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On inequalities associated with the Jordan–von Neumann functional equation

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Summary. For a group $(G, \cdot)$ and a real or complex inner product space $(E, \langle \cdot, \cdot \rangle)$ with norm $[\![.]\!]$ we consider the functional inequality

$$f : G \longrightarrow E,\ [\![2f(x) + 2f(y) - f(xy^{-1})]\!] \leq [\![f(xy)]\!] \qquad (\forall x, y \in G) \tag{I}$$

and describe situations in which (I) implies the Jordan–von Neumann parallelogram equation

$$f : G \longrightarrow E,\ 2f(x) + 2f(y) = f(xy) + f(xy^{-1}) \qquad (\forall x, y \in G). \tag{JvN}$$

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1. Introduction

Throughout the paper, G or $(G, \cdot)$ or $(G, \cdot, e)$ will denote a (not necessarily abelian) group with group operation $\cdot$ and identity element e and E or $(E, \langle \cdot, \cdot \rangle)$ an inner product space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) with inner product $\langle \cdot, \cdot \rangle$ and associated norm $[\![.]\!]$.

Starting from the Jordan–von Neumann functional equation (JvN), fourteen inequalities may be obtained by first transferring at most one summand of either side of (JvN) to the other, then taking norms, and finally replacing $=$ by $\leq$ or $\geq$. Of course, each of these inequalities is a weakening of (JvN). The question is whether this weakening is always strict. In [4] (p. 304, Satz 1; p. 307, Bemerkungen 1 und 2), A. Gilányi comes to the quite surprising conclusion that exactly one of these inequalities (namely (I) above) is equivalent to (JvN). The main result of [4] reads as follows:

Theorem 1.1. (A. Gilányi).

a) *If $(G, \cdot, e)$ is a 2-divisible abelian group and $(E, \langle \cdot, \cdot \rangle)$ an inner product space over $\mathbb{K}$, then* (I) *implies* (JvN)*, i.e., f is quadratic.*
b) *In part* a)*, the commutativity of G may be replaced by the condition*

$$f(xyz) = f(xzy) \qquad (x, y, z \in G). \tag{C}$$

It is the purpose of this paper to show that 2-divisibility of G can be deleted and that (C) can be essentially weakened and to exhibit classes of examples of groups in which we do not have to care about any commutation condition. The results were announced in [5] and [12], and variants of Theorem 1.1 were discussed on the 39th International Symposium on Functional Equations (cf. [6], [12]). The question as to whether in the general case, some substitute for commutativity is needed for (I) $\Longrightarrow$ (JvN) remains open.

Further notations. The symbol := means that the right hand side defines the left hand side. The symbols $\mathbb{N}$, $\mathbb{N}^0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational, real, and complex numbers, respectively; $\mathbb{R}_+ := \{\alpha \in \mathbb{R};\ \alpha \geq 0\}$, $\mathbb{R}_+^* := \{\alpha \in \mathbb{R}; \alpha > 0\}$. $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{K}^* := \{\alpha \in \mathbb{K}; \alpha \neq 0\}$. We denote by $\underline{c}$ the constant mapping with value c, and by i_B the identity mapping of the set B. For an easy unified treatment of the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, we put for $\mathbb{K} = \mathbb{R}$: Re $:= i_{\mathbb{R}}$, Im $:= \underline{0}$. We use 0 for the zero vector of E as well as for the number zero and for the identity element of an additively written abelian group; it will always be clear from the context what is meant.

2. Substitutes for commutativity of G

Remark 2.1. To the author's knowledge, the first occurrence of condition (C) (cf. Theorem 1.1 b) above) is in PL. Kannappan's Ph.D. Thesis work where it plays a fundamental role in the theory of d'Alembert's functional equation (cum grano salis also called the "cosine equation", [9], p. 72, Theorem 2).

