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Symmetries and Dissipation Laws on Contact Systems

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Abstract. In this article, we focus on the formulation of dissipative mechanical systems through contact Hamiltonian systems. Different forms of symmetry of a contact dynamical system (geometric, dynamic, and gage) are defined to, in the realm of Noether, find their corresponding dissipated quantities. We also address the existence of dissipated quantities associated with a general vector field X on $TQ \times \mathbb{R}$, focusing on the case where its contact Hamiltonian function is dissipative.

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1. Introduction

Physical systems are inevitably subjected to external influences, dissipation, damping, as well as a wide range of irreversible phenomenological aspects.

A model description of dissipative systems is crucial, given the wide range of contexts in which this feature determines the system evolution, such as quantum mechanics [30], general relativity [15], etc.

In this sense, the Hamiltonian principle is crucial in a Lagrangian or Hamiltonian description of a system with conservative dynamics, but it cannot address the various aspects that arise in dissipative dynamics: dispersive forces linked to linear or nonlinear velocity dependencies, history-dependence, coupling with the environment, etc. In this regard, new formulations of the Hamiltonian principle have been developed to allow for the inclusion of dissipative effects in Lagrangian and Hamiltonian dynamics (consult, for example, Galley [21]).

However, the Lagrangian approach to dissipatory processes is by no means straightforward, with multiple approaches (one may consult [17] for a survey, which aiming to be exhaustive at the time of its publication, collects hundreds of references), such as the consideration of additional variables [47], the time-dependent Lagrangian [27], or complex actions [8].

Understanding dissipative processes through a geometric framework has been a relatively recent endeavor [42]. Contact geometry, which had already been employed to describe a wide range of physical systems [32], emerged naturally in mechanics when attempting to characterize mechanical systems whose Hamiltonian explicitly depends on time. In this context, a new extended phase space (incorporating the time dimension) is endowed with a contact structure defined by the renowned Poincaré–Cartan 1-form. Despite the versatility of this framework to formulate a theory of Hamilton–Jacobi, and that beyond the point symmetries of the Lagrangian and their Noether invariants, the concept of contact symmetries was necessary to characterize a non-canonical quantization of the symplectic solution of the Solution Manifold through Hamiltonian vector fields (consult [1]), a new framework for the phase space of any mechanical system (dissipative or non-dissipative) was proposed in [5], consisting of a contact manifold which serves as a natural extension of the classical symplectic phase space, endowed with an additional dimension represented by a new dynamic variable "z" characterizing the system interaction with the environment.

The principles of heat and energy interactions have substantially impacted the development of geometric–differential models that combine the two fundamental laws of thermodynamics: the conservation of energy (free energy) and the increase of entropy (dissipation). Notable examples include the introduction of dissipation brackets and Casimir functions, which have proven instrumental in capturing the essential characteristics of thermodynamic systems within a geometric framework (see, for instance, [31])

In particular, thermodynamics has greatly benefited from its influence on contact geometry, particularly within the realm of non-equilibrium energy transformation processes. The geometric framework of contact geometry has been recognized as an appropriate approach for describing thermodynamic systems and their transformations capturing the intricacies of energy conservation and entropy production (one can consult, in an extensive literature, the references [4,10,28,29,40]).

This viewpoint crystallized as a powerful tool in describing mechanical systems with dissipation (see, for instance, [5,9,51]). In particular, contact geometry has been able to propose a generalization of Hamilton's equations through the so-called contact Hamiltonian equations. This has emerged as a potent approach in modeling systems with frictional forces or those that exchange energy or have other interactions with the environment (consult, for example, [16]).

It is noteworthy how this contact framework has also been capable of addressing dissipation processes within the realm of field theories (see [24] and [25]).

A fundamental difference between symplectic and contact Hamiltonian dynamics is that in the latter case, the Hamiltonian H is not preserved along the system's dynamics. In fact, along every curve defined by this evolutionary process, the following relation holds

$$\frac{\mathrm{d}}{\mathrm{d}t}H = H\frac{\partial H}{\partial z},$$

(one can refer to [50] for a classical variational proof of this equation. The first geometric justification of this fact may have been provided in [5]). This condition is essential in the description of irreversible processes and open systems, assuming both system invariance properties and the encoding of flows due to non-equilibrium conditions (consult [9] and [19]) and leads us to define dissipated quantities of the system as those functions on the configuration space that dissipate at the same ratio as the Hamiltonian.

In this regard, just as the analysis of conserved quantities in the Hamiltonian formulation of autonomous Hamiltonian systems is guided by a strong geometric framework (we refer, for example, in an extensive bibliography, to [43]), it has resulted that the consideration of dissipated quantities in a contact Lagrangian system is intimately linked to the new geometric framework on the configuration manifold. We refer to [23] and [36], two of the earliest seminal works in this project, as they made groundbreaking contributions to establishing the foundations of this subject. This topic has since rapidly branched out extensively, and we will briefly go over some significant developments. Notably, [22] highlights the importance of understanding these concepts in non-conservative field theories. Drawing upon the approach of the pioneering work of Skinner and Rusk [49], formulated on the Whitney sum $T^*Q \oplus TQ$, [38] highlights the importance of a combined geometric framework in advancing our understanding of contact autonomous mechanical systems. Canonoid transformations are a class of transformations for Hamiltonian systems that preserve the Hamiltonian form of equations of motion but are not necessarily canonical [3]. The main objective of studying these transformations, as discussed in [38], is to explore the existence of characteristic quantities of motion associated with them. In [2], the author examines Lie integrability by quadratures for both time-independent and time-dependent Hamiltonian systems, emphasizing the connection between Lie integrability and the existence of a solvable Lie algebra of constants of motion, enhancing our understanding of the integrability of Hamiltonian systems on various geometric structures, including contact Hamiltonian systems. Similarly, an interesting perspective is presented in [7], where the authors explore the concept of reducing systems with scaling symmetries to equivalent contact Hamiltonian systems, effectively removing physically irrelevant degrees of freedom.

In this wise, the Noether Theorem undergoes a radical transformation in the transition from a Lagrangian system to a contact Lagrangian system: infinitesimal symmetries do not yield conserved quantities but rather dissipated quantities.

Thus, the geometric description of Noether Theorem for a Lagrangian $L : TQ \to \mathbb{R}$ (see, for instance, [14]) states that the invariance condition $X^c(L) = 0$ for the lift X^c to TQ of a vector field X provides $X^v(L)$ as a conserved quantity, where X^v is a vertical field on TQ over Q, canonically associated with X.

We owe to de León and Lainz [36] the striking result that, in the analogous situation for a contact Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$, it yields $X^v(L)$ as a quantity that dissipates in the same way as the energy of the system

$H = v^i \partial L/\partial v^i - L$. Upon this crucial point, we shall return several times throughout this article.

