb{R})$ and $p \in L_2([0,1])$ are proved in [13] by variational methods.

The regular BVP (1.1), (1.2) with $g \in C^0(\mathbb{R})$, $g(\mathbb{R}) = \mathbb{R}$, g increasing and $f \in Car([a,b] \times \mathbb{R}^2)$ was considered in [2] and [3]. In [2] solutions are obtained as the limit of solutions of different nonhomogeneous mixed problems whereas existence was proved by the method of lower and upper functions in [14]. Combining the method of lower and upper functions in the reverse order together with monotone methods and a iterative technique, the existence of minimal and maximal solutions of the BVP (1.1), (1.2) lying between upper and lower functions is proved in [3]. In [4] the author discuss the existence and nonexistence of solutions to the differential equation $(g(x'))' + p(x') + h(x) = q(t)$ satisfying (1.2). Here g is an increasing homeomorphism on I_1 onto I_2, where I_1 and I_2 are open intervals containing zero, $g(0) = 0$, $p \in C^0(\mathbb{R})$, $q \in C^0([0,T])$ with $\int_0^T q(t)\,dt = 0$ and $h \in C^0(\mathbb{R})$ is bounded, $\lim_{u\to-\infty} h(u) < \lim_{u\to\infty} h(u)$. In [10] the authors stated conditions for the existence of a solution to the BVP $(|x'|^{p-2}x')' + f_1(t,x) + f_2(t,x) = 0$, (1.2) where f_1 is bounded, f_2 satisfies a one-sided growth condition, $f_1 + f_2$ has some sign condition, and the solutions to some associated homogeneous problem are not oscillatory. In [7] the BVP $-(|x'(t)|^{p-2}x'(t))' = h(t, x(t))$, (1.2) $(2 \le p < \infty)$ is considered, where $h : J \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function satisfying some extra conditions. To guarantee the existence of solutions the authors pass to a multivalued problem which is solved using variational techniques based on nonsmooth critical point theory.

Our existence result for the singular BVP (1.1), (1.2) is proved by regularization and sequential techniques. We first define a sequence of auxiliary regular BVPs to the BVP (1.1), (1.2) and give *a priori* bounds for their solutions (Lemma 2.1). Then we use twice the topological transversality principle (Theorem 1.3) to prove that the sequence of the auxiliary BVPs has a sequence $\{x_n\}$ of solutions (Lemma 2.2 and 2.3). The construction of the sequence of auxiliary BVPs guarantees that $x_n'(t) > 1/n$ for each $t \in J$ and $n \in \mathbb{N}$. In addition, we show that $g(x_n'(t)) \ge \mu t$ and $g(x_n'(t)) \ge \mu(T - t)$ on a neighbourhood of zero for each

$n \in \mathbb{N}$, where μ is a positive constant (Lemma 2.5). Applying the Arzelà–Ascoli theorem we can select a convergent subsequence of $\{x_n\}$ in $C^1(J)$, and then the Lebesgue dominated convergence theorem shows that its limit x is a solution of the BVP (1.1), (1.2) with $x' > 0$ on $(0,T)$ (Theorem 3.1). We illustrate our theory with two examples (Example 3.2 and 3.3).

Throughout this paper, the following assumptions are used.

(H_1) $g \in C^0([0,\infty))$ is increasing, $g(0) = 0$ and $\lim_{u\to\infty} g(u) = \infty$;

(H_2) $f \in Car(J \times \mathbb{R} \times (0,\infty))$ and

$$f(t,\alpha(t),y) = 0 \quad \text{for a.e. } t \in J \text{ and each } y \in (0,\infty),$$

where $\alpha \in C^0(J)$ is decreasing;

(H_3) There exist functions $q_1 \in C^0((-\infty,\alpha(0)])$, $q_2 \in C^0([\alpha(T),\infty))$ positive, $\omega_1 \in C^0([0,\infty))$ nonnegative and nondecreasing and $\omega_2 \in C^0((0,\infty))$ positive and nonincreasing, $\int_0^1 \omega_2(g^{-1}(s))\,ds < \infty$, such that

$$\begin{gathered} 0 < f(t,x,y) \le q_1(x)(\omega_1(y) + \omega_2(y)) \\ \text{for a.e. } t \in J \text{ and each } x < \alpha(t),\ y > 0 \end{gathered}$$

and

$$\begin{gathered} 0 > f(t,x,y) \ge -q_2(x)(\omega_1(y) + \omega_2(y)) \\ \text{for a.e. } t \in J \text{ and each } x > \alpha(t),\ y > 0; \end{gathered}$$

(H_4) $\lim_{u\to\infty} H(u) = \infty$ and

$$\min\left\{ \limsup_{u\to\infty} \frac{\displaystyle\int_{\alpha(T)-Tu}^{\alpha(0)} q_1(s)\,ds}{H(u)},\ \limsup_{u\to\infty} \frac{\displaystyle\int_{\alpha(T)}^{\alpha(0)+Tu} q_2(s)\,ds}{H(u)} \right\} < 1$$

where

$$H(u) = \int_0^{g(u)} \frac{g^{-1}(s)}{\omega_1(g^{-1}(s)) + \omega_2(g^{-1}(s))}\,ds \quad \text{for } u \in [0,\infty);$$

(H_5) For each $\varepsilon > 0$ there exists $\delta > 0$ such that for a.e. $t \in J$ and each $a > \varepsilon$, $y \in (0,1]$,

$$f(t,\alpha(t)-a,y) \ge \delta, \quad f(t,\alpha(t)+a,y) \le -\delta.$$

Remark 1.1. Since $g^{-1}(0) = 0$ and g^{-1} is continuous and increasing on $[0,\infty)$ which follows from (H_1) and ω_2 is a positive, nonincreasing and continuous function on $(0,\infty)$ by (H_3), the condition $\int_0^1 \omega_2(g^{-1}(s))\,ds < \infty$ implies that $\int_0^c \omega_2(g^{-1}(s))\,ds < \infty$ for each $c > 0$.

