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The Ground State Solutions of Discrete Nonlinear Schrödinger Equations with Hardy Weights

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Abstract. In this paper, we study the discrete nonlinear Schrödinger equation

$$-\Delta u+\left(V(x)-\frac{\rho}{(|x|^2+1)}\right)u=f(x,u),\quad u\in\ell^2(\mathbb{Z}^N),$$

where $N\geq 3$, V is a bounded periodic potential and 0 lies in a spectral gap of the Schrödinger operator $-\Delta+V$. The resulting problem engages two major difficulties: one is that the associated functional is strongly indefinite and the other is the lack of compactness of the Cerami sequence. We overcome these two major difficulties by the generalized linking theorem and Lions lemma. This enables us to establish the existence and asymptotic behavior of ground state solutions for small $\rho\geq 0$.

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Keywords. Schrödinger equation, spectral gap, ground state solutions, lattice graphs.

1. Introduction

The nonlinear Schrödinger equation

$$-\Delta u+\left(V(x)-\frac{\rho}{|x|^2}\right)u=f(x,u),\quad x\in\mathbb{R}^N,$$

has drawn a great deal of interest in recent years. In particular, for $\rho=0$, there is a broad literature treating the Schrödinger equation with periodic potential. For example, when the operator $-\Delta+V$ is positive definite, Pankov [27] proved an existence result by the Nehari variational principle and concentration compactness methods. (Even more general asymptotically periodic case was treated in that paper). Later, Rabinowitz [32] obtained the existence of nontrivial solutions under less restrictive assumptions on the nonlinearity

f. Moreover, in [21], the authors established the ground state solutions under a more natural super-quadratic condition [see (F4) below]. When 0 lies in a finite spectral gap and the operator $-\Delta + V$ is not positive definite, the first existence results (under very strong assumptions on the nonlinearity) were found in [1,16]. Later, Troestler and Willem [40] and Kryszewski and Szulkin [19] proved the existence of nontrivial solutions under much more natural conditions. Pankov [28] demonstrated the existence of ground state solutions by the Nehari manifold method to the case of strongly indefinite functionals. Moreover, Szulkin and Weth [38] obtained the ground state solutions based on a direct and simple reduction of the indefinite variational problem to a definite one. After that, Liu [25] improved the result of Szulkin and Weth [38] under a weaker monotonicity condition on f. Recently, for $\rho > 0$, Guo and Mederski [13] studied the existence and behavior of ground state solutions under some conditions on f. Later, the authors in [22] also established the existence and asymptotical behavior of ground state solutions under different assumptions on f. For more related results, we refer readers to [4,9,17,20,33,35,43] and the references therein.

Nowadays, many researchers turn to study differential equations on graphs, especially for the nonlinear Schrödinger equations. For example, a class of Schrödinger equations with the nonlinearity of power type have been studied on graphs, see [10–12,14,15,45]. In addition, the existence or multiplicity of gap solitons (then the associated energy functional is strongly indefinite) of periodic discrete Schrödinger equation on the lattice graph $\mathbb{Z}$ has been extensively investigated. For example, Pankov [29] obtained the existence of nontrivial solutions by a generalized linking theorem due to [19]. Pankov [30] also obtained the existence of ground state solutions by a generalized Nehari manifold and periodic approximation technique. Later, Chen and Ma [6] proved the existence of ground state solitons and the existence of infinitely many pairs of geometrically distinct solitons by the generalized Nehari manifold method developed by Szulkin and Weth [38]. Moreover, Chen and Ma [5,7] established the existence of nontrivial solutions with asymptotically or super linear terms by a variant generalized weak linking theorem. For related works, we refer readers to [23,36,37,39,41,44].

As far as we know, there is no existence results for the Schrödinger equation with hardy potential on the lattice graph $\mathbb{Z}^N$, which is a natural discrete model for the Euclidean space. Motivated by the works mentioned above, in this paper, we prove the existence and asymptotical behavior of ground state solutions for a class of strongly indefinite problems with hardy weights on $\mathbb{Z}^N$ with $N \geq 3$ by following the arguments in [13,26].

Let Ω be a subset of $\mathbb{Z}^N$, we denote by $C(\Omega)$ the space of real-valued functions on Ω. The support of $u \in C(\Omega)$ is defined as $\mathrm{supp}(u) := \{x \in \Omega \colon u(x) \neq 0\}$. Moreover, we denote by the $\ell^p(\Omega)$ the space of ℓ^p-summable functions on Ω. For convenience, for any $u \in C(\Omega)$, we always write $\int_\Omega u \,\mathrm{d}\mu := \sum_{x\in\Omega} u(x)$, where μ is the counting measure in Ω.

In this paper, we study the nonlinear Schrödinger equation

$$-\Delta u + (V(x) - \frac{\rho}{(|x|^2+1)})u = f(x,u), \qquad u \in \ell^2(\mathbb{Z}^N), \tag{1}$$

where $N \geq 3$. Here the operator Δ is the discrete Laplacian defined as $\Delta u(x) = \sum_{y \sim x}(u(y) - u(x))$. We always assume that

(H): $V \in L^\infty(\mathbb{Z}^N)$, V is T-periodic with $T \in \mathbb{Z}^N$ and

$$\sigma^- := \sup\, [\sigma(-\Delta + V) \cap (-\infty, 0)] < 0 < \sigma^+ := \inf\, [\sigma(-\Delta + V) \cap (0, +\infty)],$$

where $\sigma(-\Delta + V)$ is the spectrum of the operator $-\Delta + V$ in $\ell^2(\mathbb{Z}^N)$;

(F1): $f\colon \mathbb{Z}^N \times \mathbb{R} \to \mathbb{R}$ is T-periodic in x and continuous in $u \in \mathbb{R}$;

(F2): There are constants $a > 0$ and $p > 2$ such that

$$|f(x,u)| \leq a(1 + |u|^{p-1}), \quad (x,u) \in \mathbb{Z}^N \times \mathbb{R};$$

(F3): $f(x,u) = o(u)$ uniformly in x as $|u| \to 0$;

(F4): $\frac{F(x,u)}{u^2} \to +\infty$ uniformly in x as $|u| \to +\infty$ with $F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t$;

(F5): $u \mapsto \frac{f(x,u)}{|u|}$ is non-decreasing on $(-\infty, 0)$ and $(0, +\infty)$;

(F6): $f(x,u)$ is of C^1 class about $u \in \mathbb{R}$ and satisfies

$$f(x,u)u - 2F(x,u) \geq b|u|^q, \quad (x,u) \in \mathbb{Z}^N \times \mathbb{R},$$

where $b > 0$ and $2 < q \leq p$.

Clearly, by (F1), (F2) and (F3), for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon |u| + c_\varepsilon |u|^{p-1}, \quad (x,u) \in \mathbb{Z}^N \times \mathbb{R}. \tag{2}$$

Moreover, by (F3) and (F5), we have that

$$f(x,u)u \geq 2F(x,u) \geq 0, \quad (x,u) \in \mathbb{Z}^N \times \mathbb{R}. \tag{3}$$

Denote $A := -\Delta + V$ and $X := \ell^2(\mathbb{Z}^N)$. Then the energy functional of (1) is

$$J_\rho(u) = \frac{1}{2}(Au,u)_2 - \frac{1}{2}\int_{\mathbb{Z}^N} \frac{\rho}{(|x|^2+1)}|u|^2\,\mathrm{d}\mu - \int_{\mathbb{Z}^N} F(x,u)\,\mathrm{d}\mu,$$

where $(\cdot,\cdot)_2$ is the inner product in X. The corresponding norm in X is denoted by $\|\cdot\|_2$. Then $J_\rho(u) \in C^1(X,\mathbb{R})$ and the Gateaux derivative is given by

$$\langle J_\rho'(u), \phi\rangle = (Au,\phi)_2 - \int_{\mathbb{Z}^N} \frac{\rho}{(|x|^2+1)}u\phi\,\mathrm{d}\mu - \int_{\mathbb{Z}^N} f(x,u)\phi\,\mathrm{d}\mu, \quad u,\phi \in X.$$

By (H), we have the decomposition $X = X^+ \oplus X^-$, where X^+ and X^- are the positive and negative spectral subspaces of A in X. Then we have that

$$(Au,u)_2 \geq \sigma^+\|u\|_2^2, \quad u \in X^+, \qquad \text{and} \qquad -(Au,u)_2 \geq -\sigma^-\|u\|_2^2, \quad u \in X^-.$$

Hence the form $(Au,u)_2$ is positive definite on X^+ and negative definite on X^-.

For any $u, v \in X = X^+ \oplus X^-$, $u = u^+ + u^-$ and $v = v^+ + v^-$, we define an equivalent inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ on X by

$$(u,v) = (Au^+, v^+)_2 - (Au^-, v^-)_2 \qquad \text{and} \qquad \|u\| = (u,u)^{\frac{1}{2}},$$

respectively. Clearly, the decomposition $X = X^+ \oplus X^-$ is orthogonal with respect to both inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$. Therefore, the energy functional J_ρ and the corresponding Gateaux derivative can be rewritten as

$$J_\rho(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\int_{\mathbb{Z}^N} \frac{\rho}{(|x|^2+1)}|u|^2\,\mathrm{d}\mu - \int_{\mathbb{Z}^N} F(x,u)\,\mathrm{d}\mu,$$

and

$$\langle J_\rho'(u),\phi\rangle = (u^+,\phi) - (u^-,\phi) - \int_{\mathbb{Z}^N} \frac{\rho}{(|x|^2+1)} u\phi\,\mathrm{d}\mu - \int_{\mathbb{Z}^N} f(x,u)\phi\,\mathrm{d}\mu, \quad u,\phi \in X,$$

respectively.