This paper is organized as follows: Section 2 is devoted to introducing the concept of contact Hamiltonian systems, some calculus instruments on these systems, as well as some dissipation laws associated with symmetries of the contact system given by the action of a Lie group. In Sect. 3, we revisit the scheme outlined in [36] concerning the establishment of dissipation laws. We structure it and consider broader aspects concerning vector fields in the tangent bundle of the corresponding differentiable manifold, which generate dissipation laws. In Sect. 4, we continue, in the spirit of Noether, with the aim of obtaining dissipation laws for each element $A \in \mathfrak{g}$, the Lie algebra of a Lie group G that acts on $TQ \times \mathbb{R}$ as symmetries (in a sense we will specify) of a contact Lagrangian system. In Sect. 5, we provide several examples that exemplify some of the most noteworthy results from the theory on dissipation laws and their associated contact structure. Finally, we include a section dedicated to pointing out further research directions, drawing inspiration from the classification of symmetries of Hamiltonian systems and their corresponding conserved quantities that may be critical in the search for dissipated quantities in the dynamics of such systems. Additionally, we discuss two characteristic cases of symmetries of Lagrangian systems—gage and differential invariance—which, along with their associated conservation laws, should merit interpretation within the context of contact Lagrangian systems. These aspects forms part of a broader context, such as the development of time-dependence in contact Hamiltonian systems and a potential encounter with the Hamilton–Jacobi formulation of contact Hamiltonian systems. We have included several references that can serve as a guide for further research along these paths.

2. Contact Hamiltonian Systems: Lie Group Actions

A contact manifold (M, η) is a $(2n+1)$-dimensional orientable manifold endowed with a 1-form η, called the contact form on M, such that $\eta \wedge (d\eta)^n$ is a volume form. On (M, η) there exists a unique vector field $\mathcal{R}$, the Reeb vector field, such that

$$i_{\mathcal{R}}\left(\eta \wedge (d\eta)^n\right) = (d\eta)^n.$$

For every smooth real function H on M, there is a unique vector field X_H, its Hamiltonian vector field, verifying,

$$\eta(X_H) = -H \tag{1a}$$

$$\mathcal{L}_{X_H}\eta = \rho\eta, \tag{1b}$$

for some function $\rho \in C^\infty(M)$. From these same conditions, it follows that in fact $\rho = -\mathcal{R}(H)$.

It is conventional to employ the formula (1b) in its Cartan equivalent form

$$i_{X_H} d\eta = dH - \mathcal{R}(H) \cdot \eta. \tag{1b'}$$

When $H \in C^\infty(M)$ is fixed we shall call the triple (M, η, H) a contact Hamiltonian system.

For $f, g \in C^\infty(M)$, we define the contact bracket

$$\begin{aligned}\{f, g\} &= -i_{[X_f, X_g]}\eta = -\left(\mathcal{L}_{X_f} i_{X_g} - i_{X_g}\mathcal{L}_{X_f}\right)\eta \\ &= X_f(g) + \mathcal{R}(f) \cdot g \qquad (2)\end{aligned}$$

where X_f and X_g are the Hamiltonian vector fields corresponding to f and g respectively.

The contact bracket obeys the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

but not the Leibniz rule, we have instead the following relation

$$\{f, gh\} = \{f, g\}h + \{f, h\}g - \{f, 1\}gh. \qquad (3)$$

As a particular case of (2), we have

$$X_H(H) = -\mathcal{R}(H) \cdot H,$$

relation called the *dissipation law* of the Hamiltonian H.

In this sense, we will say that a real function f on (M, η, H) is a *dissipated quantity* if $\{H, f\} = 0$, that is, $X_H(f) = -\mathcal{R}(H) \cdot f$.

Similarly, we say that $g \in C^\infty(M)$ is a conserved quantity if $X_H(g) = 0$. It is easy to verify that if f_1 and f_2 are dissipated quantities, then f_1/f_2 is a conserved quantity in the complement of $(f_2)_0$.

De León and Lainz [36], have provided us with the following useful and elegant calculation tool.

Proposition 1. *Let us consider the contact Hamiltonian system* (M, η, H)*. Let* $f \in C^\infty(M)$ *and* X *a vector field on* M *such that* $\eta(X) = -f$*, then*

$$\eta([X_H, X]) = \eta([X_H, X_f]),$$

where X_H *and* X_f *are the Hamiltonian vector field corresponding to* H *and* f *respectively. Consequently*

$$\{H, f\} = (\mathcal{L}_X\eta)(X_H) + X(H). \qquad (4)$$

Definition 2. The vector field $Y \in \mathfrak{X}(M)$ is a dynamical symmetry of (M, η, H) if

$$\mathcal{L}_Y X_H = 0.$$

Next, we present a pair of examples applying this Proposition.

Proposition 3. *If* Y *a dynamical symmetry then* $\eta(Y)$ *is a dissipated quantity.*

Proof. It suffices to consider that

$$\{H, \eta(Y)\} = \eta([X_H, Y]). \qquad (5)$$

□

Conversely, if (5) holds, that is, Y is a vector field such that $\eta(Y)$ is a dissipated quantity then $[X_H, Y]$ is η-horizontal vector field (view [36] and refer to [23] with a specific viewpoint).

Proposition 4. *Let X be a dynamical symmetry on (M, η, H) such that $\mathcal{L}_X \eta = dg$, for some g smooth function on M. Then g is a conserved quantity if and only if H is a first integral of X.*

Proof. Writing $f = -\eta(X)$, then

$$\begin{aligned} 0 &= \{H, f\} = (\mathcal{L}_X \eta)(X_H) + X(H) \\ &= X_H(g) + X(H), \end{aligned}$$

whereby the statement of the Proposition. □

Let G be a Lie group acting by contactomorphisms on (M, η), that is, $g^*\eta = \eta$ if $g \in G$. It is plain that $\mathcal{L}_{A^*}\eta = 0$ where $A \in \mathfrak{g}$ (the Lie algebra of G) and we denote by A^* its corresponding *fundamental vector field* on M, that is,

$$A^*_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tA) \cdot x, \;\; x \in M.$$

Proposition 5. *If the Lie group G acts by contactomorphisms of the Hamiltonian system (M, η, H) and leaves invariant the Hamiltonian function H, $g^*H = H$, $\forall g \in G$, then the Hamiltonian function f_{A*} corresponding to the fundamental vector field A^* is a dissipated quantity,*

$$\{H, f_{A*}\} = 0.$$

Proof. Since $\mathcal{L}_{A^*}\eta = 0$, it follows that the vector field A^* is the Hamiltonian vector field corresponding to the function $\eta(A^*) = -f_{A^*}$ and we have

$$i_{A^*} d\eta = d\left(-i_{A^*}\eta\right) = df_{A^*} \tag{6}$$

Consequently $\mathcal{R}(f_{A^*}) = 0$. In this way,

$$\{f_{A^*}, H\} = A^*(H) + \mathcal{R}(f_{A^*}) \cdot H = 0.$$

□

Proposition 6. *Let us consider the Hamiltonian system (M, η, H) over which the Lie group G acts by contactomorphisms of (M, η) and leaves X_H invariant, $g_* X_H = X_H$, $\forall g \in G$. Then, $\{H, f_{A^*}\} = 0$.*

On the other hand, if $\mathcal{R}(H) = 0$, the relations $X_H(f_{A^*}) = 0$ and $A^*(H) = 0$ hold. Also, the product $f_{A^*} \cdot f_{B^*}$ is also a dissipated quantity for any $A, B \in \mathfrak{g}$.