Remark 1.2. By (H_3), the function $p : [0,\infty) \to [0,\infty)$,

$$p(u) = \begin{cases} 0 & \text{for } u = 0 \\ \dfrac{g^{-1}(u)}{\omega_1(g^{-1}(u)) + \omega_2(g^{-1}(u))} & \text{for } u > 0, \end{cases} \tag{1.3}$$

is continuous on $[0,\infty)$ and $H(u) = \int_0^{g(u)} p(s)\,ds$ for $u \in [0,\infty)$.

Existence results for auxiliary regular BVPs to the BVP (1.1), (1.2) are proved by the topological transversality method ([1], [5], [6]), which we state here for the convenience of the reader. Let $\mathcal{U}$ be a convex subset of a Banach space $\mathbf{X}$ and let $\Omega \subset \mathcal{U}$ be open in $\mathcal{U}$. Denote by $\mathcal{H}_{\partial\Omega}(\overline{\Omega},\mathcal{U})$ the set of compact operators $\mathcal{F} : \overline{\Omega} \to \mathcal{U}$ which are fixed point free on $\partial\Omega$. We say that $\mathcal{F} \in \mathcal{H}_{\partial\Omega}(\overline{\Omega},\mathcal{U})$ is essential if every operator in $\mathcal{H}_{\partial\Omega}(\overline{\Omega},\mathcal{U})$ which agrees with $\mathcal{F}$ on $\partial\Omega$ has a fixed point in Ω

Theorem 1.3. (Topological transversality) *Let*
(a) $\mathcal{F} \in \mathcal{H}_{\partial\Omega}(\overline{\Omega},\mathcal{U})$ *be essential,*
(b) $H : \overline{\Omega} \times [0,1] \to \mathcal{U}$ *be a compact homotopy,* $H(\cdot,0) = \mathcal{F}$ *and* $H(x,\lambda) \neq x$ *for* $x \in \partial\Omega$ *and* $\lambda \in [0,1]$.
Then $H(\cdot,1)$ *is essential and therefore it has a fixed point in* Ω.

If $p \in \Omega$ and $\mathcal{F} \in \mathcal{H}_{\partial\Omega}(\overline{\Omega},\mathcal{U})$ is a constant operator, $\mathcal{F}(x) = p$ for $x \in \overline{\Omega}$, then $\mathcal{F}$ is essential (see [1], [5], [6]).

2. Auxiliary regular BVPs

For each $n \in \mathbb{N}$, define $S_n \in C^0(\mathbb{R})$ and $\tilde{f}_n,\ f_n \in Car(J \times \mathbb{R}^2)$ by

$$S_n(u) = \begin{cases} 1 & \text{for } u > \frac{1}{n} \\ 2n(u - \frac{1}{2n}) & \text{for } \frac{1}{2n} < u \le \frac{1}{n} \\ 0 & \text{for } u \le \frac{1}{2n}, \end{cases}$$

$$\tilde{f}_n(t,x,y) = \begin{cases} f(t,x,y) & \text{for } (t,x) \in J \times \mathbb{R},\ y > \frac{1}{n} \\ f(t,x,\frac{1}{n}) & \text{for } (t,x) \in J \times \mathbb{R},\ y \le \frac{1}{n}, \end{cases}$$

$$f_n(t,x,y) = S_n(y)\tilde{f}_n(t,x,y).$$

Then (H_2) and (H_3) give

$$f_n(t,\alpha(t),y) = 0 \quad \text{for a.e. } t \in J \text{ and each } y \in \mathbb{R}, \tag{2.1}$$

$$\left.\begin{array}{l} 0 < f_n(t,x,y) \le q_1(x)[\max\{\omega_1(y),\omega_1(\frac{1}{n})\} + \omega_2(y)] \\ \text{for a.e. } t \in J \text{ and each } x < \alpha(t),\ y > \frac{1}{2n}, \end{array}\right\} \tag{2.2}$$

$$\left.\begin{array}{l} 0 > f_n(t,x,y) \geq -q_2(x)\left[\max\left\{\omega_1(y), \omega_1\left(\frac{1}{n}\right)\right\} + \omega_2(y)\right] \\ \text{for a.e. } t \in J \text{ and each } x > \alpha(t),\ y > \frac{1}{2n} \end{array}\right\} \tag{2.3}$$

and

$$f_n(t,x,y) = 0 \quad \text{for a.e. } t \in J \text{ and each } x \in \mathbb{R},\ y \leq \tfrac{1}{2n}. \tag{2.4}$$

Let $g_* \in C^0(\mathbb{R})$ be defined by the formula

$$g_*(u) = \begin{cases} g(u) & \text{for } u \in [0,\infty) \\ -g(-u) & \text{for } u \in (-\infty, 0). \end{cases}$$

Consider the family of regular BVPs

$$(g_*(x'(t)))' = \lambda f_n(t, x(t), x'(t)), \tag*{(E)$_n^\lambda$}$$

$$x'(0) = \frac{1}{n}, \quad x'(T) = \frac{1}{n} \tag*{(B)$_n$}$$

depending on the parameters $n \in \mathbb{N}$ and $\lambda \in [0,1]$.

A priori bounds for solutions of the BVPs $(\mathrm{E})_n^\lambda$, $(\mathrm{B})_n$ are presented in the following lemma.