Remark 2.2. Let for the moment M be a nonempty set and $f : G \longrightarrow M$.

a) (C) turns out to be equivalent to the invariance of $f(z_1 \cdot \ldots \cdot z_n)$ ($n \in \mathbb{N}$, $n \geq 2$; $z_1, \ldots, z_n \in G$) under all permutations of the factors $z_1, \ldots, z_n$. Therefore (C) implies each of the three conditions

$$f(xy) = f(yx) \qquad (\forall x, y \in G), \tag{C$'$}$$

$$f(xyxy^{-1}) = f(x^2) \qquad (\forall x, y \in G), \tag{C$''$}$$

$$f(y^{-1}x^{-1}yx^{-1}) = f(x^{-2}) \qquad (\forall x, y \in G). \tag{C$'''$}$$

By the way, (C$'$) is equivalent to the invariance of $f(z_1 \cdot \ldots \cdot z_n)$ ($n \in \mathbb{N}$, $n \geq 2$; $z_1, \ldots, z_n \in G$) under all circular rearrangements (i.e., under all powers of the index permutation $(1\,2 \ldots n)$) of the factors $z_1, \ldots, z_n$.

b) It is easily seen that in case of an even mapping $f : G \longrightarrow M$ (i.e., $f(x^{-1}) = f(x)$ $(\forall x \in G)$), (C$''$) and (C$'''$) are equivalent.

3. Variants of Theorem 1.1

A useful tool for later purposes will be

Lemma 3.1. *For elements a, b of the inner product space E*

$$\|a\|^2 \le \mathrm{Re}\langle a, b\rangle \tag{1}$$

and

$$\|a\| = \|b\| \tag{2}$$

imply $a = b$.

Proof. We have $\|a-b\|^2 = \|a\|^2 - 2\mathrm{Re}\langle a,b\rangle + \|b\|^2 =_{(2)}= 2\|a\|^2 - 2\mathrm{Re}\langle a,b\rangle \le_{(1)}\le 0$, so $a = b$.

The next theorem strengthens Theorem 1.1 by deleting the divisibility assumption and by weakening the commutativity requirement for G.

Theorem 3.2. *Let $(G, \cdot, e)$ be an arbitrary group, $(E, \langle\cdot,\cdot\rangle)$ an inner product space over $\mathbb{K}$, and $f : G \longrightarrow E$ a solution of* (I). *Then we have*

a) $f(e) = 0,\ f(x^{-1}) = f(x),\ f(x^2) = 4f(x)\ \ (\forall x \in G)$.

b) *If f satisfies* (C$''$) *(or, equivalently,* (C$'''$)*), then f is quadratic.*

Proof. a) We first follow the proof of ([4], Satz 1) to obtain

$$f(e) = 0 \tag{3}$$

and

$$2f(x) + 2f(x^{-1}) = f(x^2) \qquad (\forall x \in G) \tag{4}$$

and now deviate from it. Taking $x \in G$ arbitrarily, putting $y := x$ in (I) and using (3) we get

$$\|4f(x)\| \le \|f(x^2)\| \qquad (\forall x \in G). \tag{5}$$

Again let $x \in G$ be arbitrary. Then $\|4f(x)\| \le_{(5)}\le \|f(x^2)\| =_{(4)}= \|2f(x) + 2f(x^{-1})\| \le \|2f(x)\| + \|2f(x^{-1})\|$, so $\|2f(x)\| \le \|2f(x^{-1})\|$, i.e. $\|f(x)\| \le \|f(x^{-1})\|$ $(\forall x \in G)$, and for x^{-1} instead of x, we get also the reverse inequality, therefore

$$\|f(x^{-1})\| = \|f(x)\| \quad (\forall x \in G). \tag{6}$$

Furthermore $\|f(x^2)\| =_{(4)}= \|2f(x) + 2f(x^{-1})\| \le \|2f(x)\| + \|2f(x^{-1})\| =_{(6)}= 4\|f(x)\| \le_{(5)}\le \|f(x^2)\|$, so