Proof. The first statement follows from the equality

$$\{H, f_{A^*}\} = -\eta([X_H, A^*]) = 0, \tag{7}$$

The second assertion of the Proposition follows from the relations

$$\begin{aligned} \{H, f_{A^*}\} &= X_H(f_{A^*}) + \mathcal{R}(H) \cdot f_{A^*} \\ &= -(A^*(H) + \mathcal{R}(f_{A^*}) \cdot H). \end{aligned}$$

Finally, the conclusion follows from relation (3), as we have

$$\{H, f_{A^*} \cdot f_{B^*}\} = \{H, f_{A^*}\} \cdot f_{B^*} + \{H, f_{B^*}\} \cdot f_{A^*}.$$

□

Remark. We know from [23], as a consequence of considering several kinds of symmetries for contact dynamical systems, that the contact action of G, along with the G-invariance of the Hamiltonian H, implies the preservation by G of the Hamiltonian vector field X_H. Conversely, let us see that conditions $g^*\eta = \eta$ and $g_* X_H = X_H$, $\forall g \in G$, imply the G-invariance of the Hamiltonian H. In fact,

$$g^* H = g^* \left(i_{X_H}\eta\right) = i_{g_* X_H} g^*\eta = i_{X_H}\eta = H.$$

In this manner, the statements of Propositions 4 and 5 are equivalent. They have been set apart to emphasize two arguments of a different nature: the first rests on the genesis of the Hamiltonian function f_{A^*}, while the second focuses on the preservation of solutions of the Hamiltonian vector field X_H.

Within the framework of contact geometry, analogously to the symplectic case, one can define a momentum map relative to the action of a Lie group that preserves a contact structure (see [20]). A version of Proposition 5 using this context can be found in [36].

3. Contact Lagrangian Systems

Analogously to the cotangent bundle of the configuration space manifold of a mechanical system, which inherently possesses a central geometric structure in Hamiltonian theory (the canonical 1-form), the tangent bundle of a differentiable manifold is intrinsically characterized by a geometric structure which plays a crucial role in the development of Lagrangian theory, with a key function, for example, in determining the inverse problem of Lagrangian dynamics or defining symmetries of a Lagrangian system, (consult, for example, [37,41] and [14]). In fact, the integrable almost tangent structure on a manifold M, defined by a $(1,1)$-tensor field J, such that $\mathrm{Im}(S_p) = \ker(S_p)$ (for every $p \in M$) and whose Nijenhuis tensor vanishes, already embodies the full Lagrangian theory within itself (foundations for this splendid theory can be consulted in the references [12,13], and [48]).

In this way, let Q be an n-dimensional manifold, and we consider TQ its tangent bundle and the extended phase space $TQ\times\mathbb{R}$. We denote by (q^i, v^j, z) the bundle coordinates on $TQ \times \mathbb{R}$ and consider the natural extension to $TQ \times \mathbb{R}$ of the canonical almost tangent structure J on TQ locally defined by the $(1,1)$-tensor field

$$J = \frac{\partial}{\partial v^i} \otimes dq^i.$$

If w_p is a p-form on $TQ \times \mathbb{R}$, then $i_J w_p$ is the $(0,p)$-tensor field defined as

$$\left(i_J w_p\right)(X_1, ..., X_p) = w_p(JX_1, X_2, ..., X_p).$$

For a vector field X on $TQ \times \mathbb{R}$, one has

$$\mathcal{L}_X \left(i_J w_p\right) = \left(i_{\mathcal{L}_X J}\right) w_p + i_J \left(\mathcal{L}_X w_p\right) \tag{8}$$

Also we define the Liouville vector field Δ on $TQ\times\mathbb{R}$ by the local expression

$$\Delta = v^i \frac{\partial}{\partial v^i}.$$

Let us consider a regular Lagrangian function $L : TQ \times \mathbb{R} \to \mathbb{R}$, that is, its Hessian matrix

$$(W_{ij}) = \left(\frac{\partial^2 L}{\partial v^i \partial v^j} \right)$$

is nonsingular everywhere.

As a consequence, the 1-form on $TQ \times \mathbb{R}$

$$\eta_L := dz - i_J dL$$

is a contact form. In local coordinates, we write

$$\alpha_L := i_J dL = dL \circ J = \frac{\partial L}{\partial v^i} dq^i.$$

The couple $(TQ \times \mathbb{R}, \eta_L)$ will be called a contact Lagrangian system

The energy function of the system is given by

$$E_L = \Delta L - L = v^i \frac{\partial L}{\partial v^i} - L.$$

The Hamiltonian vector field X_{E_L} of the energy will be denoted by ξ_L. In local coordinates

$$\xi_L = v^i \frac{\partial L}{\partial q^i} + W^{ik} \left(\frac{\partial L}{\partial q^k} - \frac{\partial^2 L}{\partial q^j \partial v^k} v^j - L \frac{\partial^2 L}{\partial z \partial v^k} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial v^k} \right) \frac{\partial}{\partial v^i} + L \frac{\partial}{\partial z}.$$

Consequently $J(\xi_L) = \Delta$.

In the context of a contact Lagrangian system, it is interesting to observe that the conditions (1a) and (1b) of the definition of ξ_L can be alternatively replaced by the so-called Herglotz conditions,

$$\left. \begin{array}{c} \xi_L(z) = L \\ \xi_L \left(\frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial q^i} \frac{\partial L}{\partial z} \end{array} \right\}$$

This viewpoint is useful when considering the dynamics of the variational problem or to address and solve the inverse problem in the case of Lagrangian contact systems (see, for example, [37] and [18]).

The Reeb vector field $\mathcal{R}_L$ of the contact form η_L is given by the local expression

$$\mathcal{R}_L = \frac{\partial}{\partial z} - W^{ij} \frac{\partial^2 L}{\partial v^i \partial z} \frac{\partial}{\partial v^j}, \tag{9}$$

where (W^{ij}) is the inverse of the Hessian matrix (W_{ij}).

Furthermore, a direct calculation yields

$$\mathcal{R}_L(E_L) = -\frac{\partial L}{\partial z}.$$

3.1. Q- and $\mathbb{R}$-Projectable Vector Fields

We define $\mathfrak{X}_{\mathrm{proj}}(TQ \times \mathbb{R})$ as the set of vector fields on $TQ \times \mathbb{R}$ that can be projected onto both Q and $\mathbb{R}$. Its expression in the local coordinates (q^j, v^i, z) of $TQ \times \mathbb{R}$ takes the form

$$X = F_i \frac{\partial}{\partial q^i} + G_i \frac{\partial}{\partial v^i} + H \frac{\partial}{\partial z}, \tag{10}$$

with

$$F_i = F_i(q^1, ..., q^n),\ (1 \leq i \leq n),\ H = H(z).$$

Proposition 7. *A sufficient condition for a vector field X on $TQ \times \mathbb{R}$ to verify $\mathcal{L}_X \alpha_L = \alpha_{X(L)}$ is that $\mathcal{L}_X J = 0$.*

Proof. It suffices to consider that, by (8), we have

$$\begin{aligned}\mathcal{L}_X \alpha_L &= \mathcal{L}_X(i_J dL) = i_{\mathcal{L}_X J} dL + i_J d(\mathcal{L}_X L)\\ &= i_{\mathcal{L}_X J} dL + \alpha_{X(L)}.\end{aligned}$$