Lemma 2.1. *Let assumptions (H_1)–(H_4) be satisfied and $n \in \mathbb{N}$, $\lambda \in (0,1]$. Let x be a solution of the BVP $(\mathrm{E})_n^\lambda$, $(\mathrm{B})_n$. Then there exist positive constants A and Λ independent of n and λ, and a unique $\xi \in (0,T)$ (depending on x) such that*

$$\|x\| = \max\{|x(t)| : t \in J\} < A, \tag{2.5}$$

$$x(\xi) = \alpha(\xi) \tag{2.6}$$

and

$$\frac{1}{n} \leq x'(t) < \Lambda \quad \textit{for } t \in J. \tag{2.7}$$

Proof. We first prove that

$$x(0) < \alpha(0). \tag{2.8}$$

If not, then $x(0) \geq \alpha(0)$ and from $x'(0) = 1/n$ and α being decreasing on J we see that $x(t) > \alpha(t)$ on a right neighbourhood of $t = 0$. If $x(\tau) = \alpha(\tau)$ for a $\tau \in (0,T]$ and $x(t) > \alpha(t)$ for $t \in (0,\tau)$, then $x'(\tau) \leq 0$ and so for a $\nu \in (0,\tau)$ we have $x'(\nu) = 1/(2n)$ and $x' \leq 1/(2n)$ on $[\nu,\tau]$. Hence $(g_*(x'))' = 0$ a.e. on $[\nu,\tau]$ by (2.4) and then $x'(t) = 1/(2n)$ on this interval, which contradicts $x'(\tau) \leq 0$. Therefore $x(t) > \alpha(t)$ for $t \in (0,T]$ and then (2.3) and (2.4) yield

$$(g_*(x'(t)))' = \lambda f_n(t, x(t), x'(t)) \begin{cases} < 0 & \text{if } x'(t) > \frac{1}{2n} \\ = 0 & \text{if } x'(t) \leq \frac{1}{2n}. \end{cases}$$

Then

$$g_*(x'(T)) = g\left(\frac{1}{n}\right) + \int_0^T (g_*(x'(t))'\,dt = g\left(\frac{1}{n}\right) + \lambda\int_0^T f_n(t,x(t),x'(t))\,dt < g\left(\frac{1}{n}\right)$$

which contradicts $x'(T) = 1/n$. We have proved that (2.8) is true.

Since $x(0) < \alpha(0)$, $x(t) < \alpha(t)$ on a right neighbourhood of $t = 0$ and then $x'(0) = 1/n$ and (2.2) show that x' is increasing on any right neighbourhood of $t = 0$ where $x(t) < \alpha(t)$. Now from $x'(T) = 1/n$ we deduce that (2.6) holds with a $\xi \in (0, T)$, $x(t) < \alpha(t)$ for $t \in [0, \xi)$, $x'(t) > 1/n$ for $t \in (0, \xi]$ and $x(t) > \alpha(t)$ on a right neighbourhood of $t = \xi$. Arguing as in the first part of our proof we can verify that $x(t) > \alpha(t)$ for $t \in (\xi, T]$. Hence $(g_*(x'))' \le 0$ a.e. on $[\xi, T]$ by (2.3) and (2.4), and then $x'(T) = 1/n$ implies $x' \ge 1/n$ on $[\xi, T]$ which together with (2.3) yields $(g_*(x'))' < 0$ a.e. on $[\xi, T]$. As a result $x'(t) > 1/n$ for $t \in [\xi, T)$, and so

$$x'(t) > \frac{1}{n} \quad \text{for } t \in (0, T) \tag{2.9}$$

and (2.6) is satisfied with a unique $\xi \in (0, T)$. In addition, it is clear that

$$\|x'\| = \max\{x'(t) : t \in J\} = x'(\xi) \tag{2.10}$$

and

$$\|x\| = \max\{|x(0)|, |x(T)|\}. \tag{2.11}$$

We now give bounds for x on J. Clearly (see (2.9)), $x(0) \le x(t) \le x(T)$ for $t \in J$. By (2.2), (2.3) and (2.9), we have

$$(g(x'(t)))' \le q_1(x(t))(\omega_1(x'(t)) + \omega_2(x'(t))) \quad \text{for a.e. } t \in [0, \xi],$$

$$(g(x'(t)))' \ge -q_2(x(t))(\omega_1(x'(t)) + \omega_2(x'(t))) \quad \text{for a.e. } t \in [\xi, T].$$

Integrating the inequality

$$\frac{(g(x'(t)))'x'(t)}{\omega_1(x'(t)) + \omega_2(x'(t))} \le q_1(x(t))x'(t) \quad \text{for a.e. } t \in [0, \xi]$$

and

$$\frac{(g(x'(t)))'x'(t)}{\omega_1(x'(t)) + \omega_2(x'(t))} \ge -q_2(x(t))x'(t) \quad \text{for a.e. } t \in [\xi, T]$$

over $[0, t] \subset [0, \xi]$ and $[t, T] \subset [\xi, T]$, we get

$$\begin{aligned} \int_{g(1/n)}^{g(x'(t))} \frac{g^{-1}(s)}{\omega_1(g^{-1}(s)) + \omega_2(g^{-1}(s))}\, ds &\le \int_{x(0)}^{x(t)} q_1(s)\, ds \\ &< \int_{x(0)}^{\alpha(0)} q_1(s)\, ds, \quad t \in [0, \xi] \end{aligned}$$

and

$$\begin{aligned} \int_{g(x'(t))}^{g(1/n)} \frac{g^{-1}(s)}{\omega_1(g^{-1}(s)) + \omega_2(g^{-1}(s))}\, ds &\ge -\int_{x(t)}^{x(T)} q_2(s)\, ds \\ &> -\int_{\alpha(T)}^{x(T)} q_2(s)\, ds, \quad t \in [\xi, T], \end{aligned}$$

respectively. Hence

$$H(x'(t)) < H\left(\frac{1}{n}\right) + \int_{x(0)}^{\alpha(0)} q_1(s)\,ds, \quad t \in [0,\xi] \tag{2.12}$$

and

$$H(x'(t)) < H\left(\frac{1}{n}\right) + \int_{\alpha(T)}^{x(T)} q_2(s)\,ds, \quad t \in [\xi,T]. \tag{2.13}$$

Since (see (2.6))

$$x(0) = x(\xi) - \int_0^{\xi} x'(s)\,ds > \alpha(\xi) - T\|x'\| > \alpha(T) - T\|x'\|, \tag{2.14}$$

$$x(T) = x(\xi) + \int_{\xi}^{T} x'(s)\,ds < \alpha(\xi) + T\|x'\| < \alpha(0) + T\|x'\|, \tag{2.15}$$