We say that $u \in X$ is a solution of (1), if u is a critical point of the energy functional J_ρ, i.e., $J_\rho'(u) = 0$. A ground state solution of (1) means that u is a nontrivial critical point of J_ρ with the least energy, that is,

$$J_\rho(u) = \inf_{N_\rho} J_\rho > 0,$$

where

$$N_\rho = \{u \in X\backslash X^- \colon \langle J_\rho'(u),u\rangle = 0 \text{ and } \langle J_\rho'(u),v\rangle = 0 \text{ for } v \in X^-\}$$

is the Nehari manifold.

Denote

$$\rho^+ := \sup\{M > 0 \colon (Au,u)_2 \geq M \int_{\mathbb{Z}^N} |\nabla u|^2\,\mathrm{d}\mu, \quad u \in X^+\}. \tag{4}$$

Since $|\nabla u(x)|^2 = \frac{1}{2}\sum\limits_{y\sim x}(u(y)-u(x))^2$, one gets easily that

$$\int_{\mathbb{Z}^N} |\nabla u|^2\,\mathrm{d}\mu \leq C_N\|u\|_2^2.$$

Note that for $u \in X^+$, $(Au,u)_2$ is positive definite, then $\rho^+ > 0$. Let $\tilde{\rho}^+ = \min\{\rho^+, 1\}$ and $\kappa > 0$ be the constant in Lemma 2.1 below. Now we state our first main result of this paper.

Theorem 1.1. *Let $0 \leq \rho < \frac{\tilde{\rho}^+}{\kappa}$. Assume that (H) and (F1)–(F5) hold. Then the Eq. (1) has a ground state solution.*

Remark 1.2. (i) In the continuous setting, the nonlinear term f has a superlinear and subcritical growth. However, in our context, it is just a superlinear nonlinear term thanks to the embedding ℓ^s into ℓ^t for $s < t$ in the discrete setting;

(ii) The authors in [8,12,31,46] have proved the existence of nontrivial solutions to the discrete Schrödinger equations with unbounded potentials. The unbounded potential V ensures a compact embedding from a weighted subspace of ℓ^2 into ℓ^q $(q \geq 2)$, which allows to handle the lack of compactness of a Palais–Smale or Cerami sequence. In contrast to the unbounded case, in this paper, the Hardy potential V tends to zero, which has no direct compact embedding. This leads to our proof more difficult;

(iii) For the Schrödinger equations with Hardy type potentials, the existence of ground state solutions depends on the constant ρ. This fine property is well known in the continuous case, but not yet in the discrete case;

(iv) The assumption that 0 is in a finite spectral gap of the operator $-\Delta+V$ leads to the associated functional is strongly indefinite. To tackle this difficulty, we follow the lines of the continuous case to get the discrete version. It is worth noting that our conditions can be used to significantly improve the well-known results of the corresponding continuous case;

(v) The existence of nontrivial solutions to the discrete Schrödinger equation with a sign-changing periodic potential has been extensively studied on the lattice graph $\mathbb{Z}$, see for example [5,7,29,30]. However, for the higher dimensional lattice graphs $\mathbb{Z}^N$, as far as we know, there is no such existence results. This is the first attempt in the literature on the existence of a ground state solution for the strongly indefinite problem with a Harty weight.

The second main result is about the behavior of ground state solution in the limit $\rho \to 0^+$.

Theorem 1.3. *Let $0 \leq \rho < \frac{\tilde{\rho}^+}{\kappa}$. Assume that (H) and $(F1)$–$(F6)$ hold. Let u_ρ and u_0 be the ground state solutions of J_ρ and J_0. Then for $\rho_n \to 0^+$, there exists a sequence $\{x_n\} \subset \mathbb{Z}^N$ such that $u_{\rho_n}(x+x_n)$ tends to a ground state solution u_0 of J_0 as $n \to +\infty$.*

This paper is organized as follows. In Sect. 2, we present some preliminaries including settings for graphs and some auxiliary lemmas. In Sect. 3, we state a generalized linking theorem and demonstrate the functional $J_\rho \in C^1(X,\mathbb{R})$ satisfies the conditions of the linking theorem. In Sect. 4, we study the behavior of Cerami sequences. In Sect. 5, we are devoted to prove Theorems 1.1 and 1.3.

2. Preliminaries

In this section, we introduce some settings for graphs and give some useful lemmas.

Let $G = (\mathbb{V}, \mathbb{E})$ be a connected, locally finite graph, where $\mathbb{V}$ denotes the vertex set and $\mathbb{E}$ denotes the edge set. We call vertices x and y neighbors, denoted by $x \sim y$, if there is an edge connecting them, i.e., $(x,y) \in \mathbb{E}$. For any $x, y \in \mathbb{V}$, the distance $d(x,y)$ is defined as the minimum number of edges connecting x and y, i.e.,

$$d(x,y) = \inf\{k \colon x = x_0 \sim \cdots \sim x_k = y\}.$$

Let $B_r(a) = \{x \in \mathbb{V} \colon, d(x,a) \leq r\}$ be the closed ball of radius r centered at $a \in \mathbb{V}$ and denote $|B_r(a)| = \sharp B_r^S(a)$ as the volume (i.e., cardinality) of the set $B_r(a)$. For brevity, we write $B_r := B_r(0)$.

In this paper, we consider the natural discrete model of the Euclidean space, the integer lattice graph. The N-dimensional integer lattice graph, denoted by $\mathbb{Z}^N$, consists of the set of vertices $\mathbb{V} = \mathbb{Z}^N$ and the set of edges $\mathbb{E} = \{(x, y): x,\, y \in \mathbb{Z}^N,\, \sum_{i=1}^{N} d(x_i, y_i) = 1\}$. In the sequel, we write the distance $d(x, y)$, as defined in the Euclidean space, as $|x - y|$ on $\mathbb{Z}^N$.

We denote the space of real-valued functions on $\mathbb{V}$ by $C(\mathbb{V})$, and denote the subspace of functions with finite support by $C_c(\mathbb{V})$. For any $\Omega \subset \mathbb{V}$, via continuation by zero, the spaces $C(\Omega)$ and $C_c(\Omega)$ are considered to be subspaces of $C(\mathbb{V})$ and $C_c(\mathbb{V})$. For any $u \in C(\Omega)$, the $\ell^p(\Omega)$ space is given by

$$\ell^p(\Omega) = \{u \in C(\Omega): \|u\|_{\ell^p(\Omega)} < +\infty\}, \qquad p \in [1, +\infty],$$

where

$$\|u\|_{\ell^\infty(\Omega)} = \sup_{x\in\Omega} |u(x)| \quad \text{and} \quad \|u\|_{\ell^p(\Omega)} = \left(\sum_{x\in\Omega} |u(x)|^p\right)^{\frac{1}{p}}, \quad p \in [1, +\infty).$$

We shall write $\|u\|_p$ instead of $\|u\|_{\ell^p(\mathbb{V})}$ if $\Omega = \mathbb{V}$.

For $u, v \in C(\mathbb{V})$, the gradient form Γ, called the "carré du cham" operator, is defined as

$$\Gamma(u, v)(x) = \frac{1}{2}\sum_{y\sim x}(u(y) - u(x))(v(y) - v(x)) =: \nabla u \nabla v.$$

In particular, we write $\Gamma(u) = \Gamma(u, u)$ and denote the length of $\Gamma(u)$ by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2}\sum_{y\sim x}(u(y) - u(x))^2\right)^{\frac{1}{2}}.$$

The Laplacian of u at $x \in \mathbb{V}$ is defined as $\Delta u(x) = \sum_{y\sim x}(u(y) - u(x))$. For convenience, for any $u \in C(\Omega)$, we always write $\int_\Omega u \,\mathrm{d}\mu := \sum_{x\in\Omega} u(x)$, where μ is the counting measure in $\Omega \subset \mathbb{V}$.

Next, we give some useful lemmas. First, we recall a variant of Hardy type inequality, see [34].

Lemma 2.1. *Let $N \geq 3$. We have the discrete Hardy inequality*

$$\int_{\mathbb{Z}^N} \frac{|u|^2}{(|x|^2 + 1)}\, d\mu \leq \kappa \int_{\mathbb{Z}^N} |\nabla u|^2\, d\mu, \quad u \in C_c(\mathbb{Z}^N), \tag{5}$$

where κ depends only on N.

Lemma 2.2. *For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any $u \in X$,*

$$\int_{\mathbb{V}} F(x, u)\,\mathrm{d}\mu \leq \varepsilon\|u\|_2^2 + C_\varepsilon\|u\|_p^p.$$

Proof. It follows from (2) and (3) that

$$\begin{aligned}\int_{\mathbb{V}} F(x, u)\,\mathrm{d}\mu &\leq \frac{1}{2}\int_{\mathbb{V}} f(x, u)u\,\mathrm{d}\mu \\ &\leq \frac{1}{2}\left(\int_{\mathbb{V}} \varepsilon|u|^2 + c_\varepsilon|u|^p\,\mathrm{d}\mu\right) \\ &\leq \varepsilon\|u\|_2^2 + C_\varepsilon\|u\|_p^p.\end{aligned}$$

□

Lemma 2.3. *Let* $0 \le \rho < \frac{\tilde{\rho}^+}{\kappa}$. *For any* $u \in X^+$, $\|u\|_\rho^2 := \Big(\|u\|^2 - \int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|u|^2 \,\mathrm{d}\mu\Big)$ *satisfies*

$$\|u\|^2 \ge \|u\|_\rho^2 \ge \frac{1}{2}(\tilde{\rho}^+ - \kappa\rho)\|u\|^2.$$

Hence $\|\cdot\|_\rho$ *is a norm defined on* X^+ *and it is equivalent with the norm* $\|\cdot\|$.