$$\|f(x^2)\| = 4\|f(x)\| \qquad (\forall x \in G). \tag{7}$$

Using (JvN) for $\|.\|^2$, we obtain $\|f(x) - f(x^{-1})\|^2 = 2\|f(x)\|^2 + 2\|f(x^{-1})\|^2 - \|f(x) + f(x^{-1})\|^2 =_{(6),(4)}= 4\|f(x)\|^2 - \frac{1}{4}\|f(x^2)\|^2 =_{(7)}= 4\|f(x)\|^2 - 4\|f(x)\|^2 = 0$, so

$$f(x^{-1}) = f(x) \qquad (\forall x \in G), \tag{8}$$

and together with (4)

$$f(x^2) = 4f(x) \qquad (\forall x \in G). \tag{9}$$

b) By (8) and Remark 2.2 b), conditions (C$''$) and (C$'''$) are equivalent. An inspection of the part of the proof of ([4], Satz 1) following its formula (14) shows that in fact only condition (C$'''$) is needed for the simplification of its formula (18), and (C$'''$) is strictly weaker than (C) in our context as we shall see next.

Remark 3.3. a) Every quadratic mapping $f : G \longrightarrow E$ satisfies (3) since the additive group $(E, +)$ of our inner product space does not contain elements of order 2 ([11], p. 247, Lemma 4.1 a), d), therefore also (8), (C$'$), (C$''$) and (C$'''$) ([7], p. 190, (2), (3), (5)). Not every quadratic mapping satisfies (C) of Theorem 1.1 b) (cf. [8], p. 36/37; worked out in [10]). Therefore, for quadratic mappings $f : G \longrightarrow E$, conditions (C$'$), (C$''$), (C$'''$), and, e.g., (C$'$) $\wedge$ (C$''$) are strictly weaker than (C); remember Remark 2.2 a) for (C) $\Longrightarrow$ (C$'$) $\wedge$ (C$''$) $\wedge$ (C$'''$) in general.

b) To use a commutativity substitute logically "below" the target assertion (JvN) in Theorem 3.2 b) is of course more adequate than (C), which is "beyond" (JvN).

Remark 3.4. a) Other functional equations for the solutions of which (C$'$) is strictly weaker that (C) are, e.g., the d'Alembert equation

$$f : G \longrightarrow \mathbb{C},\ f(e) = 1,\ f(xy) + f(xy^{-1}) = 2f(x)f(y)\ \ (\forall x, y \in G) \tag{d'A}$$

([3], p. 339, (3); J. Lawrence, private communication), or

$$f, g, h : G \longrightarrow \mathbb{C},\ f(xy) + f(xy^{-1}) - 2f(x) = g(x)h(y)\ \ (\forall x, y \in G) \tag{ACN}$$

([1], p. 20/21, Remark 5).

b) On the other hand, there do exist functional equations for the solutions of which (C) and (C$'$) become equivalent, e.g.,

$$f : G \longrightarrow \mathbb{C}, e_0 \in G,\quad e_0 \neq e,\quad f(e_0) = 1,\quad f(xy) + f(xy)^{-1} = 2f(x)f(e_0y) \\ (\forall x, y \in G)$$

([2]), p. 51, Lemma 2).

At the end of this section, we present another stronger variant of Theorem 1.1 which is weaker than Theorem 3.2 b) but the proof of which reveals a connection to one of the thirteen inequalities besides (I) associated with (JvN), namely to (14) below.

Theorem 3.5. *If $(G, \cdot, e)$ is an arbitrary group, $(E, \langle\cdot,\cdot\rangle)$ an inner product space over $\mathbb{K}$, and $f : G \rightarrow E$ a solution of* (I) *satisfying* (C$'$) $\wedge$ (C$''$), *then f is quadratic.*

Proof. By Theorem 3.2 a), f has properties (3), (8), and (9). (I) yields

$$\|2f(x)+2f(y)\|^2+\|f(xy^{-1})\|^2-2\mathrm{Re}\langle 2f(x)+2f(y),f(xy^{-1})\rangle \le \|f(xy)\|^2 \quad (\forall x,y\in G). \tag{10}$$