(2.12) and (2.13) with $t = \xi$ and (2.10) show that

$$H(\|x'\|) < H(1) + \int_{\alpha(T)-T\|x'\|}^{\alpha(0)} q_1(s)\,ds, \tag{2.16}$$

$$H(\|x'\|) < H(1) + \int_{\alpha(T)}^{\alpha(0)+T\|x'\|} q_2(s)\,ds. \tag{2.17}$$

From (H_4) there exists a positive constant Λ such that

$$H(1) + \int_{\alpha(T)-Tu}^{\alpha(0)} q_1(s)\,ds < H(u)$$

and/or

$$H(1) + \int_{\alpha(T)}^{\alpha(0)+Tu} q_2(s)\,ds < H(u)$$

for $u \geq \Lambda$. Consequently (see (2.16) and (2.17)), $\|x'\| < \Lambda$. We have proved that (2.7) is satisfied, and then (see (2.11), (2.14) and (2.15)) (2.5) holds with $A = \max\{|\alpha(0)|, |\alpha(T)|\} + T\Lambda$. □

The solvability of the BVP $(E)_n^1$, $(B)_n$, $n \in \mathbb{N}$, will be proved by the topological transversality method. For this we denote $\mathbf{X} = C^1(J) \times \mathbb{R}$ the Banach space equipped with the norm $\|(x,a)\|_* = \|x\| + \|x'\| + |a|$. Set

$$\mathcal{U} = \{(x,a) : (x,a) \in \mathbf{X}, x(0) = 0\}$$

and (for $n \in \mathbb{N}$)

$$\begin{aligned} \Omega_n = \Big\{ &(x,a) : (x,a) \in \mathcal{U},\ \|x\| < 2A + T,\ \|x'\| < \Lambda, \\ &x'(t) > \frac{3}{4n} \text{ for } t \in J,\ |a| < A + |\alpha(0)| + |\alpha(T)| + T \Big\}, \end{aligned}$$

where positive constants A and Λ are given in Lemma 2.1, and the function α appears in (H_2). Then $\mathcal{U}$ is a closed and convex subset of $\mathbf{X}$ and Ω_n is an open subset of $\mathcal{U}$ for each $n \in \mathbb{N}$.

We now give the lemma which will be used in the proof of Lemma 2.3 which gives conditions for the solvability to the BVP $(\mathrm{E})_n^1$, $(\mathrm{B})_n$.

Lemma 2.2. *Let assumptions (H_2) and (H_3) be satisfied and $n \in \mathbb{N}$. Let the operator $\mathcal{K}_n : \overline{\Omega}_n \to \mathcal{U}$ be defined by*

$$\mathcal{K}_n(x, a) = \Big\{ \frac{t}{n},\, a + \int_0^T f_n(t, x(t) + a, x'(t))\, dt \Big\}. \tag{2.18}$$

Then $\mathcal{K}_n$ is essential.

Proof. Let $\mathcal{F}_n : \overline{\Omega}_n \times [0,1] \to \mathcal{U}$ be given by

$$\mathcal{F}_n(x, a, \lambda) = \Big\{ \frac{t}{n},\, a + (1-\lambda)(\alpha(0) - a) + \lambda \int_0^T f_n(t, x(t) + a, x'(t))\, dt \Big\}.$$

Then $\mathcal{F}_n(\cdot,\cdot,1) = \mathcal{K}_n(\cdot,\cdot)$ and $\mathcal{F}_n(x,a,0) = p$ for $(x,a) \in \overline{\Omega}_n$ where $p = (t/n, \alpha(0)) \in \Omega_n$. If we show that $\mathcal{F}_n$ is compact and $\mathcal{F}_n(x,a,\lambda) \neq (x,a)$ for $(x,a) \in \partial\Omega_n$ and $\lambda \in (0,1]$, then Lemma 2.2 follows from Theorem 1.3. Since $f_n \in Car(J \times \mathbb{R}^2)$, there exists $\gamma \in L_1(J)$ such that

$$|f_n(t, x, y)| \le \gamma(t) \tag{2.19}$$

for a.e. $t \in J$ and each $|x| \le 3A + |\alpha(0)| + |\alpha(T)| + 2T$, $|y| \le \Lambda$, and so $\mathcal{F}_n$ is continuous on $\overline{\Omega}_n \times [0,1]$ by the Lebesgue dominated convergence theorem and also it is easy to check (use the Arzelà–Ascoli theorem and the compactness criterion on $\mathbb{R}$) that $\mathcal{F}_n(\overline{\Omega}_n \times [0,1])$ is compact in $\mathcal{U}$. Suppose that $\mathcal{F}_n(x_0, a_0, \lambda_0) = (x_0, a_0)$ for some $(x_0, a_0) \in \partial\Omega_n$ and $\lambda_0 \in (0,1]$. Then $x_0(t) = t/n$ and

$$(1-\lambda_0)(\alpha(0) - a_0) + \lambda_0 \int_0^T f_n(t, x_0(t) + a_0, x_0'(t))\, dt = 0.$$

Set

$$r(a) = (1-\lambda_0)(\alpha(0) - a) + \lambda_0 \int_0^T f_n(t, x_0(t) + a, x_0'(t))\, dt$$

for $a \in \mathbb{R}$. Then $r \in C^0(J)$, $r(a_0) = 0$, and since $f_n(t, x_0(t) + a, x_0'(t)) = f_n(t, (t/n) + a, 1/n) = f(t, (t/n) + a, 1/n)$, we deduce from (H_2) and (H_3) that $r(a) < 0$ for $a \ge \alpha(0)$ and $r(a) > 0$ for $a < \alpha(T) - T/n$. Hence $a_0 \in (\alpha(T) - T/n, \alpha(0))$ which contradicts $(x_0, a_0) \in \partial\Omega_n$. □

Lemma 2.3. *Let assumptions (H_1)–(H_4) be satisfied. Then for each $n \in \mathbb{N}$, the BVP $(\mathrm{E})_n^1$, $(\mathrm{B})_n$ has a solution x_n and*

$$\|x_n\| < A, \tag{2.20}$$

$$x_n(\xi_n) = \alpha(\xi_n) \tag{2.21}$$

and

$$\frac{1}{n} \le x'(t) < \Lambda \quad \textit{for } t \in J, \tag{2.22}$$

where the constants A, Λ *are given in Lemma* 2.1 *and* $\xi_n \in (0,T)$ *is unique.*