Proof. For any $u \in X^+$, clearly, we have that $\|u\|^2 \ge \|u\|_\rho^2$. In the following, we prove that

$$\|u\|_\rho^2 \ge \frac{1}{2}(\tilde{\rho}^+ - \kappa\rho)\|u\|^2.$$

By (4) with $\rho^+ \ge \tilde{\rho}^+$ and the Hardy inequality (5), one has that

$$\begin{aligned}
\|u\|_\rho^2 &= \|u\|^2 - \int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|u|^2 \,\mathrm{d}\mu \\
&= \int_{\mathbb{V}} \left(|\nabla u|^2 + V(x)|u|^2\right) - \frac{\rho}{(|x|^2+1)}|u|^2 \,\mathrm{d}\mu \\
&\ge \int_{\mathbb{V}} \tilde{\rho}^+|\nabla u|^2 - \frac{\rho}{(|x|^2+1)}|u|^2 \,\mathrm{d}\mu \\
&\ge (\tilde{\rho}^+ - \kappa\rho)\int_{\mathbb{V}} |\nabla u|^2 \,\mathrm{d}\mu. \qquad (6)
\end{aligned}$$

If $\int_{\mathbb{V}} V(x)|u|^2 \,\mathrm{d}\mu \le 0$, then it follows from (6) that

$$\begin{aligned}
\|u\|_\rho^2 &\ge (\tilde{\rho}^+ - \kappa\rho)\int_{\mathbb{V}} |\nabla u|^2 \,\mathrm{d}\mu \\
&\ge (\tilde{\rho}^+ - \kappa\rho)\int_{\mathbb{V}} (|\nabla u|^2 + V(x)|u|^2) \,\mathrm{d}\mu \\
&\ge \frac{1}{2}(\tilde{\rho}^+ - \kappa\rho)\int_{\mathbb{V}} (|\nabla u|^2 + V(x)|u|^2) \,\mathrm{d}\mu \\
&= \frac{1}{2}(\tilde{\rho}^+ - \kappa\rho)\|u\|^2.
\end{aligned}$$

If $\int_{\mathbb{V}} V(x)|u|^2 \,\mathrm{d}\mu \ge 0$, by the Hardy inequality (5) and the fact $\kappa\rho < \tilde{\rho}^+ \le 1$, we have that

$$\int_{\mathbb{V}} |\nabla u|^2 - \frac{\rho}{(|x|^2+1)}|u|^2 \,\mathrm{d}\mu \ge (1-\kappa\rho)\int_{\mathbb{V}} |\nabla u|^2 \,\mathrm{d}\mu > 0.$$

This implies that

$$\begin{aligned}
\|u\|_\rho^2 &= \|u\|^2 - \int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|u|^2 \,\mathrm{d}\mu \\
&= \int_{\mathbb{V}} \left(|\nabla u|^2 + V(x)|u|^2\right) - \frac{\rho}{(|x|^2+1)}|u|^2 \,\mathrm{d}\mu \\
&\ge \int_{\mathbb{V}} V(x)|u|^2 \,\mathrm{d}\mu \\
&\ge (\tilde{\rho}^+ - \kappa\rho)\int_{\mathbb{V}} V(x)|u|^2 \,\mathrm{d}\mu, \qquad (7)
\end{aligned}$$

where we have used the fact that $(\tilde{\rho}^+ - \kappa\rho) < 1$ in the last inequality. Summing (6) and (7), we get that

$$\|u\|_\rho^2 \geq \frac{1}{2}(\tilde{\rho}^+ - \kappa\rho)\|u\|^2.$$

In a summary, we have $\|u\|_\rho^2 \geq \frac{1}{2}(\tilde{\rho}^+ - \kappa\rho)\|u\|^2$. □

Lemma 2.4. *If* $\lim_{n\to+\infty} |x_n| = +\infty$, *then for any* $u \in X$, *as* $n \to +\infty$,

$$\int_{\mathbb{V}} \frac{1}{(|x|^2+1)} |u(x-x_n)|^2 \,\mathrm{d}\mu \to 0.$$

Proof. Let $\phi_m \in C_c(\mathbb{V})$ and $\phi_m \to u$ in X as $m \to +\infty$. Assume that $\mathrm{supp}(\phi_m) \subset B_{r_m}$ with $r_m \geq 1$. Since $\lim_{n\to+\infty} |x_n| = +\infty$, for any m, there exists $n = n(m)$ such that $|x_n| - r_m \geq m$ and $\{n(m)\}$ is an increasing sequence. Then

$$\begin{aligned}
\int_{\mathbb{V}} \frac{1}{(|x|^2+1)} |\phi_m(x-x_n)|^2 \,\mathrm{d}\mu &= \int_{\mathbb{V}} \frac{1}{(|x+x_n|^2+1)} |\phi_m|^2 \,\mathrm{d}\mu \\
&= \int_{B_{r_m}} \frac{1}{(|x+x_n|^2+1)} |\phi_m|^2 \,\mathrm{d}\mu \\
&\leq \frac{1}{(|x_n|-r_m)^2} \int_{B_{r_m}} |\phi_m|^2 \,\mathrm{d}\mu \\
&\leq \frac{1}{m^2} \|\phi_m\|_2^2 \to 0, \qquad \text{as } m \to +\infty.
\end{aligned}$$

Then by the Hardy inequality (5), we get the result. □

Let (Ω, Σ, τ) be a measure space, which consists of a set Ω equipped with a $\sigma-$algebra Σ and a Borel measure $\tau\colon \Sigma \to [0, +\infty]$. We introduce the classical Brézis—Lieb lemma [3].

Lemma 2.5. (Brézis–Lieb lemma) *Let* (Ω, Σ, τ) *be a measure space and* $\{u_n\} \subset L^p(\Omega, \Sigma, \tau)$ *with* $0 < p < +\infty$. *If*

(a) $\{u_n\}$ *is uniformly bounded in* $L^p(\Omega)$,

(b) $u_n \to u, \tau-$*almost everywhere in* Ω,

then we have that

$$\lim_{n\to+\infty} (\|u_n\|_{L^p(\Omega)}^p - \|u_n - u\|_{L^p(\Omega)}^p) = \|u\|_{L^p(\Omega)}^p.$$

Remark 2.6. If Ω is countable and τ is the counting measure μ in Ω, then we get a discrete version of the Brézis–Lieb lemma.

We give a discrete Lions lemma corresponding to Lions [24] on $\mathbb{R}^N$, which denies a sequence $\{u_n\}$ to distribute itself over $\mathbb{V}$.

Lemma 2.7. (Lions lemma) *Let* $1 \leq p < +\infty$. *Assume that* $\{u_n\}$ *is bounded in* $\ell^p(\mathbb{V})$ *and* $\|u_n\|_\infty \to 0$, *as* $n \to \infty$. *Then for any* $p < q < +\infty$, *as* $n \to \infty$,

$$u_n \to 0, \qquad \text{in } \ell^q(\mathbb{V}).$$

Proof. For $p < q < +\infty$, this result follows from the interpolation inequality

$$\|u_n\|_q^q \le \|u_n\|_p^p \|u_n\|_\infty^{q-p}.$$

□

Finally, we prove that the direct sum $X^+ \oplus X^-$ in X associated to a decomposition of the spectrum of the operator A remains "topologically direct" in the $\ell^p(\mathbb{V})$ space.

Lemma 2.8. *Let $X^+ \oplus X^-$ be the decomposition of $X = \ell^2(\mathbb{V})$ according to the positive and negative part of the spectrum $\sigma(A)$. Assume that $P, Q: X \to X$ are the projectors onto X^- along X^+ and onto X^+ along X^-, respectively. Then for any $p \in [1, +\infty]$, the restrictions of P and Q to $X \cap \ell^p(\mathbb{V})$ are ℓ^p-continuous.*

Proof. Assume that $\ell^p(\mathbb{V}; \mathbb{C}) = \ell^p(\mathbb{V}) + i\,\ell^p(\mathbb{V})$ is the complexification of $\ell^p(\mathbb{V})$. Let A_p be the operator

$$A_p: \ell^p(\mathbb{V}; \mathbb{C}) \to \ell^p(\mathbb{V}; \mathbb{C}): u \mapsto -\Delta u + V(x)u$$

with domain $D(A_p) := \{u \in \ell^p(\mathbb{V}; \mathbb{C}) |\, A_p u \in \ell^p(\mathbb{V}; \mathbb{C})\}$. Since the potential V is bounded, it follows from [2] that the spectrum $\sigma(A_p) \subset \mathbb{R}$ is independent of $p \in [1, +\infty]$, and moreover, for any $\lambda \notin \sigma(A_p) = \sigma(A_2) = \sigma(A)$,

$$(A_p - \lambda)^{-1} = (A_2 - \lambda)^{-1}, \quad \text{on } \ell^p(\mathbb{V}; \mathbb{C}) \cap \ell^2(\mathbb{V}; \mathbb{C}).$$

Then $0 \notin \sigma(A_p)$ and we assume that P_p, Q_p are the projectors on the negative and positive eigenspaces of A_p. Since $\sigma(A_p)$ is bounded below, by Theorem 6.17 of [18], we can define the projector P_p as follows:

$$P_p = \frac{1}{2\pi i} \int_\Gamma (A_p - \lambda)^{-1}\, \mathrm{d}\lambda,$$

where Γ is a right-oriented curve around the negative part of $\sigma(A_p)$ but not crossing the spectrum. This yields that

$$P_p = P_2, \quad \text{on } \ell^p(\mathbb{V}; \mathbb{C}) \cap \ell^2(\mathbb{V}; \mathbb{C}).$$

Then we get the desired result since $P = P_2|_X$ and $Q = I - P$. □

3. Generalized Linking Theorem

In this section, we first introduce a new topology $\mathcal{T}$ on the space X so as to provide a generalized linking theorem involving the Nehari–Pankov manifold, then we demonstrate that the functional $J_\rho \in C^1(X, \mathbb{R})$ satisfies the conditions of the linking theorem.