Replacing y by y^{-1} and using (8) gives

$$\|2f(x)+2f(y)\|^2+\|f(xy)\|^2-2\mathrm{Re}\langle 2f(x)+2f(y),f(xy)\rangle \le \|f(xy^{-1})\|^2 \quad (\forall x,y\in G). \tag{11}$$

By adding (10) and (11) and dividing by 2 we obtain

$$\|2f(x)+2f(y)\|^2 \le \mathrm{Re}\langle 2f(x)+2f(y),f(xy)+f(xy^{-1})\rangle \quad (\forall x,y\in G). \tag{12}$$

(12) together with $\mathrm{Re}\,\beta \le |\mathrm{Re}\,\beta| \le |\beta|$ $(\forall \beta\in\mathbb{K})$ and the Cauchy–Schwarz Inequality (CSI) lead to

$$\|2f(x)+2f(y)\|^2 \le \|2f(x)+2f(y)\|\cdot\|f(xy)+f(xy^{-1})\| \quad (\forall x,y\in G). \tag{13}$$

If $\|2f(x)+2f(y)\|>0$, we get from (13) $\|2f(x)+2f(y)\|\le\|f(xy)+f(xy^{-1})\|$, which holds also for $\|2f(x)+2f(y)\|=0$, so

$$\|2f(x)+2f(y)\|\le\|f(xy)+f(xy^{-1})\| \quad (\forall x,y\in G). \tag{14}$$

Let $x,y\in G$ be arbitrary and put $u:=xy$, $v:=xy^{-1}$. Then $\|2f(xy)+2f(xy^{-1})\| = \|2f(u)+2f(v)\| \le_{(14)} \le \|f(uv)+f(uv^{-1})\| = \|f(xyxy^{-1})+f(xy^2x^{-1}\| =_{(\mathrm{C}''),(\mathrm{C}')}= \|f(x^2)+f(y^2)\| =_{(9)}= \|4f(x)+4f(y)\|$, briefly $\|f(xy)+f(xy^{-1})\|\le \|2f(x)+2f(y)\|$ and together with (14)

$$\|2f(x)+2f(y)\|=\|f(xy)+f(xy^{-1})\| \qquad (\forall x,y\in G). \tag{15}$$

Now (12), (15) and Lemma 3.1 with $a:=2f(x)+2f(y)$, $b:=f(xy)+f(xy^{-1})$ imply $2f(x)+2f(y)=f(xy)+f(xy^{-1})$. Since $x,y\in G$ were arbitrary, this is (JvN).

4. Homogeneity of degree 2

Remark 4.1. Theorem 3.2 a) guarantees (3), (8), and (9) for every solution $f: G\to E$ of (I). These three properties alone do not imply homogeneity of f of degree 2, i.e. the validity of

$$f(x^k)=k^2f(x) \qquad (\forall k\in\mathbb{Z},\ \forall x\in G): \tag{16}$$

Choose $(G,\cdot):=(\mathbb{R},+)$, $E:=\mathbb{R}$, $f:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $f(x):=4^{n_x}$ for $2^{n_x}\le x< 2^{n_x+1}$, $f(0):=0$, $f(x):=f(-x)$ for $x<0$, where $n_x\in\mathbb{Z}$ is uniquely determined for every $x\in\mathbb{R}_+^*$. Then f has the three properties above, (9) of course in the form $f(2x)=4f(x)$ $(\forall x\in\mathbb{R})$, but $f(3\cdot 1)=f(3)=4$ while $9f(1)=9\cdot 1=9$. On the other hand, with the aid of (I), (16) can be established, so that the solutions of (I) and those of (JvN) share property (16):

Theorem 4.2. *If $(G, \cdot, e)$ is an arbitrary group, $(E, \langle \cdot, \cdot \rangle)$ an inner product space over $\mathbb{K}$, and $f : G \to E$ a solution of* (I)*, then we have for every $k \in \mathbb{Z}$*

$$f(x^k) = k^2 f(x) \qquad (\forall x \in G). \tag{16_{k}}$$

First proof. Let be $x \in G$ arbitrary and C_x the subgroup of G generated by x. Since C_x is abelian, the restriction $f|C_x$ satisfies (I) on C_x and condition (C$''$) and therefore is quadratic by Theorem 3.2 b). Since $(f|C_x)(e) = f(e) = 0$ by Theorem 3.2 a), $f|C_x$ is homogeneous of degree 2 ([11], p. 247, Lemma 4.1c). As $x \in G$ was arbitrary, (16) holds.