Proof. Fix $n \in \mathbb{N}$ and define the operator $\mathcal{A} : \overline{\Omega}_n \times [0,1] \to \mathcal{U}$ by

$$\mathcal{A}(x,a,\lambda) = \Big\{ \int_0^t g_*^{-1}\Big(g\Big(\frac{1}{n}\Big) + \lambda \int_0^s f_n(v, x(v)+a, x'(v))\, dv \Big)\, ds,$$
$$a + \int_0^T f_n(t, x(t)+a, x'(t))\, dt \Big\}.$$

Suppose that (x_*, a_*) is a fixed point of $\mathcal{A}(\cdot,\cdot,1)$. Then

$$x_*(t) = \int_0^t g_*^{-1}\Big(g\Big(\frac{1}{n}\Big) + \int_0^s f_n(v, x_*(v)+a_*, x_*'(v))\, dv \Big)\, ds, \quad t \in J$$

and

$$\int_0^T f_n(t, x_*(t)+a_*, x_*'(t))\, dt = 0. \tag{2.23}$$

It follows that

$$g_*(x_*'(t)) = g\left(\frac{1}{n}\right) + \int_0^t f_n(s, x_*(s)+a_*, x_*'(s))\, ds, \quad t \in J$$

and (see (2.23)) $x_*'(0) = x_*'(T) = 1/n$. Setting $x_n(t) = x_*(t) + a_*$ for $t \in J$, we see that x_n is a solution of the BVP $(\mathrm{E})_n^1$, $(\mathrm{B})_n$ and the validity of (2.20)–(2.22) now follows from Lemma 2.1. Therefore to prove the existence of a solution of the BVP $(\mathrm{E})_n^1$, $(\mathrm{B})_n$ satisfying (2.20)–(2.22), we have to show that the operator $\mathcal{A}(\cdot,\cdot,1)$ has a fixed point. Since $\mathcal{A}(\cdot,\cdot,0) = \mathcal{K}_n(\cdot,\cdot)$ and $\mathcal{K}_n$ is essential by Lemma 2.2, for the existence of a fixed point of $\mathcal{A}(\cdot,\cdot,1)$ it is sufficient to verify, by Theorem 1.3, that

(i) $\mathcal{A}$ is a compact operator, and

(ii) for each $\lambda \in [0,1]$, $\mathcal{A}(\cdot,\cdot,\lambda)$ is fixed point free on $\partial\Omega_n$.

Using (2.19), $\mathcal{A}$ is continuous by the Lebesgue dominated convergence theorem and also $\mathcal{A}(\overline{\Omega}_n \times [0,1])$ is compact in $\mathcal{U}$. Thus (i) is satisfied. Let $\mathcal{A}(x_0, a_0, \lambda_0) = (x_0, a_0)$ for some $(x_0, a_0) \in \overline{\Omega}_n$ and $\lambda_0 \in [0,1]$. If $\lambda_0 = 0$ then $(x_0, a_0) \notin \partial\Omega_n$ which has been proved in the proof of Lemma 2.2. Let $\lambda_0 \in (0,1]$. Then

$$x_0(t) = \int_0^t g_*^{-1}\Big(g\Big(\frac{1}{n}\Big) + \lambda_0 \int_0^s f_n(v, x_0(v)+a_0, x_0'(v))\, dv \Big)\, ds, \quad t \in J$$

and

$$\int_0^T f_n(t, x_0(t)+a_0, x_0'(t))\, dt = 0. \tag{2.24}$$

Hence

$$g_*(x_0'(t)) = g\left(\frac{1}{n}\right) + \lambda_0 \int_0^t f_n(s, x_0(s) + a_0, x_0'(s))\, ds, \quad t \in J,$$

and (see (2.24)) $x_0'(0) = x_0'(T) = 1/n$. Setting $x_n(t) = x_0(t) + a_0$ for $t \in J$, we see that x_n is a solution of BVP $(\mathrm{E})_n^{\lambda_0}$, $(\mathrm{B})_n$. Consequently,

$$\|x_0 + a_0\| = \|x_n\| < A, \quad \frac{1}{n} \le x_n'(t) = x_0'(t) < \Lambda \quad \text{for } t \in J \tag{2.25}$$

by Lemma 2.1. Since $x_0(0) = 0$, (2.25) yields $|a_0| < A$, and so $\|x_0\| < A + |a_0| < 2A$. Hence $(x_0, a_0) \notin \partial\Omega_n$ and (ii) is verified. □

Remark 2.4. Let $n \in \mathbb{N}$. By Lemma 2.3, there exists a solution x_n of the BVP $(\mathrm{E})_n^1$, $(\mathrm{B})_n$ satisfying (2.20) and (2.22). From the definition of the functions f_n and g_* we see that $f_n(t, x_n(t), x_n'(t)) = f(t, x_n(t), x_n'(t))$, $g_*(x_n'(t)) = g(x_n'(t))$, and so

$$(g(x_n'(t)))' = f(t, x_n(t), x_n'(t)) \quad \text{for a.e. } t \in J. \tag{2.26}$$

Lemma 2.3 guarantees that for each $n \in \mathbb{N}$, there exists a solution x_n of the BVP $(\mathrm{E})_n^1$, $(\mathrm{B})_n$ and that $x_n' \ge 1/n$ on J. Since we consider solutions of the BVP (1.1), (1.2) in the class $C^1(J)$ and having positive derivative on $(0, T)$, it is important for limiting processes to know properties of $\{x_n'\}$ on neighbourhoods of the points $t = 0$ and $t = T$. This is done in Lemma 2.5.