Let $X = X^+ \oplus X^-$ with $X^+ \perp X^-$. For any $u \in X$, we write $u = u^+ + u^-$, where $u^+ \in X^+$ and $u^- \in X^-$, as the direct sum decomposition.

Clearly, we have the norm topology $\|\cdot\|$ on X. Now we introduce a new topology $\mathcal{T}$ on X which is introduced by the norm

$$\|u\|_{\mathcal{T}} = \max\left\{\|u^+\|, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} |\langle u^-, e_k\rangle|\right\},$$

where $\{e_k\}_{k=1}^{+\infty}$ is a complete orthonormal system in X^- [19,42]. Observe that for any $u \in X$,

$$\|u^+\| \leq \|u\|_{\mathcal{T}} \leq \|u\|.$$

The convergence of a sequence $\{u_n\} \subset X$ in $\mathcal{T}$ will be denoted by $u_n \xrightarrow{\mathcal{T}} u$. Obviously, the new topology $\mathcal{T}$ is closely related to the topology on X which is strong on X^+ and weak on X^-. More precisely, if $\{u_n\} \subset X$ is bounded, then

$$u_n \xrightarrow{\mathcal{T}} u \quad \Leftrightarrow \quad u_n^+ \to u^+ \quad \text{and} \quad u_n^- \rightharpoonup u^-. \tag{8}$$

We will show that the functional J_ρ satisfies the following conditions:

(A1) For $\rho \geq 0$, J_ρ is $\mathcal{T}-$upper semicontinuous, i.e., $J_\rho^{-1}([t, +\infty))$ is $\mathcal{T}$-closed for any $t \in \mathbb{R}$;

(A2) For $\rho \geq 0$, J_ρ' is $\mathcal{T}-$to-weak* continuous, i.e. $J_\rho'(u_n) \rightharpoonup J_\rho'(u)$ as $u_n \xrightarrow{\mathcal{T}} u_0$;

(A3) For $0 \leq \rho < \tilde{\rho}^+$, there exists $r > 0$ such that $m := \inf_{u \in X^+ : \|u\| = r} J_\rho(u) > 0$;

(A4) For $0 \leq \rho < \tilde{\rho}^+$, if $u \in X \backslash X^-$, then there exists $R(u) > r$ such that

$$\sup_{\partial M(u)} J_\rho \leq J_\rho(0) = 0,$$

where $M(u) = \{tu + v \in X | v \in X^-, \|tu + v\| \leq R(u), t \geq 0\} \subset \mathbb{R}^+ u \oplus X^- = \mathbb{R}^+ u^+ \oplus X^-$ with $\mathbb{R}^+ = [0, +\infty)$;

(A5) For $\rho \geq 0$, if $u \in N_\rho$, then $J_\rho(u) \geq J_\rho(tu + v)$ for $t \geq 0$ and $v \in X^-$.

Note that the conditions (A3) and (A4) imply that the functional J_ρ satisfies the linking geometry. Hence, we introduce a generalized linking theorem. For any $A \subset X$, $I \subset [0, +\infty)$ such that $0 \in I$, and $h : A \times I \to X$, we collect the following assumptions:

(h1): h is $\mathcal{T}$-continuous (with respect to norm $\|\cdot\|_{\mathcal{T}}$);

(h2): $h(u, 0) = u$ for all $u \in A$;

(h3): $J_\rho(u) \geq J_\rho(h(u, t))$ for all $(u, t) \in A \times I$;

(h4): each $(u, t) \in A \times I$ has an open neighborhood W in the product topology of $(X, \mathcal{T})$ and I such that the set $\{v - h(v, s) : (v, s) \in W \cap (A \times I)\}$ is contained in a finite-dimensional subspace of X.

Now we state the linking theorem, which can be seen in [26,42].

Theorem 3.1. *If $J_\rho \in C^1(X, \mathbb{R})$ satisfies (A1)–(A4), then there exists a Cerami sequence $\{u_n\}$ at level c_ρ, that is, $J_\rho(u_n) \to c$ and $(1 + \|u_n\|) J_\rho'(u_n) \to 0$,* *where*

$$c_\rho := \inf_{u\in X\setminus X^-}\ \inf_{h\in\Gamma(u)}\ \sup_{u'\in M(u)} J_\rho(h(u',1)) \geq m > 0,$$

$$\Gamma(u) := \{h\colon M(u)\times[0,1]\to X \ \textit{satisfies}\ (h1)-(h4)\}.$$

Suppose that (A5) holds, then $c_\rho \leq \inf_{N_\rho} J_\rho$. *If* $c_\rho \geq J_\rho(u)$ *for some critical point* $u\in X\setminus X^-$, *then* $c_\rho = \inf_{N_\rho} J_\rho$.

Now we are devoted to verify the conditions (A1)–(A5) so as to apply the linking theorem 3.1. First, we show that J_ρ satisfies (A1)–(A2).

Lemma 3.2. *Let* $\rho \geq 0$. *Then* J_ρ *is* $\mathcal{T}-$*upper semicontinuous and* J'_ρ *is* $\mathcal{T}-$*to-weak* continuous.*

Proof. Assume that $u_n \xrightarrow{\mathcal{T}} u$. Let $t\in\mathbb{R}$ such that

$$J_\rho(u_n) = \frac{1}{2}(\|u_n^+\|^2 - \|u_n^-\|^2) - \frac{1}{2}\int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|u_n|^2\,\mathrm{d}\mu - \int_{\mathbb{V}} F(x,u_n)\,\mathrm{d}\mu \geq t.$$

It is clear that $\|u_n^+\|$ is bounded; Since $\|u_n^-\|^2 \leq \|u_n^+\|^2 - 2t$, $\|u_n^-\|$ is bounded and hence $\|u_n\|$ is bounded. Passing to a subsequence if necessary,

$$u_n \rightharpoonup u, \quad \text{in } X, \qquad \text{and} \qquad u_n \to u, \quad \text{pointwise in } \mathbb{V}.$$

(i) By (8), the weak lower semicontinuity of $\|\cdot\|$ and the Fatou lemma, we obtain that

$$J_\rho(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{1}{2}\int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|u|^2\,\mathrm{d}\mu - \int_{\mathbb{V}} F(x,u)\,\mathrm{d}\mu \geq t.$$

(ii) It is sufficient to show that for any $\phi\in C_c(\mathbb{V})$, $\lim_{n\to+\infty}\langle J'_\rho(u_n),\phi\rangle = \langle J'_\rho(u),\phi\rangle$.

Assume that $\mathrm{supp}(\phi)\subset B_r$ with $r\geq 1$. Since B_{r+1} is a finite set in $\mathbb{V}$, $u_n\to u$ pointwise in $\mathbb{V}$ as $n\to+\infty$ and the assumption (F2), we get that

$$\begin{aligned}\langle J'_\rho(u_n),\phi\rangle - \langle J'_\rho(u),\phi\rangle &= \frac{1}{2}\sum_{x\in B_{r+1}}\sum_{y\sim x}[(u_n-u)(y)-(u_n-u)(x)](\phi(y)-\phi(x))\\ &\quad + \sum_{x\in B_r}(V(x) - \frac{\rho}{(|x|^2+1)})(u_n-u)(x)\phi(x)\\ &\quad - \sum_{x\in B_r}(f(x,u_n)-f(x,u))\phi(x)\\ &\to 0, \qquad n\to+\infty.\end{aligned}$$

□

Then, for $0\leq\rho<\frac{\tilde{\rho}^+}{\kappa}$, we prove that J_ρ satisfies (A3)-(A4).

Lemma 3.3. *Let* $0\leq\rho<\frac{\tilde{\rho}^+}{\kappa}$. *Then for any* $u_0\in X\setminus X^-$, *there exist* $R(u_0) > r > 0$ *such that*

$$m := \inf_{u\in X^+\colon \|u\|=r} J_\rho(u) > J_\rho(0) = 0 \geq \sup_{\partial M(u_0)} J_\rho(u),$$

where $M(u_0) = \{u = tu_0 + v \in X\colon v\in X^-,\ \|u\|\leq R(u_0),\ t\geq 0\} \subset \mathbb{R}^+u_0 \oplus X^-$.

Proof. For $u \in X^+$, by Lemmas 2.2 and 2.3, we get that

$$J_\rho(u) \geq \frac{1}{4}(\tilde{\rho}^+ - \kappa\rho)\|u\|^2 - \int_{\mathbb{V}} F(x,u)\,\mathrm{d}\mu$$
$$\geq \frac{1}{4}(\tilde{\rho}^+ - \kappa\rho)\|u\|^2 - \varepsilon\|u\|_2^2 - C_\varepsilon\|u\|_p^p.$$

Note that $\|\cdot\|$ is equivalent to $\|\cdot\|_2$ on X^+ and $\|u\|_p \leq \|u\|_2$ for $p > 2$. Hence, for $\varepsilon > 0$ small enough, there exists $r > 0$ small enough such that

$$m := \inf_{u\in X^+ :\, \|u\|=r} J_\rho(u) > J_\rho(0) = 0.$$

Now we prove that $\sup_{\partial M(u_0)} J_\rho(u) \leq 0$. For $u_0 \in X\backslash X^-$, since $\mathbb{R}^+u_0 \oplus X^- = \mathbb{R}^+u_0^+ \oplus X^-$, we may assume that $u_0 \in X^+$. Arguing indirectly, assume that for some sequence $\{u_n\} \subset \mathbb{R}^+u_0 \oplus X^-$ with $\|u_n\| \to +\infty$ such that $J_\rho(u_n) > 0$. Let $z_n = \frac{u_n}{\|u_n\|} = s_nu_0 + z_n^-$, then $\|s_nu_0 + z_n^-\| = 1$. Passing to a subsequence, we assume that $s_n \to s$, $z_n^- \rightharpoonup z^-$ and $z_n^- \to z^-$ pointwise in $\mathbb{V}$. Hence,

$$\begin{aligned} 0 &< \frac{J_\rho(u_n)}{\|u_n\|^2} \\ &= \frac{1}{2}(s_n^2\|u_0\|^2 - \|z_n^-\|^2 - \int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|z_n|^2\,\mathrm{d}\mu) - \int_{\mathbb{V}} \frac{F(x,u_n)}{|u_n|^2}z_n^2\,\mathrm{d}\mu \\ &\leq \frac{1}{2}(s_n^2\|u_0\|^2 - \|z_n^-\|^2) - \int_{\mathbb{V}} \frac{F(x,u_n)}{|u_n|^2}z_n^2\,\mathrm{d}\mu. \end{aligned} \tag{9}$$

If $s = 0$, then it follows from (9) that

$$0 \leq \frac{1}{2}\|z_n^-\|^2 + \int_{\mathbb{V}} \frac{F(x,u_n)}{|u_n|^2}z_n^2\,\mathrm{d}\mu \leq \frac{1}{2}s_n^2\|u_0\|^2 \to 0,$$

which yields that $\|z_n^-\| \to 0$, and hence $1 = \|s_nu_0 + z_n^-\|^2 \to 0$. This is a contradiction.