□

A brief calculation shows that for a vector field $X \in \mathfrak{X}_{\text{proj}}(TQ \times \mathbb{R})$ given by the coordinate expression (10), the condition $\mathcal{L}_X J = 0$ means

$$\frac{\partial G_k}{\partial v^j} = \frac{\partial F_k}{\partial q^j},\ 1 \leq j, k \leq n. \tag{11}$$

Using the usual notation,

$$X^v = J(X) = F_i \frac{\partial}{\partial v_i},$$

if we write $f = -\eta_L(X)$, then

$$f = -(H - X^v L) = -\left(H - \sum_i F_i \frac{\partial L}{\partial v_i}\right)$$

Lemma 8. *If $X \in \mathfrak{X}_{\text{proj}}(TQ \times \mathbb{R})$ then*

$$\mathcal{R}_L(f) = -\frac{\partial H}{\partial z}.$$

Proof. An straightforward computation gives

$$\sum_{i,j} W^{ij} \frac{\partial^2 L}{\partial v^i \partial z} \frac{\partial}{\partial v^j}\left(\sum_k F_k \frac{\partial L}{\partial v^k}\right) = 0.$$

With this, the local expression of the vector field $\mathcal{R}_L$ in (9) gives the result. □

Definition 9. The vector field $X \in \mathfrak{X}_{\text{proj}}(TQ \times \mathbb{R})$ is called an infinitesimal symmetry of L if

$$X(L) = -\mathcal{R}_L(f) \cdot L,$$

where $f = -\eta_L(X)$.

Remark. This definition is the natural generalization of the case $\mathcal{R}_L(f) = 0$, which includes the scenario of an infinitesimal symmetry of L given by the lift X^c of a field $X \in \mathfrak{X}(Q)$, in such a way that the relation $X^c(L) = 0$, is equivalent to the dissipative property of the function $X^v(L)$.

Proposition 10. *Every infinitesimal symmetry* $X \in \mathfrak{X}_{proj}(TQ \times \mathbb{R})$ *with* $\mathcal{L}_X J = 0$ *is a Hamiltonian vector field for* η_L.

Proof. Writing $\eta_L(X) = -f$, we have

$$\begin{aligned}\mathcal{L}_X \eta_L &= \frac{\partial h}{\partial z} dz - \mathcal{L}_X \left(i_J dL\right) = \frac{\partial h}{\partial z} dz - i_J d(XL) \\ &= \frac{\partial h}{\partial z} dz + i_J d\left(\mathcal{R}_L(f) \cdot L\right) \\ &= \frac{\partial h}{\partial z} dz - \frac{\partial h}{\partial z} i_J dL = -\mathcal{R}_L(f)\eta_L.\end{aligned}$$

□

In this way, henceforth we will write $X = X_f$ for $f = -\eta_L(X)$ if $X \in \mathfrak{X}_{\mathrm{proj}}(TQ \times \mathbb{R})$ is an infinitesimal symmetry.

Theorem 11. *With the previous notations, for* $X_f \in \mathfrak{X}_{proj}(TQ \times \mathbb{R})$ *it holds that* f *is a dissipated quantity.*

Proof. We have

$$\begin{aligned}\{E_L, f\} &= \left(\mathcal{L}_{X_f}\eta_L\right)\xi_L + X_f(E_L) = -\mathcal{R}_L(f)\eta_L(\xi_L) + X_f(E_L) \\ &= -\mathcal{R}_L(f)(L - dL(J(\xi_L)) + X_f(E_L) \\ &= -\mathcal{R}_L(f)(L - dL(\Delta) + X_f(E_L) \\ &= \mathcal{R}_L(f)E_L + X_f(E_L) = \{f, E_L\}.\end{aligned}$$

In consequence $\{E_L, f\} = 0$. □

Remark. In [36], vector fields on $TQ \times \mathbb{R}$ are considered of the form

$$Y = F_i \frac{\partial}{\partial q^i} + v^j \frac{\partial F_i}{\partial q^j}\frac{\partial}{\partial v^i} + H\frac{\partial}{\partial z} \text{ with } F_i = f_i(q, z),\ H = h(z). \tag{12}$$

obtained by restricting lifts to $T(Q \times \mathbb{R})$ of fields in $Q \times \mathbb{R}$. However, their calculations (Theorem 10) require that Y be, in turn, Q-projectable (to ensure that $\mathcal{R}_L(f) = -\partial H/\partial z$), a condition not satisfied by the field given by expression (12). In any case, (12) represents a more restrictive condition than that given by (11), thus satisfying $[\Delta, Y] = 0$; using this property, the authors conclude (addressing some inadequacies) that

$$\{E_L, f\} = \{L, f\}.$$

Now,

$$\{L, f\} = X_f(L) + \mathcal{R}_L(f)L = -\mathcal{R}_L(f) \cdot L + \mathcal{R}_L(f) \cdot L = 0.$$

3.2. Vector Fields over $TQ \times \mathbb{R}$ in General Position

We are now seeking dissipation laws similar to those obtained in the previous scenarios, applicable to fields on $TQ \times \mathbb{R}$ that lack a significant connection with the base $Q \times \mathbb{R}$.

Given the strong dependence of the previous techniques on the geometric nature of the fields, we will now adopt a different approach in our calculations.

In this manner, consider

$$X = F_i \frac{\partial}{\partial q^i} + G_j \frac{\partial}{\partial v^j} + H \frac{\partial}{\partial z}, \tag{13}$$

where (q^i, v^j, z) are the fiber coordinates in $T(Q) \times \mathbb{R}$.

First, we have

$$\begin{aligned} d\left(i_J dL\right)(\xi_L, X) &= \xi_L \left\{i_J dL(X)\right\} - X\left\{i_J dL(\xi_L)\right\} - i_J dL\left[\xi_L, X\right] \\ &= \xi_L(X^v L) - X(\Delta L) - \left[\xi_L, X\right]^v L. \end{aligned} \tag{14}$$

Moreover, utilizing the relation (1b')

$$i_{\xi_L} d\eta_L = dE_L - (\mathcal{R}_L E_L)\, \eta_L \tag{15}$$

and since $d\eta_L = -d(d_J f)$, we derive

$$\begin{aligned} d\left(d_J L\right)(\xi_L, X) &= -X(E_L) + (\mathcal{R}_L E_L)\, \eta_L(X) \\ &= -X(\Delta L) + X(L) + (\mathcal{R}_L E_L)\,(H - X^v L) \end{aligned}$$

thus,

$$\{E_L, X^v(L)\} = X(L) + \left[\xi_L, X\right]^v L - \frac{\partial L}{\partial z} H. \tag{16}$$

Summarizing, we have

Proposition 12. *Let us consider the vector field $X \in \mathfrak{X}(TQ \times \mathbb{R})$ in the form (13), then $X(L) + [\xi_L, X]^v L + (\mathcal{R}_L E_L)\, H$ represents the obstruction for the function $X^v(L)$ to be a dissipated quantity for the contact Lagrangian system defined by L.*

Remark. Let us examine the expression (16) for the particular case where the vector field X on $TQ \times \mathbb{R}$ is the lift Y^c of a field $Y \in \mathfrak{X}(Q)$. In this situation, (16) provides the expression

$$\{E_L, Y^v(L)\} = Y^c(L)$$

which defines $Y^v(L)$ as the dissipated quantity linked to the symmetry condition $Y^c(L) = 0$. This recovers the translation of Noether Theorem in conservative Lagrangian systems to the case of dissipative contact Lagrangian systems.