Second proof. (Induction on k; direct work with (I)). The statements (16_0), (16_{-1}), (16_2) are ensured by Theorem 3.2 a), (3), (8), (9), respectively, and (16_1) is trivial. By (8), it is sufficient to consider the case $k \in \mathbb{N}^0$. So let be $k \in \mathbb{N}$, $k \geq 2$, and assume (16_ℓ) to hold for $\ell = 0, 1, \ldots, k-1$. For completing the proof, we have to show (16_k) (second principle of induction). Case 1: k is even, say $k = 2q$ with $q \geq 1$. Then (16_k) easily follows from (16_q) and (9). – Case 2: k is odd, say $k = 2q+1$ with $q \geq 1$. Then $k+1 = 2(q+1)$, and (16_{k+1}) is obtained as in Case 1. Let $x \in G$ be arbitrary. We first replace the pair (x, y) in (I) by (x^{k-1}, x), and (16_{k-1}), (16_{k-2}) yield

$$\|k^2 f(x)\| \leq \|f(x^k)\| \qquad (\forall x \in G). \tag{17}$$

Now (x, y) is replaced in (I) by (x^k, x), and (16_{k-1}), (16_{k+1}) and squaring lead to

$$\|f(x^k)\|^2 \leq ((k-1)^2 - 2)\mathrm{Re}\langle f(x^k), f(x)\rangle + k^2(2k+1)\|f(x)\|^2 \qquad (\forall x \in G). \tag{18}$$

Next, (x, y) is substituted in (I) by (x^k, x^{-1}), and (8), (16_{k+1}), (16_{k-1}) similarly imply

$$\|f(x^k)\|^2 \leq ((k+1)^2 - 2)\mathrm{Re}\langle f(x^k), f(x)\rangle + k^2(1-2k)\|f(x)\|^2 \qquad (\forall x \in G). \tag{19}$$

The combination $(2k-1) \cdot (18) + (2k+1) \cdot (19)$ provides

$$\|f(x^k)\|^2 \leq k^2 \mathrm{Re}\langle f(x^k), f(x)\rangle = \mathrm{Re}\langle f(x^k), k^2 f(x)\rangle \qquad (\forall x \in G). \tag{20}$$

From (20) and the CSI we get

$$\|f(x^k)\|^2 \leq \|f(x^k)\| \cdot \|k^2 f(x)\| \qquad (\forall x \in G). \tag{21}$$

If $f(x) = 0$, we have equality in (21). If $f(x) \neq 0$, (17) enforces $f(x^k) \neq 0$, so that (21) implies $\|f(x^k)\| \leq \|k^2 f(x)\|$, and this is now true for all $x \in G$. Together with (17) we obtain

$$\|f(x^k)\| = \|k^2 f(x)\| \qquad (\forall x \in G). \tag{22}$$

Finally, (20), (22) and Lemma 3.1 with $a := f(x^k)$, $b := k^2 f(x)$ ensure $f(x^k) = k^2 f(x)$ $(\forall x \in G)$, i.e., again (16_k), and the second proof is complete.