If $s \neq 0$, since $\|u_n\| \to +\infty$, by (9) and (F4), we get that

$$\begin{aligned} 0 &\leq \limsup_{n\to+\infty}\,[\frac{1}{2}s_n^2\|u_0\|^2 - \int_{\mathbb{V}} \frac{F(x,u_n)}{|u_n|^2}z_n^2\,\mathrm{d}\mu] \\ &\leq \frac{1}{2}s^2\|u_0\|^2 - \liminf_{n\to+\infty} \int_{\mathbb{V}} \frac{F(x,u_n)}{|u_n|^2}z_n^2\,\mathrm{d}\mu \\ &\leq \frac{1}{2}s^2\|u_0\|^2 - \int_{\mathbb{V}} \liminf_{n\to+\infty} \frac{F(x,u_n)}{|u_n|^2}z_n^2\,\mathrm{d}\mu \\ &\to -\infty. \end{aligned}$$

This is impossible. Hence we complete the proof. □

Lemma 3.3 implies that the Nehari Manifold $N_\rho \neq \emptyset$.

Corollary 3.4. *If* $0 \leq \rho < \frac{\tilde{\rho}^+}{\kappa}$, *then for any* $u_0 \in X\backslash X^-$, *there exist* $t > 0$ *and* $v \in X^-$ *such that* $tu_0 + v \in N_\rho$.

Proof. For $u_0 \in X \setminus X^-$, since $\mathbb{R}^+ u_0 \oplus X^- = \mathbb{R}^+ u_0^+ \oplus X^-$, we may assume that $u_0 \in X^+$, then $tu_0 \in X^+$. Consider a map $\xi : \mathbb{R}^+ \times X^- \to \mathbb{R}$ in the form

$$\xi(t, v) = -J_\rho(tu_0 + v),$$

where

$$J_\rho(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|u|^2 \, \mathrm{d}\mu - \int_{\mathbb{V}} F(x, u) \, \mathrm{d}\mu.$$

Observe that ξ is bounded from below, coercive and weakly lower semi-continuous for $\rho \geq 0$. Hence there exist $t \geq 0$ and $v \in X^-$ such that $J_\rho(tu_0 + v) = \sup_{\mathbb{R}^+ u_0 \oplus X^-} J_\rho(u)$. By Lemma 3.3, one gets that $t > 0$, and hence $tu_0 + v \in N_\rho$.

□

The following lemma implies the condition (A5).

Lemma 3.5. *Let $\rho \geq 0$. For any $u \in X \setminus X^-$,*

$$J_\rho(u) \geq J_\rho(tu + v) - \langle J_\rho'(u), (\frac{t^2-1}{2}u + tv)\rangle, \qquad t \geq 0, \quad v \in X^-.$$

Proof. For $u \in X \setminus X^-$, $v \in X^-$ and $t \in [0, +\infty)$, we have that $tu + v = tu^+ + (tu^- + v)$, where $tu^+ \in X^+$ and $(tu^- + v) \in X^-$. Direct calculation yields that

$$\begin{aligned} & J_\rho(tu+v) - J_\rho(u) - \left\langle J_\rho'(u), \left(\frac{t^2-1}{2}u + tv\right)\right\rangle \\ & = -\frac{1}{2}\|v\|^2 - \frac{1}{2}\int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|v|^2 \, \mathrm{d}\mu + \int_{\mathbb{V}} \varphi(t,x) \, \mathrm{d}\mu \\ & \leq \int_{\mathbb{V}} \varphi(t,x) \, \mathrm{d}\mu, \end{aligned}$$

where $\varphi(t,x) := (\frac{t^2-1}{2}u + tv)f(x,u) + F(x,u) - F(x, tu+v)$. We only need to prove, for any $x \in \mathbb{V}$, that

$$F(x,u) - F(x, tu+v) \leq -\left(\frac{t^2-1}{2}u + tv\right) f(x,u), \quad t \geq 0, \quad v \in \mathbb{R}, \tag{10}$$

since this implies that $\varphi(t,x) \leq 0$ for $t \geq 0$ and $x \in \mathbb{V}$.

Now we prove (10). In fact, for any $x \in \mathbb{V}$ and $u \neq 0$, the condition (F5) implies that

$$f(x,s) \geq \frac{f(x,u)}{|u|}|s|, \quad s \geq u. \tag{11}$$

To show (10), without loss of generality, we assume that $u \leq tu + v$. Note that

$$F(x, tu+v) - F(x,u) = \int_u^{tu+v} f(x,s) \, \mathrm{d}s,$$

if $0 < u \le tu+v$ or $u \le tu+v \le 0$, by (3) and (11),

$$\int_u^{tu+v} f(x,s)\,\mathrm{d}s \ge \frac{f(x,u)}{|u|}\int_u^{tu+v} |s|\,ds \ge (\frac{t^2-1}{2}u+tv)f(x,u);$$

if $u < 0 \le tu+v$, by (3) and (11),

$$\int_u^{tu+v} f(x,s)\,\mathrm{d}s \ge \int_u^0 f(x,s)\,\mathrm{d}s \ge \frac{f(x,u)}{|u|}\int_u^0 |s|\,\mathrm{d}s \ge (\frac{t^2-1}{2}u+tv)f(x,u).$$

Hence (10) holds. The proof is completed. □

4. The Behavior of Cerami Sequences

In this section, we study the behavior of Cerami sequences, which are useful in the proof of Theorems 1.1 and 1.3.

Lemma 4.1. *Let $\{\rho_n\} \subset [0,+\infty)$, $\rho_n \le \rho < \frac{\tilde{\rho}^+}{\kappa}$. If $\{u_n\} \subset X\backslash X^-$ satisfies $(1+\|u_n\|)J'_{\rho_n}(u_n) \to 0$ and $J_{\rho_n}(u_n)$ is bounded from above, then $\{u_n\}$ is bounded. In particular, any Cerami sequence of J_ρ at level $c \ge 0$ is bounded for $0 \le \rho < \frac{\tilde{\rho}^+}{\kappa}$.*

Proof. Let $J_{\rho_n}(u_n) \le M$. Suppose that $\|u_n\| \to +\infty$ as $n \to +\infty$. Let $v_n := \frac{u_n}{\|u_n\|}$, then up to a subsequence, we have that

$$v_n \rightharpoonup v, \quad \text{in } X, \quad \text{and} \quad v_n \to v, \quad \text{pointwise in } \mathbb{V}.$$

We first claim that $\{v_n^+\}$ does not converge to 0 in $\ell^q(\mathbb{V})$ with $q>2$. In fact, by contradiction, we assume that $v_n^+ \to 0$ in $\ell^q(\mathbb{V})$. Then it follows from Lemma 2.2 that, for any $s>0$,

$$\int_{\mathbb{V}} F(x, sv_n^+)\,\mathrm{d}\mu \to 0.$$

Moreover, by (3) and the fact that $\langle J'_{\rho_n}(u_n), u_n\rangle \to 0$ as $n \to +\infty$, we have that

$$\|u_n^+\|^2 - \|u_n^-\|^2 \ge \langle J'_{\rho_n}(u_n), u_n\rangle,$$

and hence

$$2\|u_n^+\|^2 \ge \|u_n^+\|^2 + \|u_n^-\|^2 + \langle J'_{\rho_n}(u_n), u_n\rangle = \|u_n\|^2 + \langle J'_{\rho_n}(u_n), u_n\rangle.$$

Since $v_n^+ = \frac{u_n^+}{\|u_n\|}$, passing to a subsequence if necessary, one gets thatpg $\liminf_{n\to\infty} \|v_n^+\|^2 = C > 0$. As a consequence, by Lemmas 2.3 and 3.5,

$$\begin{aligned} M &\ge \limsup_{n\to\infty} J_{\rho_n}(u_n) \ge \limsup_{n\to\infty} J_{\rho_n}(sv_n^+) \\ &= \frac{s^2}{2}\limsup_{n\to\infty} \|v_n^+\|^2_{\rho_n} \\ &\ge \frac{s^2}{4}(\tilde{\rho}^+ - \rho\kappa)\limsup_{n\to\infty}\|v_n^+\|^2 \\ &\ge \frac{s^2}{4}C(\tilde{\rho}^+ - \rho\kappa). \end{aligned} \tag{12}$$

We obtain a contradiction since s is arbitrary. We complete the claim.