Definition 13. A Cartan symmetry is a vector field $X \in \mathfrak{X}(TQ \times \mathbb{R})$ satisfying

$$\mathcal{L}_X \eta_L = df$$

for a certain function $f \in C^\infty(TQ \times \mathbb{R})$.

Proposition 14. *Let X be a Cartan symmetry of the contact system defined by L, given by the expression (13) in local coordinates. Then, the quantity $f - H + X^v L$ is a first integral of the Reeb field $\mathcal{R}_L$.*

Proof. We have

$$df = i_X d\eta_L + d i_X \eta_L = i_X d\eta_L + d(H - X^v L),$$

consequently,

$$\mathcal{R}_L(f - H + X^v L) = 0.$$

□

4. Gauge Symmetries on Lagrangian Contact Systems

In this section, we continue the objective initiated in Sect. 2 of defining dissipation laws for each element $A \in \mathfrak{g}$, the Lie algebra of a Lie Group G acting on the phase space. Specifically, we focus on G acting on $TQ \times \mathbb{R}$, which corresponds to lifting its operation via diffeomorphisms of the base Q.

This purpose has its roots in the consideration of symplectomorphisms of a classical symplectic Lagrangian system on TQ, defined by the lifting of diffeomorphisms of the base manifold Q.

Let us briefly revisit the case of a classical mechanical system characterized by a regular Lagrangian function $L : TQ \to \mathbb{R}$, where the Hessian matrix is regular in every coordinate neighborhood (q^i, v^j). It is crucial to point out that, under these conditions, the exact 2-form $w_L = -d(\alpha_L) = -d(i_J dL)$ defines a symplectic structure on TQ. Notably, the associated Poisson algebra encodes the primary dynamic properties of the system. In this way, we can define the Hamiltonian vector field X_L associated with the energy function $E_L = \Delta L - L$ as the unique vector field determined by the dynamical equation

$$i_{X_L} w_L = dE_L.$$

Let us now consider dispersion laws associated with transformations of the configuration manifold Q that define gage-contact symmetries on the extended tangent bundle $TQ \times \mathbb{R}$ of a given contact Lagrangian system.

In this context, it is significant to consider the moduli of Lagrangian $L : TQ \to \mathbb{R}$ with respect to the class of X_L-preserving symplectomorphisms of w_L. This is already a result considered classic, which we restate in our own terms. One may refer, for example, to reference [11], which includes a reflection on this point in relation to working on a closely related subject.

Proposition 15. *Let $\widetilde{L}$ and L be regular Lagrangian on TQ. Let $X_{\widetilde{L}}$,and X_L be the corresponding Lagrangian vector fields. The following assertions are equivalent:*

(a) $\widetilde{L} = L + \alpha + k$ *where α is a closed 1-form on Q, understood as a function $\alpha : TQ \to \mathbb{R}$ and $k \in \mathbb{R}$.*
(b) $X_{\widetilde{L}} = X_L$ *and* $w_{\widetilde{L}} = w_L$.

Proof. Assume (a). The first crucial step is to take into account that 1-forms on the manifold Q can be identified with linear functions on the tangent bundle TQ. This can be seen naturally in local coordinates (q^i, v^i) by defining a function f on TQ from the 1-form α as follows

$$\alpha = \sum_i f_i(q) dq^i \equiv \sum_i f_i v^i := f.$$

In this way, it is clear that $\Delta(f) = f$, hence $\Delta(\widetilde{L}) - \widetilde{L} = \Delta(L) - L - k$, that is,

$$\widetilde{E} = E - k.$$

With the same interpretation for the 1-form $i_J df$ as a function on TQ, we can write

$$i_J df = \frac{\partial f}{\partial v^i} dq^i = f_i dq^i \equiv f_i v^i = f.$$

In this way,

$$\alpha_{\widetilde{L}} = i_J d\widetilde{L} = i_J d(L + f + k) = i_J dL + f \equiv \alpha_L + \alpha.$$

As α is a closed 1-form, we have

$$d\eta_{\widetilde{L}} = d\eta_L$$

that is, $w_{\widetilde{L}} = w_L$. Accordingly, we also have $X_{\widetilde{L}} = X_L$.

If (b) holds, then $\widetilde{E} = E + k$, hence

$$L - \widetilde{L} = \Delta(L - \widetilde{L}) + k.$$

Expressing the function $\Delta(L - \widetilde{L})$ on TQ using 1-form notation,

$$\Delta(L - \widetilde{L}) \equiv \left(\frac{\partial L}{\partial v^i} - \frac{\partial \widetilde{L}}{\partial v^i} \right) dq^i := \Theta. \tag{17}$$

Since $w_{\widetilde{L}} = w_L$, we have $d\Theta = 0$ and we can write locally

$$\Delta(L - \widetilde{L}) = dg = \frac{\partial g}{\partial q^i} dq^i + \frac{\partial g}{\partial v^i} dv^i. \tag{18}$$

for some local function $g : TQ \to \mathbb{R}$. From (17) and (18), it follows that g does not depend on the coordinates $\{v^i\}$ and hence $\Delta(L - \widetilde{L})$ is a closed 1-form on Q and we have (a). □

Considering now again the case of a contact Lagrangian system defined by a regular Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$, we aim to investigate the behavior of the corresponding contact Lagrangian form $\eta_L = dz - i_J dL$ under the action of lifting of diffeomorphisms of the base space Q.

Inspired by the symplectic case, we shall provide the following definition.

Definition 16. Let φ be a diffeomorphism of Q and $L \in C^\infty(TQ \times \mathbb{R})$. Denote by $\overline{\varphi} : TQ \times \mathbb{R} \to TQ \times \mathbb{R}$ the product application of the tangent lift of φ and the identify on $\mathbb{R}$. We say that φ is a gage symmetry of the contact Lagrangian system defined by L, if $\overline{\varphi}^* L = L + \alpha$, where α is a closed 1-form on Q.

Henceforth, we will use the notation

$$\widetilde{L} = \overline{\varphi}^* L = L + \alpha \equiv L + f_\alpha$$

where $f_\alpha : TQ \to \mathbb{R}$ is the function associated with the 1-form α on Q. Taking into account the proof of Proposition 15, we have

$$\begin{aligned} \eta_L &= dz - i_J dL \\ \eta_{\widetilde{L}} &= dz - i_J d\widetilde{L} = dz - i_J dL - f_\alpha \equiv dz - i_J dL - \alpha. \end{aligned} \tag{19}$$

Consequently, we have on $TQ \times \mathbb{R}$ two different contact structures given by the contact Lagrangian forms η_L y $\eta_{\widetilde{L}}$, such that their Lagrangian energies are identical,

$$E_{\widetilde{L}} = \Delta(\widetilde{L}) - \widetilde{L} = \Delta(L) + f_\alpha - (L + f_\alpha) = E_L. \tag{20}$$

Moreover, by the local expression (9) of the Reeb vector fields, we have

$$\mathcal{R}_L = \mathcal{R}_{\widetilde{L}}, \tag{21}$$

while the closed character of the 1-form α implies

$$d\eta_L = d\eta_{\widetilde{L}}. \tag{22}$$

We gather these facts in the following proposition:

Proposition 17. *With the notations of Definition 16, let η_L be the contact form associated with the regular Lagrangian $L : TQ\times\mathbb{R} \to \mathbb{R}$ and consider the contact form $\eta_{\widetilde{L}}$ associated with the Lagrangian $\widetilde{L} = \varphi^* L = L + \alpha$ where α is a closed 1-form on Q. Both forms share their Lagrangian energies, $E_{\widetilde{L}} = E_L$, their Reeb vector fields $\mathcal{R}_L = \mathcal{R}_{\widetilde{L}}$ and satisfy $d\eta_L = d\eta_{\widetilde{L}}$.*

The following result entails a suitable position for a Lagrangian contact system within the framework of Proposition 17 for gage symmetries.