Lemma 2.5. *Let assumptions (H_1)–(H_5) be satisfied and let x_n be a solution of the BVP $(\mathrm{E})_n^1$, $(\mathrm{B})_n$. Then there exist positive constants Δ and μ such that*

$$g(x_n'(t)) \ge \mu t, \quad g(x_n'(T - t)) \ge \mu(T - t) \quad \text{for } t \in [0, \Delta],\ n \in \mathbb{N}. \tag{2.27}$$

Proof. By Lemma 2.3 and Remark 2.4, there exist $\{\xi_n\} \subset (0, T)$ and a positive constant Λ such that

$$x_n(\xi_n) = \alpha(\xi_n) \tag{2.28}$$

and

$$\frac{1}{n} \le x_n'(t) < \Lambda \tag{2.29}$$

for $t \in J$ and $n \in \mathbb{N}$, and also (2.26) is true. In addition (see (2.10)),

$$\max\{x_n'(t) : t \in J\} = x_n'(\xi_n) \tag{2.30}$$

and, by (H_3) and (2.26), x_n' is increasing on $[0, \xi_n]$ and decreasing on $[\xi_n, T]$.

We now show that

$$\chi \le \xi_n \le T - \chi \quad \text{for } n \in \mathbb{N}, \tag{2.31}$$

where χ is a positive number. First, assume on the contrary that $\lim_{n\to\infty} \xi_{k_n} = 0$ for a subsequence $\{\xi_{k_n}\}$ of $\{\xi_n\}$. Then from $\lim_{n\to\infty} x_{k_n}(\xi_{k_n}) = \lim_{n\to\infty} \alpha(\xi_{k_n}) = \alpha(0)$ and

$$0 < x_{k_n}(\xi_{k_n}) - x_{k_n}(0) = x_{k_n}'(\tau_n)\xi_{k_n} < \Lambda \xi_{k_n}$$

where $\tau_n \in (0, \xi_{k_n})$, we deduce that $\lim_{n\to\infty} x_{k_n}(0) = \alpha(0)$ and then using the inequalities (see (2.12) with x_{k_n} and ξ_{k_n} instead of x and t, respectively)

$$\frac{1}{k_n} \le x'_{k_n}(\xi_{k_n}) < H^{-1}\Big(H\Big(\frac{1}{k_n}\Big) + \int_{x_{k_n}(0)}^{\alpha(0)} q_1(s)\,ds\Big), \quad n \in \mathbb{N},$$

we get $\lim_{n\to\infty} x'_{k_n}(\xi_{k_n}) = 0$. Now (2.29) and (2.30) give

$$\lim_{n\to\infty} x'_{k_n}(t) = 0 \quad \text{uniformly on } J. \tag{2.32}$$

Since $x_{k_n}(T) > \alpha(\xi_{k_n})$ and α is decreasing on J by (H_2), we can assume (going if necessary to a subsequence) that

$$x_{k_n}(t) > \alpha\left(\frac{T}{2}\right) + \varepsilon \ge \alpha(t) + \varepsilon, \quad 0 < x'_{k_n}(t) \le 1 \quad \text{for } t \in \left[\frac{T}{2}, T\right],\ n \in \mathbb{N}, \tag{2.33}$$

where ε is a positive constant. By (H_5), there is a $\delta > 0$ such that $f(t, x, y) \le -\delta$ for a.e. $t \in [T/2, T]$ and each $x \ge \alpha(t) + \varepsilon$, $y \in (0, 1]$. Therefore (see (2.26) and (2.33))

$$(g(x'_{k_n}(t)))' \le -\delta \quad \text{for a.e. } t \in \left[\frac{T}{2}, T\right] \text{ and each } n \in \mathbb{N},$$

which implies

$$g\left(x'_{k_n}\left(\frac{T}{2}\right)\right) - g(x'_{k_n}(T)) = -\int_{T/2}^{T} (g(x'_{k_n}(t)))'\,dt \ge \frac{\delta T}{2}, \quad n \in \mathbb{N},$$

contrary to (2.32). Analogously we can show that $\limsup_{n\to\infty} \xi_n < T$. We have verified that (2.31) is true with a positive constant χ. Then $x_n(0) - \alpha(0) < x_n(\xi_n) - \alpha(0) = \alpha(\xi_n) - \alpha(0) \le \alpha(\chi) - \alpha(0)$ and $x_n(T) - \alpha(T) > x_n(\xi_n) - \alpha(T) = \alpha(\xi_n) - \alpha(T) \ge \alpha(T - \chi) - \alpha(T)$. Hence

$$x_n(0) < \alpha(0) - \phi, \quad x_n(T) > \alpha(T) + \phi \quad \text{for } n \in \mathbb{N}. \tag{2.34}$$

where $\phi = \min\{\alpha(0) - \alpha(\chi), \alpha(T - \chi) - \alpha(T)\} > 0$. We now claim that there exists $\Delta_1 > 0$ such that

$$x_n(t) < \alpha(t) - \frac{\phi}{2} \quad \text{for } t \in [0, \Delta_1],\ n \in \mathbb{N}. \tag{2.35}$$

If not, there are a subsequence $\{l_n\}$ of $\{n\}$ and a sequence $\{t_n\} \subset (0, T)$, $\lim_{n\to\infty} t_n = 0$, such that

$$x_{l_n}(t_n) = \alpha(t_n) - \frac{\phi}{2}, \quad n \in \mathbb{N}.$$

Then (see (2.34))

$$x_{l_n}(t_n) - x_{l_n}(0) > \alpha(t_n) - \frac{\phi}{2} - \alpha(0) + \phi = \alpha(t_n) - \alpha(0) + \frac{\phi}{2}, \quad n \in \mathbb{N},$$

contrary to

$$\lim_{n\to\infty} (x_{l_n}(t_n) - x_{l_n}(0)) = \lim_{n\to\infty} x'_{l_n}(\eta_n) t_n \le \Lambda \lim_{n\to\infty} t_n = 0, \quad \eta_n \in (0, t_n).$$