Then by Lions Lemma 2.7, there exist a sequence $\{y_n\} \subset \mathbb{V}$ and a positive constant c such that $|v_n^+(y_n)| \geq c > 0$. Let $w_n(x) = v_n(x+y_n)$. Then for some $w \in X$,

$$w_n \rightharpoonup w, \qquad \text{in } X, \qquad \text{and} \qquad w_n \to w, \qquad \text{pointwise in } \mathbb{V},$$

where $|w^+(0)| \geq c > 0$. This implies that $w \neq 0$.

Denote $\tilde{u}_n(x) = u_n(x+y_n)$, then $|\tilde{u}_n(x)| = |w_n(x)|\|u_n\| \to +\infty$ since $w(x) \neq 0$. It follows from (F4) that

$$\frac{F(x, \tilde{u}_n(x))}{\|u_n\|^2} = \frac{F(x, \tilde{u}_n(x))}{|\tilde{u}_n(x)|^2}|w_n(x)|^2 \to +\infty.$$

Since $\langle J'_{\rho_n}(u_n), u_n\rangle \to 0$ as $n \to +\infty$, for n large enough,

$$\|u_n^+\|^2 - \|u_n^-\|^2 - \int_{\mathbb{V}} \frac{\rho_n}{(|x|^2+1)}|u_n|^2 \,\mathrm{d}\mu \geq 0.$$

This implies that $0 \leq \frac{1}{\|u_n\|^2}\int_{\mathbb{V}} \frac{\rho_n}{(|x|^2+1)}|u_n|^2 \,\mathrm{d}\mu \leq 1$ for n large enough. Therefore by the periodicity of F in $x \in \mathbb{V}$ and the Fatou lemma,

$$\begin{aligned} 0 = \limsup_{n\to\infty} \frac{J_{\rho_n}(u_n)}{\|u_n\|^2} &= \limsup_{n\to\infty} \left[\frac{1}{2}\left(\|v_n^+\|^2 - \|v_n^-\|^2 - \frac{1}{\|u_n\|^2}\int_{\mathbb{V}} \frac{\rho_n}{(|x|^2+1)}|u_n|^2 \,\mathrm{d}\mu\right)\right. \\ &\quad \left. - \int_{\mathbb{V}} \frac{F(x, \tilde{u}_n(x))}{\|u_n\|^2} \,\mathrm{d}\mu\right] \\ &= -\infty. \end{aligned}$$

We get a contradiction. □

Lemma 4.2. *Let $0 \leq \rho < \frac{\tilde{\rho}^+}{\kappa}$. Assume that $\{u_n\} \subset X$ is a bounded Palais–Smale sequence of the functional J_ρ at level $c_\rho \geq 0$, that is, $J'_\rho(u_n) \to 0$ and $J_\rho(u_n) \to c_\rho$. Passing to a subsequence if necessary, there exists some $u \in X$ such that*

(i) $\lim_{n\to+\infty} J_\rho(u_n - u) = c_\rho - J_\rho(u)$;

(ii) $\lim_{n\to+\infty} J'_\rho(u_n - u) = 0, \qquad$ *in X.*

Proof. Since $\{u_n\}$ is bounded in X, we assume that for some $u \in X$,

$$u_n \rightharpoonup u, \qquad \text{in } X, \qquad \text{and} \qquad u_n \to u, \qquad \text{pointwise in } \mathbb{V}.$$

(i) By the Brézis–Lieb lemma 2.5, we obtain that

$$\|u_n^+\|^2 - \|u_n^+ - u^+\|^2 = \|\bar{u}^+\|^2 + o(1), \qquad \|u_n^-\|^2 - \|u_n^- - u^-\|^2 = \|u^-\|^2 + o(1), \tag{13}$$

$$\int_{\mathbb{V}} \frac{|u_n|^2}{(|x|^2+1)} \,\mathrm{d}\mu - \int_{\mathbb{V}} \frac{|u_n - u|^2}{(|x|^2+1)} \,\mathrm{d}\mu = \int_{\mathbb{V}} \frac{|u|^2}{(|x|^2+1)} \,\mathrm{d}\mu + o(1). \tag{14}$$

We claim that

$$\int_{\mathbb{V}} F(x, u_n) \,\mathrm{d}\mu = \int_{\mathbb{V}} F(x, u_n - u) \,\mathrm{d}\mu + \int_{\mathbb{V}} F(x, u) \,\mathrm{d}\mu + o(1). \tag{15}$$

In fact, direct calculation yields that

$$\begin{aligned}\int_{\mathbb{V}} F(x,u_n)\,\mathrm{d}\mu - \int_{\mathbb{V}} F(x,u_n-u)\,\mathrm{d}\mu &= -\int_{\mathbb{V}}\int_0^1 \frac{\mathrm{d}}{d\theta}F(x,u_n-\theta u)\,\mathrm{d}\theta\,\mathrm{d}\mu\\ &= \int_0^1\int_{\mathbb{V}} f(x,u_n-\theta u)u\,\mathrm{d}\mu\,\mathrm{d}\theta.\end{aligned}$$

Since $\{u_n-\theta u\}$ is bounded in X, by (2), we obtain that the sequence $\{f(x,u_n-\theta u)u\}$ is uniformly summable and tight over $\mathbb{V}$, that is, for any $\varepsilon>0$, there is a $\delta>0$ such that, for any $\Omega\subset\mathbb{V}$ with the measure $\mu(\Omega)<\delta$,

$$\int_{\Omega} |f(x,u_n-\theta u)u|\,\mathrm{d}\mu<\varepsilon$$

with any $n\in\mathbb{N}$; and there exists Ω_0 with $\mu(\Omega_0)<+\infty$ such that, for any $n\in\mathbb{N}$,

$$\int_{\mathbb{V}\setminus\Omega_0} |f(x,u_n-\theta u)u|\,\mathrm{d}\mu<\varepsilon.$$

Note that

$$f(x,u_n-\theta u)u\to f(x,u-\theta u)u, \qquad \text{pointwise in } \mathbb{V},$$

by the Vitali convergence theorem, we get that $f(x,u-\theta u)u$ is summable and

$$\int_{\mathbb{V}} f(x,u_n-\theta u)u\,\mathrm{d}\mu\to\int_{\mathbb{V}} f(x,u-\theta u)u\,\mathrm{d}\mu, \quad n\to+\infty.$$

Then as $n\to+\infty$, we get that

$$\int_{\mathbb{V}} F(x,u_n)\,\mathrm{d}\mu-\int_{\mathbb{V}} F(x,u_n-u)\,\mathrm{d}\mu\to\int_0^1\int_{\mathbb{V}} f(x,u-\theta u)u\,\mathrm{d}\mu\,\mathrm{d}\theta=\int_{\mathbb{V}} F(x,u)\,\mathrm{d}\mu.$$

Hence (15) holds. Therefore, by (13)–(15), one has that

$$J_\rho(u_n)=J_\rho(u_n-u)+J_\rho(u)+o(1).$$

Note that $\lim_{n\to+\infty} J_\rho(u_n)=c_\rho$, hence

$$J_\rho(u_n-u)=c_\rho-J_\rho(u)+o(1).$$

(ii) For any $\phi\in C_c(\mathbb{V})$, assume that $\operatorname{supp}(\phi)\subset B_r$, where r is a positive constant. Since B_{r+1} is a finite set in $\mathbb{V}$ and $u_n\to u$ pointwise in $\mathbb{V}$ as $n\to+\infty$, we get that

$$\begin{aligned}|\langle J'_\rho(u_n-u),\phi\rangle| &\le \sum_{x\in B_{r+1}} |\nabla(u_n-u)||\nabla\phi|+\sum_{x\in B_r}|V(x)||u_n-u|\phi|\\ &\quad+\sum_{x\in B_r}\frac{\rho}{(|x|^2+1)}|u_n-u||\phi|\\ &\quad+\sum_{x\in B_r}|f(x,u_n-u)||\phi|\\ &\le C\xi_n\|\phi\|,\end{aligned}$$

where C is a constant not depending on n and $\xi_n \to 0$ as $n \to +\infty$. Hence

$$\lim_{n\to+\infty} \|J'_\rho(u_n - u)\|_X = \lim_{n\to+\infty} \sup_{\|\phi\|=1} |\langle J'_\rho(u_n - u), \phi\rangle| = 0.$$

□

We give a decomposition of bounded Palais–Smale sequence of J_ρ in discrete version.

Lemma 4.3. *Let $0 \leq \rho < \frac{\tilde{\rho}^+}{\kappa}$. Assume that $\{u_n\} \subset X$ is a bounded Palais–Smale sequence of J_ρ at level $c_\rho \geq 0$. Then there exist sequences $\{\bar{u}_i\}_{i=0}^k \subset X$ and $\{x_n^i\}_{0\leq i\leq k} \subset \mathbb{V}$ with $x_n^0 = 0$, $|x_n^i| \to +\infty$, $|x_n^i - x_n^j| \to +\infty$, $i \neq j$, $i, j = 1, 2, \ldots, k$, such that, up to a subsequence,*

- (i) $J'_\rho(\bar{u}_0) = 0$;
- (ii) $J'_0(\bar{u}_i) = 0$ *with* $\bar{u}_i \neq 0$ *for* $i = 1, 2, \cdots, k$;
- (iii) $u_n - \Sigma_{i=0}^k \bar{u}_i(x - x_n^i) \to 0, \qquad \|u_n\|^2 \to \Sigma_{i=0}^k \|\bar{u}_i\|^2, \qquad n \to \infty$;
- (iv) $c_\rho = J_\rho(\bar{u}_0) + \Sigma_{i=1}^k J_0(\bar{u}_i)$.