Proposition 18. *Let G be a Lie group of transformations on Q such that its lift to $TQ \times \mathbb{R}$ is such that*

$$g^* L = L + \alpha,\ \ g \in G,$$

where α is a closed 1-form on Q and in such a way that the action of G leaves ξ_L invariant,

$$g_*\xi_L = \xi_L, \forall g \in G.$$

For each $A \in \mathfrak{g}$, the function $\eta_L(A^*)$ is a dissipated quantity on $TQ \times \mathbb{R}$. Additionally, if $H^1(Q) = 0$, there exists a function h_A on Q such that $\mathcal{L}^*_A\eta_L = dh_A$; furthermore, h_A is a first integral of ξ_L.

Proof. Let $g_t = \exp(tA)$ with $A \in \mathfrak{g}$. Then

$$g_t^* L = L + \beta_t$$

where β_t is a closed form on Q. Thus, $A^* L = \beta_A$ for a certain 1-form closed on Q, and let us denote its corresponding function on TQ as f_A.

In this manner, we have

$$A^*(\Delta L) = \Delta A^* L \equiv \Delta f_A = f_A,$$

consequently

$$A^*(E_L) = A^*(\Delta L - L) = 0.$$

Thus, by the proof of Proposition 17, we have

$$g_t^*\eta_L = \eta_L - \beta_t$$

which implies

$$\mathcal{L}_{A^*}\eta_L = \beta_A.$$

The closedness of β_A implies, by Poincaré's Lemma and considering that $H^1(Q) = 0$ as stated in the Proposition, that β_A on Q can be expressed as the differential of a function h_A, that is, $\beta_A = dh_A$.

By Proposition 1, we have

$$[E_L, \eta_L(A^*)] = -\left(\mathcal{L}_{A^*}\eta_L\right)(\xi_L) - A^*(E_L)$$

On the other hand,

$$\langle \mathcal{L}_{A^*}\eta_L, \xi_L\rangle = \mathcal{L}_{A^*}\langle \eta_L, \xi_L\rangle - \langle \eta_L, \mathcal{L}_{A^*}\xi_L\rangle,$$

and since $\langle \mathcal{L}_{A^*}\eta_L, \xi_L\rangle = \xi_L(h_A)$, $\mathcal{L}_{A^*}\langle \eta_L, \xi_L\rangle = -A^*E_L = 0$ and $\mathcal{L}_{A^*}\xi_L = 0$, the statements of the proposition follow. □

5. Some Examples

5.1. Gravity, Friction, and a Body in Motion

Let us consider a body subjected to the action of a constant gravitational field, as well as experiencing the influence of a resistive force or friction which opposes the motion and dissipates energy, typically in the form of heat. Consider a simplified Lagrangian given by,

$$L = \frac{1}{2}(v_x^2 + v_y^2) - y - \gamma z$$

where the first term accounts for the kinetic energy associated with velocities v_x and v_y, the second term represents the potential energy related to position y, and the final term incorporates the energy dissipation γz due to the damping force.

The contact structure of the system is defined by the following tensors

$$\eta_L = dz - v_x dx - v_y dy,\ d\eta_L = dx \wedge dv_x + dy \wedge dv_y$$
$$R_L = \frac{\partial}{\partial z},\ E_L = \frac{1}{2}(v_x^2 + v_y^2) + y + \gamma z.$$

Now, let us consider the vector field

$$X = \gamma\frac{\partial}{\partial y} - \frac{\partial}{\partial z} \in \mathfrak{X}_{\text{proj}}(T\mathbb{R}^2 \times \mathbb{R})$$

We have

$$f = -\eta_L(X) = 1 + \gamma v_y.$$

Since $X(L) = 0$ and $R_L(f) = 0$, the conditions for X to be an infinitesimal symmetry of L (according to Definition 9) are satisfied. As a consequence of Theorem 11, it follows that

$$\{E_l, f\} = 0$$

what defines f as a dissipated quantity.

5.2. Emden Model and Interstellar Gas Clouds

We will now develop an approximation to Robert Emden model (1907) for an interstellar gas cloud. In this endeavor, we shall first put into operation our derived formula (16) for vector fields in general position which respect to the projection $TQ \times \mathbb{R} \to Q \times \mathbb{R}$. Thus, we aim to put the theoretical foundation supporting our approach into practice.

In his model, Emden introduced a Lagrangian that we have modified by excluding the time dependency and incorporating an additional term to account for the energy dissipated due to dispersion forces, represented by

$$L = \frac{1}{2}v^2 - \frac{1}{6}q^6 - \gamma z.$$

The sixth-degree dependence of the potential energy of gas within the interstellar cloud can be attributed to the internal structure of the star and its asymptotic behavior: as we move away from the center of the star, the density and pressure decrease, affecting the interactions between the particles that compose it. This includes the van der Waals force potential for molecular interactions and also captures the distribution of electric charge that induces interaction between molecules.

The Emden model contact structure is characterized by the following elements

$$\eta_L = dz - vdq, \; d\eta_L = dq \wedge dv$$
$$R_L = \frac{\partial}{\partial z}, \; E_L = \frac{1}{2}v^2 + \frac{1}{2}q^2 + \gamma z.$$

The Hamiltonian vector field $\xi_L$arises from the dynamical equations,

$$\left.\begin{array}{r} \eta_L(\xi_L) = -E_L \\ i_{\xi_L} d\eta_L = dE_L - R_L(E_L)\eta_L \end{array}\right\}$$

Solving these equations provides:

$$\xi_L = v\frac{\partial}{\partial q} - (q^5 + \gamma v)\frac{\partial}{\partial v} + L\frac{\partial}{\partial z}.$$

Consider the field $X = \gamma\frac{\partial}{\partial q} - q^5\frac{\partial}{\partial z}$ which satisfies the condition $X(L) = 0$. A computation leads us to

$$[\xi_L, X]^v = \left[v\frac{\partial}{\partial q} - (q^5 + \gamma v)\frac{\partial}{\partial v} + L\frac{\partial}{\partial z}, \gamma\frac{\partial}{\partial q} - q^5\frac{\partial}{\partial z}\right]^v = 0.$$

As a result, based on Eq. (16), we obtain

$$\{E_l, X^v(L)\} = X(L) + \left[v\frac{\partial}{\partial q} - (q^5 + \gamma v)\frac{\partial}{\partial v} + L\frac{\partial}{\partial z}, \gamma\frac{\partial}{\partial q} - q^5\frac{\partial}{\partial z}\right]^v L - \gamma q^5.$$

This leads to the following law

$$\{E_l, \gamma v\} = -\gamma q^5.$$

This author has made consistent efforts, using the result expressed in (16), to identify dissipated quantities of the form $X^v(L)$ for this particular case; however, these attempts have thus far proven unsuccessful.