Since, by (H_5), there is a $\delta_1^* > 0$ such that $f(t,x,y) \geq \delta_1^*$ for a.e. t where $x \leq \alpha(t) - \phi/2$ and $y \in (0,1]$, (2.35) shows that $(g(x_n'))' \geq \delta_1^*$ a.e. on any subinterval of $[0,\Delta_1]$ where $x_n' \leq 1$. Hence (note $x_n'(0) = 1/n$)

$$g(x_n'(t)) \geq \delta_1 t, \quad t \in [0,\Delta_1],\ n \in \mathbb{N}$$

with $\delta_1 = \min\{g(1)/\Delta_1, \delta_1^*\}$. Similar arguments show that there exist positive constants δ_2 and Δ_2 such that

$$g(x_n'(T-t)) \geq \delta_2(T-t), \quad t \in [0,\Delta_2],\ n \in \mathbb{N}.$$

Therefore (2.27) is true with $\mu = \min\{\delta_1, \delta_2\}$ and $\Delta = \min\{\Delta_1, \Delta_2\}$. □

3. Existence results and examples

Theorem 3.1. *Let assumptions (H_1)–(H_5) be satisfied. Then the BVP (1.1), (1.2) has a solution x such that $x'(t) > 0$ for $t \in (0,T)$.*

Proof. From Lemma 2.3 it follows that the BVP $(\mathrm{E})_n^1$, $(\mathrm{B})_n$ has a solution x_n for each $n \in \mathbb{N}$,

$$\|x_n\| < A, \tag{3.1}$$

$$\frac{1}{n} \leq x_n'(t) < \Lambda \tag{3.2}$$

for $t \in J$, where A and Λ are positive constants and, by Remark 2.4, (2.26) is satisfied. In addition, by Lemma 2.5, there exist positive constants Δ and μ such that (2.27) is true. From (2.27) and the properties of x_n we deduce that

$$x_n'(t) \geq \eta(t) \quad \text{for } t \in J,\ n \in \mathbb{N} \tag{3.3}$$

where

$$\eta(t) = \begin{cases} g^{-1}(\mu t) & \text{for } t \in [0,\Delta] \\ g^{-1}(\mu\Delta) & \text{for } t \in (\Delta, T-\Delta) \\ g^{-1}(\mu(T-t)) & \text{for } t \in [T-\Delta, T]. \end{cases}$$

Now (H_3), (3.1) and (3.2) yield

$$|f(t,x_n(t),x_n'(t))| \leq \gamma(t) \quad \text{for a.e. } t \in J \text{ and each } n \in \mathbb{N} \tag{3.4}$$

with

$$\gamma(t) = (\omega_1(\Lambda) + \omega_2(\eta(t))) \max\Big\{ \max_{-A \leq u \leq \alpha(0)} q_1(u), \max_{\alpha(T) \leq u \leq A} q_2(u) \Big\}.$$

Since $\int_0^1 \omega_2(g^{-1}(s))\,ds < \infty$ in (H_3) and Remark 1.1 imply $\omega_2(\eta(t)) \in L_1(J)$, we have $\gamma \in L_1(J)$.

Now $\{g(x_n'(t))\}$ is equicontinuous on J which follows from

$$|g(x_n'(t_2)) - g(x_n'(t_1))| = \Big| \int_{t_1}^{t_2} f(t,x_n(t),x_n'(t))\,dt \Big| \leq \Big| \int_{t_1}^{t_2} \gamma(t)\,dt \Big|$$

for $t_1, t_2 \in J$ and $n \in \mathbb{N}$. The equalities

$$|x_n'(t_2) - x_n'(t_1)| = |g^{-1}(g(x_n'(t_2))) - g^{-1}(g(x_n'(t_1)))|, \quad t_1, t_2 \in J, \ n \in \mathbb{N}$$

and g^{-1} being continuous and increasing on $[0, \infty)$ imply that also $\{x_n'(t)\}$ is equicontinuous on J. From (3.1) and (3.2), $\{x_n\}$ is bounded in $C^1(J)$, and therefore the Arzelà–Ascoli theorem guarantees that without loss of generality we can assume that $\{x_n\}$ is convergent in $C^1(J)$. Let $\lim_{n\to\infty} x_n = x$. Then $x \in C^1(J)$, $x'(0) = x'(T) = 0$ and (see (3.3)) $x'(t) \geq \eta(t)$ for $t \in J$. Therefore $x'(t) > 0$ for $t \in (0, T)$,

$$\lim_{n\to\infty} f(t, x_n(t), x_n'(t)) = f(t, x(t), x'(t)) \quad \text{for a.e. } t \in J$$

and letting $n \to \infty$ in

$$g(x_n'(t)) = g\left(\frac{1}{n}\right) + \int_0^t f_n(s, x_n(s), x_n'(s))\, ds, \quad t \in J, \ n \in \mathbb{N},$$

we have (see (3.4))

$$g(x'(t)) = \int_0^t f(s, x(s), x'(s))\, ds, \quad t \in J,$$

by the Lebesgue dominated convergence theorem. Hence $g(x') \in AC(J)$ and x satisfies (1.1) a.e. on J. We have proved that x is a solution of the BVP (1.1), (1.2) and $x'(t) > 0$ for $t \in (0, T)$. □

Example 3.2. Consider the differential equation

$$((x')^p)' = r(t)(\alpha(t) - x)^{2n-1}\Big((x')^\gamma + \frac{m}{(x')^\varrho}\Big) \tag{3.5}$$

where $r \in L_1(J)$, $0 < a \leq r(t) \leq b$ for a.e. $t \in J$, $\alpha \in C^0(J)$ is decreasing, p, γ, ϱ and m are positive constants, $\varrho < p$, $1 + p - \gamma > 0$, and $n \in \mathbb{N}$. Set

$$f(t, x, y) = r(t)(\alpha(t) - x)^{2n-1}\Big(y^\gamma + \frac{m}{y^\varrho}\Big)$$