Proof. We assume that for some $\bar{u}_0 \in X$,

$$u_n \rightharpoonup \bar{u}_0, \qquad \text{in } X, \qquad \text{and} \qquad u_n \to \bar{u}_0, \qquad \text{pointwise in } \mathbb{V}.$$

Similar to the proof of (ii) in Lemma 3.2, we obtain that $J'_\rho(\bar{u}_0) = 0$ since $J'_\rho(u_n) \to 0$ as $n \to +\infty$.

Let $v_n(x) = u_n(x) - \bar{u}_0(x)$. Then, we have that

$$v_n \rightharpoonup 0, \qquad \text{in } X, \qquad \text{and} \qquad v_n \to 0, \qquad \text{poinwise in}\mathbb{V}.$$

By Lemma 4.2, one has that

$$\begin{aligned} J_\rho(v_n) &= c_\rho - J_\rho(\bar{u}_0) + o(1), \\ J'_\rho(v_n) &= o(1), \quad \text{in } X. \end{aligned} \tag{16}$$

For $\{v_n\}$, we discuss two cases:

Case 1 $\limsup_{n\to+\infty} \|v_n\|_\infty = 0$. By the boundedness of $\{v_n\}$ in X and Lemma 2.7, we have that $\|v_n\|_t \to 0$ as $n \to +\infty$ for $t > 2$. By Lemma 2.8, for $t > 2$, we have that

$$v_n^+ \to 0, \qquad v_n^- \to 0, \qquad \text{in } \ell^t(\mathbb{V}). \tag{17}$$

Since $J_\rho'(u_n) = o(1)$, $J_\rho'(\bar{u}_0) = 0$, $u_n = v_n + \bar{u}_0$ and $u_n^+ = v_n^+ + \bar{u}_0^+$, we obtain that

$$\begin{aligned}
o(1) &= \langle J_\rho'(u_n), v_n^+\rangle \\
&= (u_n^+, v_n^+) - \rho \int_{\mathbb{V}} \frac{u_n v_n^+}{(|x|^2+1)}\,\mathrm{d}\mu - \int_{\mathbb{V}} f(x,u_n)v_n^+\,\mathrm{d}\mu \\
&= \|v_n^+\|^2 - \rho \int_{\mathbb{V}} \frac{v_n v_n^+}{(|x|^2+1)}\,\mathrm{d}\mu + \langle J_\rho'(\bar{u}_0), v_n^+\rangle + \int_{\mathbb{V}} f(x,\bar{u}_0)v_n^+\,\mathrm{d}\mu \\
&\qquad - \int_{\mathbb{V}} f(x,u_n)v_n^+\,\mathrm{d}\mu \\
&= \|v_n^+\|^2 - \rho \int_{\mathbb{V}} \frac{|v_n^+|^2}{(|x|^2+1)}\,\mathrm{d}\mu - \rho \int_{\mathbb{V}} \frac{v_n^- v_n^+}{(|x|^2+1)}\,\mathrm{d}\mu \\
&\quad + \int_{\mathbb{V}} f(x,u_0)v_n^+\,\mathrm{d}\mu - \int_{\mathbb{V}} f(x,u_n)v_n^+\,\mathrm{d}\mu \\
&\geq \frac{1}{2}(\tilde{\rho}^+ - \rho\kappa)\|v_n^+\|^2 - \rho \int_{\mathbb{V}} \frac{v_n^- v_n^+}{(|x|^2+1)}\,\mathrm{d}\mu + \int_{\mathbb{V}} f(x,\bar{u}_0)v_n^+\,\mathrm{d}\mu \\
&\qquad - \int_{\mathbb{V}} f(x,u_n)v_n^+\,\mathrm{d}\mu,
\end{aligned}$$

which means that

$$\begin{aligned}
\frac{1}{2}(\tilde{\rho}^+ - \rho\kappa)\|v_n^+\|^2 &\leq \rho \int_{\mathbb{V}} \frac{v_n^- v_n^+}{(|x|^2+1)}\,\mathrm{d}\mu \\
&\quad + \int_{\mathbb{V}} f(x,u_n)v_n^+\,\mathrm{d}\mu - \int_{\mathbb{V}} f(x,\bar{u}_0)v_n^+\,\mathrm{d}\mu + o(1).
\end{aligned} \tag{18}$$

Similarly, we have that

$$\begin{aligned}
o(1) &= \langle J_\rho'(u_n), v_n^-\rangle \\
&= -\|v_n^-\|^2 - \rho \int_{\mathbb{V}} \frac{|v_n^-|^2}{(|x|^2+1)}\,\mathrm{d}\mu - \rho \int_{\mathbb{V}} \frac{v_n^- v_n^+}{(|x|^2+1)}\,\mathrm{d}\mu \\
&\quad + \int_{\mathbb{V}} f(x,\bar{u}_0)v_n^-\,\mathrm{d}\mu - \int_{\mathbb{V}} f(x,u_n)v_n^-\,\mathrm{d}\mu \\
&\leq -\|v_n^-\|^2 - \rho \int_{\mathbb{V}} \frac{v_n^- v_n^+}{(|x|^2+1)}\,\mathrm{d}\mu + \int_{\mathbb{V}} f(x,\bar{u}_0)v_n^-\,\mathrm{d}\mu - \int_{\mathbb{V}} f(x,u_n)v_n^-\,\mathrm{d}\mu,
\end{aligned}$$

which yields that

$$\|v_n^-\|^2 \leq -\rho \int_{\mathbb{V}} \frac{v_n^- v_n^+}{(|x|^2+1)}\,\mathrm{d}\mu + \int_{\mathbb{V}} f(x,\bar{u}_0)v_n^-\,\mathrm{d}\mu - \int_{\mathbb{V}} f(x,u_n)v_n^-\,\mathrm{d}\mu + o(1). \tag{19}$$

Then it follows from (17)–(19) that

$$\begin{aligned}
\frac{1}{2}(\tilde{\rho}^+ - \rho\kappa)\|v_n\|^2 \leq\ & \int_{\mathbb{V}} f(x,\bar{u}_0)v_n^-\,\mathrm{d}\mu - \int_{\mathbb{V}} f(x,u_n)v_n^-\,\mathrm{d}\mu + \int_{\mathbb{V}} f(x,u_n)v_n^+\,\mathrm{d}\mu \\
& - \int_{\mathbb{V}} f(x,\bar{u}_0)v_n^+\,\mathrm{d}\mu + o(1) \\
\to\ & 0,
\end{aligned}$$

that is $v_n \to 0$ in X. Then the proof ends with $k=0$.

Case 2 $\liminf_{n\to+\infty} \|v_n\|_\infty = \delta > 0$. Then there exists a sequence $\{x_n^1\} \subset \mathbb{V}$ such that $|v_n(x_n^1)| \geq \frac{\delta}{2} > 0$. Denote $u_{n,1}(x) = v_n(x + x_n^1)$ and assume that, for some $\bar{u}_1 \in X$,

$$u_{n,1} \rightharpoonup \bar{u}_1, \qquad \text{in } X, \qquad \text{and} \qquad u_{n,1} \to \bar{u}_1, \qquad \text{pointwise in } \mathbb{V},$$

where $|\bar{u}_1(0)| \geq \frac{\delta}{2} > 0$. Since $v_n \to 0$ pointwise in $\mathbb{V}$, one gets easily that $|x_n^1| \to +\infty$ as $n \to +\infty$.

For any $\phi \in X$, by the Hölder inequality, Lemma 2.4 and (5), we get that

$$\int_{\mathbb{V}} \frac{1}{(|x|^2+1)} v_n(x)\phi(x - x_n^1)\,\mathrm{d}\mu \to 0, \qquad \text{as } n \to +\infty.$$

Therefore, by the periodicity of f in $x \in \mathbb{V}$, we obtain that

$$\begin{aligned} o(1) &= \langle J_\rho'(v_n), \phi(x - x_n^1)\rangle \\ &= (v_n^+, \phi(x-x_n^1)) - (v_n^-, \phi(x-x_n^1)) - \int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)} v_n \phi(x-x_n^1)\,\mathrm{d}\mu \\ &\quad - \int_{\mathbb{V}} f(x, v_n)\phi(x-x_n^1)\,\mathrm{d}\mu \\ &= (u_{n,1}^+, \phi) - (u_{n,1}^-, \phi) - \int_{\mathbb{V}} f(x, u_{n,1})\phi\,\mathrm{d}\mu + o(1) \\ &= \langle J_0'(u_{n,1}), \phi\rangle + o(1). \end{aligned} \tag{20}$$

This means that $\langle J_0'(\bar{u}_1), \phi\rangle = 0$ and $\bar{u}_1$ is a nontrivial critical point of J_0.