5.3. Dynamics of a Charged Particle under Dispersion Forces and Electromagnetic Fields: Analysis in a Specific Gage

Let us now observe that points (19) through (22) as a direct consequence of Definition 16 in Sect. 4, naturally emerge within the framework of Herglotz classical mechanics.

Consider a particle with mass m and charge q under the influence of an electromagnetic field defined by a gage $(\phi, V = A_i \frac{\partial}{\partial x_i})$. The Lagrangian for this particle is given by

$$L = \frac{m}{2} \sum_{i=1}^{3} v_i^2 + q \sum_{i=1}^{3} A_i v_i - q\phi - \gamma z$$

where we assume that the particle is subject to a dissipative force, and γz denotes the corresponding dispersion energy associated with this force.

The contact structure of this system is defined by the following tensors:

$$\eta_L = dz - \frac{\partial L}{\partial v_i} dx_i = dz - (mv_i + qA_i)dx_i,$$

$$d\eta_L = m dx_i \wedge dv_i + q \frac{\partial A_i}{\partial x_j} dx_j \wedge dx_i, \ R_L = \frac{\partial}{\partial z}$$

$$E_L = \frac{\partial L}{\partial v_i} v_i - L = \frac{m}{2} \sum_{i=1}^{3} v_i^2 + q\phi + \gamma z.$$

Now, since

$$dE_L - R_L(E_L)\eta_L = mv_i dv_i + q\frac{\partial \phi}{\partial x_i} + \gamma(mv_i + qA_i)dx_i$$

the equations

$$\left.\begin{array}{c} \eta_L(\xi_L) = -E_L \\ i_{\xi_L} d\eta_L = dE_L - R_L(E_L)\eta_L \end{array}\right\}$$

yield

$$\xi_L = v_i \frac{\partial}{\partial x_i} + F_i \frac{\partial}{\partial v_i} + L\frac{\partial}{\partial z}.$$

where

$$mF_i = qv_i \sum_{j=1}^{3} \left(\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} \right) - q\frac{\partial \phi}{\partial x_i} - \gamma(mv_i + qA_i).$$

Let us now consider an alternative gage of the form $(\phi, V + \nabla f)$, where f a smooth function on Q. Under this new gage, the Lagrangian takes the form:

$$\widetilde{L} = \frac{m}{2} \sum_{i=1}^{3} v_i^2 + q \sum_{i=1}^{3} \left(A_i + \frac{\partial f}{\partial q_i} \right) v_i - q\phi - \gamma z$$

that is

$$\widetilde{L} = L + q \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} v_i.$$

According to (19),

$$\eta_{\widetilde{L}} = \eta_L - q df.$$

Furthermore, based on (20), (21) and (22),

$$E_{\widetilde{L}} = E_L,\ d\eta_{\widetilde{L}} = d\eta_L,\ R_{\widetilde{L}} = R_L,$$

In this manner, the equations for the field $\xi_{\widetilde{L}}$ which govern the new equations of motion, can be written as

$$\left.\begin{array}{r}\eta_{\widetilde{L}}(\xi_{\widetilde{L}}) = -E_{\widetilde{L}}\\ i_{\xi_{\widetilde{L}}} d\eta_{\widetilde{L}} = dE_{\widetilde{L}} - R_{\widetilde{L}}(E_{\widetilde{L}})\eta_{\widetilde{L}}\end{array}\right\}$$

Under these conditions, we have

$$\xi_{\widetilde{L}} = \xi_L = v_i \frac{\partial}{\partial x_i} + \widetilde{F}_i \frac{\partial}{\partial v_i} + L\frac{\partial}{\partial z}$$

where

$$m\widetilde{F}_i = qv_i \sum_{j=1}^{3}\left(\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j}\right) - q\frac{\partial \phi}{\partial x_i} - \gamma(mv_i + qA_i) - \gamma q\frac{\partial f}{\partial x_i},$$

As a consequence, we find that the equations of motion are dependent upon the specific gage definition chosen.

6. Some Future Outlook and Perspectives

In this paper, we have only addressed some aspects of what can potentially become a fertile framework for research if these initial steps follow the scheme created by the symmetries of autonomous Hamiltonian systems as a geometric support in the search for conserved quantities and in the integration of the dynamical equations (one may delve, for example, into [43], so that a general program on symmetries of symplectic Hamiltonian systems can serve as an archetype in the search for dissipated quantities associated with different forms of understanding symmetries).

- From this perspective, a vector field $Y \in \mathfrak{X}(TQ \times \mathbb{R})$ on the contact Hamiltonian system $(TQ\times\mathbb{R}, \eta_L, E_L)$ is a Noether symmetry if $\mathcal{L}_Y \eta_L = 0$ and $\mathcal{L}_Y E_L = 0$.

  A Noether symmetry is in fact a dynamical symmetry (Definition 2). Indeed, to see $[Y, \xi_L] = 0$ it suffices to verify

  $$i_{[Y,\xi_L]}\eta_L = 0,\ i_{[Y,\xi_L]}d\eta_L = 0.$$

  Now

  $$i_{[Y,\xi_L]}\eta_L = Y\left\langle \eta_L, \xi_L\right\rangle - \eta_L([Y,\xi_L]) = 0$$

  and

  $$\begin{aligned} i_{[Y,\xi_L]}d\eta_L &= [\mathcal{L}_Y, i_{\xi_L}]d\eta_L = \mathcal{L}_Y i_{\xi_L} d\eta_L\\ &= \mathcal{L}_Y(dE_L - \mathcal{R}(E_L)\cdot E_L) \end{aligned} \tag{23}$$

  On the other hand, it is not difficult to see that Y preserves the Reeb vector field, $[Y, \mathcal{R}_L] = 0$, and thus, from (23), we obtain $i_{[Y,\xi_L]}d\eta_L = 0$. It follows then that if a Hamiltonian vector field X_f, $(f \in C^\infty(TQ\times\mathbb{R}))$

is a Noether symmetry then f is a dissipated quantity. In fact, the relation

$$\{E_L, f\} = \mathcal{L}_{X_f}\eta_L(\xi_L) + X_f(E_L)$$

provides the result.
However, for dissipated quantities, the converse of Noether theorem does not hold, in the sense that if $[E_L, f] = 0$ then X_f is a Noether symmetry. It is clear that adding the condition $\mathcal{R}(f) = 0$ does indeed solve the inverse problem, but the issue persists outside of the first integrals of the Reeb vector field.