Corollary 4.3. *If $f : G \to E$ is a solution of* (I) *and H a proper subgroup of G such that f is bounded on $G \setminus H$, then $f = \underline{0}$, so f is quadratic.*

Proof. Let $\llbracket f(y) \rrbracket < c$ $(\forall y \in G \setminus H)$ for some $c \in \mathbb{R}_+^*$. Let $y \in G \setminus H$ be arbitrary and assume $f(y) \neq 0$. Then there would exist $n_0 \in \mathbb{N}$ with $n_0^2 \llbracket f(y) \rrbracket \geq c$, i.e., by Theorem 4.2, $\llbracket f(y^{n_0}) \rrbracket \geq c$, so $y^{n_0} \in H$. $y^{n_0+1} \in H$ would imply $y = y^{n_0+1}(y^{n_0})^{-1} \in H$, contradicting the definition of y. Therefore $y^{n_0+1} \in G \setminus H$, hence $(n_0+1)^2 \llbracket f(y) \rrbracket = \llbracket f(y^{n_0+1} \rrbracket < c \leq n_0^2 \llbracket f(y) \rrbracket$, which is impossible. So $f(y) = 0$. Since $y \in G \setminus H$ was arbitrary, we have

$$f(y) = 0 \qquad (\forall y \in G \setminus H). \tag{23}$$

Now let $x \in H$ be arbitrary and $y \in G \setminus H$ (notice that $H \neq G$). $xy \in H$ would imply $y = x^{-1}xy \in H$. $xy^{-1} \in H$ would imply $y^{-1} = x^{-1}xy^{-1} \in H$, so $y \in H$. Therefore xy, $xy^{-1} \in G \setminus H$, and it follows from (I) and (23) that $\llbracket 2f(x) + 0 - 0 \rrbracket \leq \llbracket 0 \rrbracket$, i.e. $f(x) = 0$ $(\forall x \in H)$, and together with (23) finally $f = \underline{0}$.

5. Other situations where (I) does imply (JvN)

Lemma 5.1. *Let G be an arbitrary group, $(E, \langle\cdot,\cdot\rangle)$ an inner product space over $\mathbb{K}$, and $f : G \to E$ a solution of* (I). *Then we have:*

a) *If $x \in G$ has finite order, then $f(x) = 0$.*

b) *If G is a torsion group, specifically if G is finite, then $f = \underline{0}$, so f is quadratic.*

Proof. a) There exists $m \in \mathbb{N}$ with $x^m = e$, so by (3) and Theorem 4.2 $m^2 f(x) = f(x^m) = f(e) = 0$, and the torsionfreeness of $(E, +)$ implies $f(x) = 0$.

b) immediately follows from a).

Remark 5.2. Lemma 5.1 is now extended to a special class of possibly mixed groups G (i.e., G has elements of possibly infinite order). The concept is that of a special semidirect product of two groups A and B. Since we wish that any two automorphisms of B be summable in End(B), we require that B be an abelian group ([13], p. 52/53) and then accordingly use the additive notation $(B, +, 0)$ for B. If in addition $(A, \cdot, e_A)$ is a group and g a given element of $\mathrm{Hom}((A,\cdot), (\mathrm{Aut(B)}, \circ))$, then the semidirect product of A and B with respect to g is the group $G = A \times_g B$ with underlying set $A \times B$ and the operation

$$(a, b) \cdot (a', b') := (aa', b + g_a(b')) \qquad (\forall (a, b), (a', b') \in A \times B), \tag{24}$$

where $g_a \in \mathrm{Aut(B)}$ denotes the image of $a \in A$ under g. (In the literature, depending on the context, the notation $B \times_g A$ is also used instead of $A \times_g B$). $e := (e_A, 0)$ turns out to be the identity element of G. If g is constant, i.e., $g_a = i_B$ $(\forall a \in A)$, then $A \times_g B$ coincides with the ordinary direct product of A and B. If, however, g is not constant, i.e., if there exists $a_0 \in A$ with $g_{a_0} \neq i_B$, then there exists $b_0 \in B$ with $g_{a_0}(b_0) \neq b_0$, and it follows that necessarily card $A \geq 2$, card $B \geq 3$, and that $A \times_g B$ is certainly non-abelian, even if A is abelian.