for $(t, x, y) \in J \times \mathbb{R} \times (0, \infty)$. Then (H_1) is satisfied with $g(u) = u^p$ and f satisfies (H_2). (H_3) is satisfied with $q_1(x) = b(\alpha(0) - x)^{2n-1}$, $q_2(x) = b(x - \alpha(T))^{2n-1}$, $\omega_1(y) = y^\gamma$, $\omega_2(y) = m/y^\varrho$ and (H_5) with $\delta = am\varepsilon^{2n-1}$. It remains to look at (H_4). For that purpose define $h \in C^0([0, \infty))$ by $h(u) = u/(m + u)$ for $u \in [0, \infty)$. Then h is increasing, $h(0) = 0$ and $\lim_{u\to\infty} h(u) = 1$. Thus for $0 < v < u$ we have

$$\begin{aligned} H(u) &= \int_0^{g(u)} \frac{g^{-1}(s)}{\omega_1(g^{-1}(s)) + \omega_2(g^{-1}(s))}\, ds = \int_0^{u^p} \frac{s^{1/p}}{s^{\gamma/p} + ms^{-\varrho/p}}\, ds \\ &= H(v) + \int_{v^p}^{u^p} h(s^{(\gamma+\varrho)/p}) s^{(1-\gamma)/p}\, ds \geq H(v) + h(v^{\gamma+\varrho}) \int_{v^p}^{u^p} s^{(1-\gamma)/p}\, ds \\ &= H(v) + \frac{ph(v^{\gamma+\varrho})}{1 + p - \gamma}(u^{1+p-\gamma} - v^{1+p-\gamma}). \end{aligned}$$

Since

$$\int_{\alpha(T)-Tu}^{\alpha(0)} q_1(s)\,ds = b\int_{\alpha(T)-Tu}^{\alpha(0)} (\alpha(0)-s)^{2n-1} = \frac{b}{2n}(Tu+\alpha(0)-\alpha(T))^{2n}$$

and

$$\int_{\alpha(T)}^{\alpha(0)+Tu} q_2(s)\,ds = b\int_{\alpha(T)}^{\alpha(0)+Tu} (s-\alpha(T))^{2n-1} = \frac{b}{2n}(Tu+\alpha(0)-\alpha(T))^{2n},$$

we get

$$\begin{aligned} S &:= \min\left\{ \limsup_{u\to\infty} \frac{\int_{\alpha(T)-Tu}^{\alpha(0)} q_1(s)\,ds}{H(u)},\ \limsup_{u\to\infty} \frac{\int_{\alpha(T)}^{\alpha(0)+Tu} q_2(s)\,ds}{H(u)} \right\} \\ &\le \limsup_{u\to\infty} \frac{b(Tu+\alpha(0)-\alpha(T))^{2n}}{2n\Big(H(v)+\dfrac{ph(v^{\gamma+\varrho})}{1+p-\gamma}(u^{1+p-\gamma}-v^{1+p-\gamma})\Big)} \end{aligned}$$

for $0<v<u$. Hence

$$S \le \begin{cases} 0 & \text{if } 2n+\gamma-1<p \\ \dfrac{bT^{2n}}{ph(v^{\gamma+\varrho})} & \text{if } 2n+\gamma-1=p \end{cases}$$

for $v\in(0,\infty)$, and consequently (letting $v\to\infty$)

$$S \le \begin{cases} 0 & \text{if } 2n+\gamma-1<p \\ \dfrac{bT^{2n}}{p} & \text{if } 2n+\gamma-1=p. \end{cases}$$

We see that f satisfies (H_4) if either $2n+\gamma-1<p$ or $2n+\gamma-1=p$ and $bT^{2n}<p$. Summarizing, by Theorem 3.1, the BVP (3.5), (1.2) with positive constants p,γ,ϱ,m and $n\in\mathbb{N}$ in (3.5) satisfying $\varrho<p$ has a solution x with $x'>0$ on $(0,T)$ if either $2n+\gamma-1<p$ or $2n+\gamma-1=p$ and $bT^{2n}>p$.

Example 3.3. Consider the differential equation

$$((x')^p)' = \frac{e^{\alpha(t)-x}-1}{h(xx')(x')^\gamma} \tag{3.6}$$

where p and γ are positive constants, $h\in C^0(\mathbb{R})$, $1\le h(u)\le M$ for $u\in\mathbb{R}$ and $\alpha\in C^0(J)$ is decreasing. Set

$$f(t,x,y) = \frac{e^{\alpha(t)-x}-1}{h(xy)y^\gamma}$$

for $(t,x,y)\in J\times\mathbb{R}\times(0,\infty)$. Then f satisfies (H_2) and (H_1) is true with $g(u)=u^p$. (H_3) is satisfied with $q_1(x)=e^{\alpha(0)-x}-1$, $q_2(x)=1-e^{\alpha(T)-x}$, $\omega_1(y)\equiv 0$ and

$\omega_2(y) = y^{-\gamma}$, and (H_5) with $\delta = (1 - e^{-\varepsilon})/M$. Since

$$\int_{\alpha(T)}^{\alpha(0)+Tu} q_2(s)\,ds = \int_{\alpha(T)}^{\alpha(0)+Tu} (1 - e^{\alpha(T)-s})\,ds = Tu + e^{\alpha(T)-\alpha(0)-Tu} + \alpha(0) - \alpha(T) - 1$$

and

$$H(u) = \int_0^{g(u)} \frac{g^{-1}(s)}{\omega_1(g^{-1}(s)) + \omega_2(g^{-1}(s))}\,ds = \int_0^{u^p} s^{(\gamma+1)/p}\,ds = \frac{pu^{p+\gamma+1}}{p+\gamma+1}$$

for $u \in [0, \infty)$, we have $\lim_{u\to\infty} H(u) = \infty$,

$$\lim_{u\to\infty} \frac{\int_{\alpha(T)}^{\alpha(0)+Tu} q_2(s)\,ds}{H(u)} = \lim_{u\to\infty} \frac{(p+\gamma+1)(Tu + e^{\alpha(T)-\alpha(0)-Tu} + \alpha(0) - \alpha(T) - 1)}{pu^{p+\gamma+1}} = 0,$$

and therefore (H_4) is satisfied. Hence Theorem 3.1 guarantees the existence of a solution x of BVP (3.6), (1.2) with $x' > 0$ on $(0, T)$.

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