- We now address symmetries that are not of Noether type. In this regard, we will say that Y is a non-Hamiltonian symmetry on $(TQ\times\mathbb{R}, \eta_L, E_L)$, if $\mathcal{L}_Y E_L \neq 0$, and $i_{[\xi_L,Y]}\eta_L = 0$.
From basic calculus, we obtain

$$0 = i_{[\xi_L,Y]}\eta_L = \mathcal{L}_{\xi_L} i_Y\eta_L - i_Y\mathcal{L}_{\xi_L}\eta_L,$$

leading to $\xi_L\left(\eta_L(Y)\right) = -R(E_L)\eta_L(Y)$ which is Proposition 3.
It is a classical result for Hamiltonian symplectic dynamics that the system energy subject to a dynamical symmetry is, in turn, a conservation law. However, in the dissipative framework there exists an obstruction to this fact. Indeed, a straightforward calculation leads us to

$$\{E_L, \mathcal{L}_Y E_L\} = -\mathcal{L}_Y\left(\mathcal{R}_L(E_L)\right)\cdot E_L.$$

In this manner, by also imposing the condition $\mathcal{L}_Y\left(R_L(E_L)\right) = 0$ on the field Y we obtain not only $\mathcal{L}_Y E_L$ as the dissipated quantity but also $\mathcal{L}_Y(\eta_L(Y))$:

$$\begin{aligned}\{E_L, \mathcal{L}_Y(\eta_L(Y))\} &= \xi_L(\mathcal{L}_Y(\eta_L(Y))) + \mathcal{L}_Y(\eta_L(Y))\mathcal{R}(E_L)\\ &= \mathcal{L}_Y\left(-R_L(E_L)\eta_L(Y)\right) + \mathcal{L}_Y(\eta_L(Y))\mathcal{R}(E_L)\\ &= -\mathcal{L}_Y\left(R_L(E_L)\right)\eta_L(Y) = 0.\end{aligned}$$

Furthermore, we have

$$\mathcal{L}_{\xi_L}(\mathcal{L}_Y\eta_L) = \mathcal{L}_Y(\mathcal{L}_{\xi_L}\eta_L) = \mathcal{L}_Y(-R_L(E_L)\eta_L) = -R_L(E_L)\mathcal{L}_Y\eta_L$$

that is,

$$\mathcal{L}_{\xi_L}(\mathcal{L}_Y\eta_L) = -R_L(E_L)\mathcal{L}_Y\eta_L$$

leading to the concept of a dissipated 1-form $\mathcal{L}_Y\eta_L$ on $(TQ\times\mathbb{R}, \eta_L, E_L)$. From this perspective, it is worthwhile to deepen our understanding of significant facets of the theory, such us Noether symmetries (for instance, [6], [45] and [46]), or hidden symmetries [33,44] which can be of great importance.

- As we are finalizing our article, we will now return to its very original germ: just as the symmetries of a classical Lagrangian system yield conserved quantities (considering the lifting, both vertical and complete, to the phase space of its symmetry group), the analogous situation within the framework of a contact Lagrangian system remarkably yields dissipation laws using the same duality between motions and their corresponding vertical components in phase space.

First, let us gather, with a certain level of basic detail, the previously mentioned result as a mean of framing two other ways to understand the invariance of a Lagrangian system in the phase space of velocities that may potentially serve as a guide in directing forthcoming research pertaining to dissipated quantities in contact dynamics (as the first of them has become a paradigmatic example).

In fact, if we have a one-parameter group $q' = Q_\varepsilon(q,t)$ of transformations of Q such that its extension $(q' = Q_\varepsilon(q,t), v' = V_\varepsilon(v,t))$ to TQ leaves the Lagrangian $L(q,v,t)$ invariant,

$$L(Q_\varepsilon(q,t), V_\varepsilon(v,t)), t) = L(q,v,t).$$

Thus, for the infinitesimal generator $X \in \mathfrak{X}(TQ)$

$$X = \left.\frac{\partial Q_\varepsilon(q,t)}{\partial \varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial q} + \left.\frac{\partial V_\varepsilon(q,t)}{\partial \varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial q} = a\frac{\partial}{\partial q} + \frac{\mathrm{d}a}{\mathrm{d}t}\frac{\partial}{\partial v}$$

$(q = (q_1, ..., q_n)$, etc.) the Euler–Lagrange equations $\partial L/\partial q = \partial/\partial t$ $(\partial L/\partial v)$ makes the condition $X(L) = a \cdot \partial L/\partial q + \mathrm{d}a/\mathrm{d}t \cdot \partial L/\partial v = 0$ into

$$\frac{\mathrm{d}\,(X^v L)}{\mathrm{d}t} = 0$$

where $X^V = a\partial/\partial v$.

However, it soon became clear that the way of understanding the invariance of a Lagrangian system substantially altered the nature of the conserved quantities

If the one-parameter transformation $Q_\varepsilon(q,t)$ de Q produces a gage invariance in L, in the sense that there exists a function $\psi(q,t)$ such that

$$L(Q_\varepsilon(q,t), V_\varepsilon(q,t), t) + \frac{\mathrm{d}\psi(Q_\varepsilon(q,t),t)}{\mathrm{d}t} = L(q,v,t)$$

(thus, $L(q,v,t)$ and $L'(q,v,t) = L(Q_\varepsilon(q,t), V_\varepsilon(q,t), t)$ define identical Euler–Lagrange equations), so a calculation similar to the previous one shows that.

$$X^v L + X_c \psi$$

(where $X_c = a_i \partial/\partial q_i$) , is a conserved quantity for the system.

As a final situation, let us consider the case where τ_ε is a one-parameter group of transformations of $Q \times \mathbb{R}_t$,

$$q' = Q_\varepsilon(q,t), v' = Q_\varepsilon(q,t), t' = \tau_\varepsilon(q,t)$$

such that its infinitesimal generator is

$$X = a\frac{\partial}{\partial q} + \frac{\mathrm{d}a}{\mathrm{d}t}\frac{\partial}{\partial v} + a_o\frac{\partial}{\partial t}.$$

A calculation proofs that as a consequence of the differential invariance of the 1-form

$$L(q', v', t')\mathrm{d}t' = L(q,v,t)\mathrm{d}t,$$

we have the conservation law

$$X^v L - a_o E_L$$

where E_L is, as usual, the energy of the system.

- This last issue must be addressed within the context of time-dependence in contact Hamiltonian systems. It is a topic that warrants further research and development. Here, we will outline some key advances and milestones achieved so far. Following an excellent introduction in [5], the authors of [26] delve deeper into the relationship between symmetries and dissipated quantities in time-dependent contact Hamiltonian systems and they introduce new classes of symmetries for time-dependent contact Lagrangian systems. In [35], the authors introduce a new geometric structure called cocontact manifolds to describe time-dependent contact systems. This study contributes to describing dissipative systems and paves the way for further advancements in understanding time-dependent contact mechanics.
- Hamilton–Jacobi Theory holds deep significance when constructing conservation laws, bearing implications on the Liouville integrability of the system. This is a topic that warrants attention beyond canonical symplectic geometry, potentially including dissipated quantities. A possible first encounter with the Hamilton–Jacobi formulation of contact Hamiltonian systems might be in the reference [5]. In particular, it is shown that, as in the classical case, a complete solution to the Hamilton–Jacobi PDE can be used to solve the contact Hamiltonian equations. Another seminal article in this subject is [34], which focuses on finding solutions for classical Hamiltonian systems that align with cosymplectic or contact structures, thereby incorporating explicit time-dependence and dissipative effects. A recent article, [39] explores the integrability of contact Hamiltonian systems, including both time-dependent and independent cases. It presents various Hamilton–Jacobi equations according to specific contexts.
- As we conclude this article, let us propose revisiting one of our own objectives by drawing attention to the result of Proposition 18, exposing the dispersive nature of the quantity $\eta_L(A^*)$ to specific problems.

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