Theorem 5.3. *Hypotheses:* (i) $(A,\cdot,e_A)$ *is a torsion group,* $\mathrm{card} A \geq 2$. (ii) $(B,+,0)$ *is an abelian group.* (iii) $g \in \mathrm{Hom}(A, \mathrm{Aut(B)})$ *is such that* $\mathrm{card}\, g(A) \geq 2$ *and* $a \in A$, $\mathrm{ord}\, a = n \geq 2 \Longrightarrow i_B + g_a + \ldots + g_{a^{n-1}} = \underline{0}$. (iv) $G = A \times_g B$. (v) E *is an inner product space over* $\mathbb{K}$. *Assertion: Every solution* $f : G \to E$ *of* (I) *is* $\underline{0}$, *so is quadratic.*

Proof. 1) Let be $a \in A \setminus \{e_A\}$, $b \in B$ arbitrary, say $\mathrm{ord}\, a = n \geq 2$. Then $(a,b)^1 = (a^1, i_B(b))$, and if for a $k \in \mathbb{N}$ we have $(a,b)^k = (a^k, i_B(b) + g_a(b) + \ldots + g_{a^{k-1}}(b))$, then by (24) $(a,b)^{k+1} = (a,b)^k \cdot (a,b) = (a^{k+1}, i_B(b) + \ldots + g_{a^{k-1}}(b) + g_{a^k}(b))$, so

$$(a,b)^k = (a^k, i_B(b) + g_a(b) + \ldots + g_{a^{k-1}}(b)) \qquad (\forall k \in \mathbb{N},\ k \geq 2). \tag{25}$$

By hypothesis (ii) and Remark 5.2, the automorphisms $i_B, g_a, \ldots, g_{a^{k-1}}$ of B are summable in $\mathrm{End}(B)$, so by (25)

$$(a,b)^k = (a^k, (i_B + g_a + \ldots + g_{a^{k-1}})(b)) \qquad (\forall a \in A \setminus \{e_A\},\ b \in B,\ k \geq 2). \tag{26}$$

$\mathrm{ord}\, a = n \geq 2$, hypothesis (iii) and (26) imply $(a,b)^n = (a^n, \underline{0}(b)) = (e_A, 0) = e$, so that (a,b) has order n in G. If $f : G \to E$ is a solution of (I), Lemma 5.1a guarantees that $f(a,b) = 0$. Let $H := \{e_A, b);\ b \in B\}$. So we have proved

$$f(G \setminus H) = \{0\}. \tag{27}$$

2) H forms a subgroup of G isomorphic to B under the embedding $b \mapsto (e_A, b)$ $(b \in B)$. (By the way, this shows that G may very well contain elements of infinite order, and since also A is embeddable into G, this will then be a mixed group.) $\mathrm{card}\, A \geq 2$ in hypothesis (i) ensures that H is proper. By (27) and Corollary 4.3, $f = \underline{0}$, which ends the proof.

Of course, examples for the situation of Theorem 5.3 are of interest.

Example 5.4. The general dihedral group. Let B be an abelian group of exponent > 2 (i.e., there exists $b_0 \in B$ with $\mathrm{ord}\, b_0 > 2$), and $A := \mathbb{Z}_2 = \{1, -1\}$, the cyclic group in multiplicative notation. If χ_B is the mapping $b \mapsto b^{-1}$ $(b \in B)$, then $\chi_B \neq i_B$, $\mathrm{ord}\, \chi_B = 2$ in Aut(B). $g \in \mathrm{Hom}(A, \mathrm{Aut(B)})$ is defined by $g_1 := i_B$, $g_{-1} := \chi_B$. Now the dihedral group associated to B is $\mathrm{Dih}\, B := \mathbb{Z}_2 \times_g B$. In our situation, $\mathrm{Dih}\, B$ is always non-abelian. For $B = Z_3$ we get $\mathrm{Dih}\, B = S_3$, the smallest non-abelian group. For $B = \mathbb{Z}_n$ $(n \geq 3)$ we obtain as $\mathrm{Dih}\, B$ the finite dihedral groups, for $B = \mathbb{Z}$ the infinite dihedral group, and for $B = \mathbb{R}$ the group of rigid motions of $\mathbb{R}^1$. Hypotheses (i), (ii), (iv) of Theorem 5.3 clearly are satisfied, but also (iii): $a = -1 \in \mathbb{Z}_2$, $\mathrm{ord}\, a = 2$, $i_B + g_a = i_B + g_{-1} = i_B + \chi_B = \underline{0}$ since $(i_B + \chi_B)(x) = x + (-x) = 0$ $(\forall x \in B)$. So the assertion of Theorem 5.3 holds in all these cases.

Example 5.5. Let $(B,+,\cdot)$ be a ring with $1 \neq 0$, not necessarily commutative, having no zero divisors, and let A be a torsion subgroup of the group $U(B)$ of units of $B, \mathrm{card}\, A \geq 2$. $g \in \mathrm{Hom}(A, \mathrm{Aut(B,+)})$ is defined by $g_a(b) := ab$ $(\forall b \in B,$

$a \in A$) and $G := A \times_g B$. Now the hypotheses (i), (ii), (iv) of Theorem 5.3 are fulfilled, but also (iii): If $a_0 \in A \setminus \{1\}$, then $g_{a_0}(1) = a_0 \neq 1 = g_1(1)$, so $g_{a_0} \neq g_1$, so $\operatorname{card} g(A) \geq 2$. Let $a \in A$, $\operatorname{ord} a = n \geq 2$. Then $0 = a^n - 1 = (a-1)(1+\ldots+a^{n-1})$, and $a - 1 \neq 0$ implies $1 + a + \ldots + a^{n-1} = 0$, so $(1 + a + \ldots + a^{n-1})b = 0$, i.e., $(i_B + g_a + \ldots + g_{a^{n-1}})(b) = 0$ $(\forall b \in B)$, i.e., $i_B + g_a + \ldots + g_{a^{n-1}} = \underline{0}$. So Theorem 5.3 is applicable.

a) If our ring B has characteristic $\neq 2$, then $A = \{1, -1\}$ becomes possible, and we return to Example 5.4; notice that $b \mapsto (-1)b$ $(b \in B)$ is nothing else than χ_B.
b) If $B = \mathbb{C}$, then A may be the locally cyclic group $\mathbb{Q}/\mathbb{Z}$ of all roots of unity of all orders or a subgroup thereof.

Example 5.6 (modification of Example 5.5)**.** For the ring $(B, +, \cdot)$ we now allow the existence of zero divisors, leave everything else unchanged, choose A, g, and G as before but require now the condition

$$a \in A,\ a \neq 1 \Longrightarrow (a-1) \in U(B). \tag{28}$$

The only crucial point is again hypothesis (iii) of Theorem 5.3: Let $a \in A$, $\operatorname{ord} a = n \geq 2$. We have again $0 = a^n - 1 = (a-1)(1 + \ldots + a^{n-1})$. This and (28) imply $1 + \ldots + a^{n-1} = (a-1)^{-1}(a^n - 1) = 0$ and then, as in Example 5.5 $i_B + \ldots + g_{a^{n-1}} = \underline{0}$, and Theorem 5.3 can be applied in this situation. An illustration: $n \in \mathbb{N}$, $B := \mathbb{F}^{n \times n}$, the ring of all $n \times n$ matrices over the field $\mathbb{F}$, A a torsion subgroup of $U(B) = GL(n, \mathbb{F})$ satisfying (28). Specifically, if $n = 2$, $\mathbb{F} = \mathbb{R}$, $a_r := \begin{pmatrix} \cos(2\pi r) & -\sin(2\pi r) \\ \sin(2\pi r) & \cos(2\pi r) \end{pmatrix}$ $(\forall r \in \mathbb{Q})$, $A := \{a_r;\ r \in \mathbb{Q}\}$ is the image of $\varphi \in \operatorname{Hom}((\mathbb{Q}, +), (A, \cdot))$ given by $\varphi(r) := a_r$ $(\forall r \in \mathbb{Q})$. A is a torsion group and satisfies (28) since $(\cos(2\pi r) - 1)^2 + (\sin(2\pi r))^2 = 2 - 2\cos(2\pi r) \neq 0$ for $r \in \mathbb{Q} \setminus \mathbb{Z}$. Of course, A is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

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