Let

$$z_n(x) = u_n(x) - \bar{u}_0(x) - \bar{u}_1(x - x_n^1). \tag{21}$$

Then we have that

$$z_n \rightharpoonup 0, \qquad \text{in } X, \qquad \text{and} \qquad z_n \to 0, \qquad \text{pointwise in } \mathbb{V}.$$

Observe that $v_n(x) = \bar{u}_1(x - x_n^1) + z_n(x)$, by (13) and the Brézis–Lieb lemma,

$$\begin{aligned} \|u_n^+\|^2 &= \|\bar{u}_0^+\|^2 + \|\bar{u}_1^+\|^2 + \|z_n^+\|^2 + o(1), \\ \|u_n^-\|^2 &= \|\bar{u}_0^-\|^2 + \|\bar{u}_1^-\|^2 + \|z_n^-\|^2 + o(1). \end{aligned}$$

Then one has that

$$\|u_n\|^2 = \|\bar{u}_0\|^2 + \|\bar{u}_1\|^2 + \|z_n\|^2 + o(1).$$

By (16), one sees that $\{v_n\}$ is a Palais–Smale sequence of J_ρ at level $c_\rho - J_\rho(\bar{u}_0)$. Then it follows from Lemma 4.2 that

$$\begin{aligned} &\lim_{n\to+\infty} J_\rho(z_n) = \lim_{n\to+\infty} (c_\rho - J_\rho(\bar{u}_0) - J_\rho(\bar{u}_1(x - x_n^1))), \\ &\lim_{n\to+\infty} J_\rho'(z_n) = 0, \qquad \text{in } X. \end{aligned}$$

Note that $\lim_{n\to+\infty}|x_n^1| = +\infty$, by the invariance of $\|\cdot\|$ with respect to translations, Lemma 2.4 and the periodicity of F in $x \in \mathbb{V}$, we have that

$$\begin{aligned}\lim_{n\to+\infty} J_\rho(\bar{u}_1(x-x_n^1)) &= \lim_{n\to+\infty}[\frac{1}{2}(\|\bar{u}_1^+(x-x_n^1)\|^2 - \|\bar{u}_1^-(x-x_n^1)\|^2 \\ &\quad - \int_{\mathbb{V}} \frac{\rho}{(|x|^2+1)}|\bar{u}_1(x-x_n^1)|^2 \,\mathrm{d}\mu) \\ &\quad - \int_{\mathbb{V}} F(x,\bar{u}_1(x-x_n^1))\,\mathrm{d}\mu] \\ &= \frac{1}{2}(\|\bar{u}_1^+\|^2 - \|\bar{u}_1^-\|^2) - \int_{\mathbb{V}} F(x,\bar{u}_1)\,\mathrm{d}\mu \\ &= J_0(\bar{u}_1).\end{aligned}$$

Hence we obtain that

$$\begin{aligned}&J_\rho(z_n) = c_\rho - J_\rho(\bar{u}_0) - J_0(\bar{u}_1) + o(1),\\ &J_\rho'(z_n) = o(1), \qquad \text{in } X.\end{aligned}$$

This implies that $\{z_n\}$ is a Palais–Smale sequence of J_ρ at level $c_\rho - J_\rho(\bar{u}_0) - J_0(\bar{u}_1)$.

For $\{z_n\}$, if the vanishing case occurs for $\|z_n\|_\infty$, by similar arguments as in **Case 1**, we obtain that $z_n \to 0$ in X, and the proof ends with $k=1$.

If the non-vanishing occurs for $\|z_n\|_\infty$, by analogous discussions as in **Case 2**, there exists a sequence $\{x_n^2\} \subset \mathbb{V}$ such that $|z_n(x_n^2)| \geq \frac{\delta}{2} > 0$. If we denote $u_{n,2}(x) = z_n(x+x_n^2)$ and assume that

$$u_{n,2} \rightharpoonup \bar{u}_2, \qquad \text{in } X, \qquad \text{and} \qquad u_{n,2} \to \bar{u}_2, \qquad \text{pointwise in } \mathbb{V}.$$

Then one gets that $|\bar{u}_2(0)| > 0$. This implies that $|x_n^2| \to +\infty$. In fact, we also have that $|x_n^2 - x_n^1| \to +\infty$ as $n \to +\infty$. By contradiction, assume that $\{x_n^2 - x_n^1\}$ is bounded in $\mathbb{V}$. Thus there exists a point $x_0 \in \mathbb{V}$ such that $(x_n^2 - x_n^1) \to x_0$ as $n \to +\infty$. For $x_0 \in \mathbb{V}$, it follows from (21) that

$$u_{n,2}(x_0 + x_n^1 - x_n^2) = u_{n,1}(x_0) - \bar{u}_1(x_0).$$

Since $(x_n^2 - x_n^1) \to x_0$ as $n \to +\infty$ and $|\bar{u}_2(0)| > 0$, one sees that $u_{n(2)}(x_0 - x_n^2 + x_n^1) \not\to 0$ in the left hand side of the above equality. While the right hand side tends to zero as $n \to +\infty$. We get a contradiction.

Since $\langle J_\rho'(z_n), \phi(x - x_n^2)\rangle = o(1)$, similar arguments to (20), we can prove that $\langle J_0'(\bar{u}_2), \phi\rangle = 0$, and hence $\bar{u}_2$ is a nontrivial critical point of J_0.

Let

$$w_n(x) = u_n(x) - \bar{u}_0(x) - \bar{u}_1(x - x_n^1) - \bar{u}_2(x - x_n^2).$$

Then we have that

$$w_n \rightharpoonup 0, \qquad \text{in } X, \qquad \text{and} \qquad w_n \to 0, \qquad \text{pointwise in } \mathbb{V}.$$

Note that $z_n(x) = \bar{u}_2(x - x_n^2) + w_n(x)$, similarly, we have that

$$\begin{aligned}&\|u_n\|^2 = \|\bar{u}_0\|^2 + \|\bar{u}_1\|^2 + \|\bar{u}_2\|^2 + \|w_n\|^2 + o(1),\\ &J_\rho(w_n) = c_\rho - J_\rho(\bar{u}_0) - J_0(\bar{u}_1) - J_0(\bar{u}_2) + o(1),\\ &J_\rho'(w_n) = o(1).\end{aligned}$$

We repeat the process again. We claim that the iterations must stop after finite steps.

We only need to prove that, for any $u \neq 0$ with $J_0'(u) = 0$, there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\|u\| \geq \varepsilon_1$ and $J_0(u) \geq \varepsilon_2$.

In fact, for $u \neq 0$ satisfying $\langle J_0'(u), u^+\rangle = 0$, by (F2), (F3) and the fact $\|u\|_p \leq \|u\|_2 \leq C\|u\|$ for $p > 2$, we have that for any $\varepsilon > 0$, there is a constant $C_1 > 0$ such that

$$\|u^+\|^2 = \int_{\mathbb{V}} f(x,u)u^+ \,\mathrm{d}\mu \leq \varepsilon\|u^+\|\|u\| + C_1\|u^+\|\|u\|^{p-1}.$$

Analogously, for $\langle J_0'(u), u^-\rangle = 0$, we get a constant $C_2 > 0$ such that

$$\|u^-\|^2 = -\int_{\mathbb{V}} f(x,u)u^- \,\mathrm{d}\mu \leq \varepsilon\|u^-\|\|u\| + C_2\|u^-\|\|u\|^{p-1}. \tag{22}$$

The above two inequalities yield that

$$\|u\|^2 \leq 2\varepsilon\|u\|^2 + 2\max\{C_1, C_2\}\|u\|^p.$$

Let ε be small enough, then there exists $\varepsilon_1 > 0$ such that $\|u\| \geq \varepsilon_1$.

By Lemma 2.2 and (22), we get that

$$\begin{aligned} J_0(u) = &\ \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{V}} F(x,u)\,\mathrm{d}\mu \\ = &\ \frac{1}{2}\|u\|^2 - \|u^-\|^2 - \int_{\mathbb{V}} F(x,u)\,\mathrm{d}\mu \\ \geq &\ \frac{1}{2}\|u\|^2 - \varepsilon\|u\|^2 - C\|u\|^p - \varepsilon\|u\|^2 - C\|u\|^p \\ = &\ \frac{1}{2}\|u\|^2 - \varepsilon\|u\|^2 - C\|u\|^p. \end{aligned}$$

Since $p > 2$, there exists $\varepsilon_2 > 0$ small enough such that $J_0(u) \geq \varepsilon_2 > 0$.

The proof is completed. □

5. Proofs of Theorems 1.1 and 1.3

In this section, we are devoted to prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. It follows from Theorem 3.1 and Lemma 4.1 that there exists a bounded Cerami sequence $\{u_n\}$ of J_ρ at level $c_\rho > 0$ in X. If $\rho = 0$, by Theorem 3.1, we obtain that

$$\inf_{N_0} J_0 \geq c_0 > 0.$$

Lemma 4.3 implies that there is a nontrivial critical point $u_0 \in N_0$ of J_0 such that $J_0(u_0) = c_0$. Hence u_0 is a ground state solution of J_0, i.e. $J_0(u_0) = \inf_{N_0} J_0$. In the following, we assume that $0 < \rho < \frac{\tilde{\rho}^+}{\kappa}$ and consider

$$M(u_0) = \{u = tu_0 + v \in X | v \in X^-, \|u\| \leq R(u_0), t \geq 0\} \subset \mathbb{R}^+ u_0^+ \oplus X^-.$$

For $u_n = t_n u_0 + v_n \in M(u_0)$, let $u_n \rightharpoonup u = t_0 u_0 + v_0$ in X. Passing to a subsequence if necessary, we may assume that

$$t_n \to t_0, \quad \text{in } \mathbb{R}^+, \quad v_n \rightharpoonup v_0, \quad \text{in } X^-, \quad v_n \to v_0, \quad \text{pointwise in} \quad \mathbb{V}. \tag{23}$$

Then we have that $u \in M(u_0)$, which implies that $M(u_0)$ is weakly closed. By (23) and the Fatou lemma, we can prove that J_ρ is weakly upper semi-continuous. Then J_ρ attains its maximum in $M(u_0)$, namely, there exists $t_0u_0 + v_0 \in M(u_0)$ such that

$$J_\rho(t_0u_0 + v_0) \geq J_\rho(w)$$

for any $w \in M(u_0)$. By Lemma 3.3, we have that $J_\rho(t_0u_0 + v_0) > 0$ and $t_0u_0 + v_0 \neq 0$. Define $h(u, s) = u$ for $u \in M(u_0)$ and $s \in [0, 1]$. Note that (h1)-(h4) in Theorem 3.1 are satisfied, that is $h \in \Gamma(u_0)$. Then by Theorem 3.1 and Lemma 3.5, we have that

$$c_0 = J_0(u_0) \geq J_0(t_0u_0 + v_0) > J_\rho(t_0u_0 + v_0) = \max_{u \in M(u_0)} J_\rho(h(u, 1)) \geq c_\rho. \tag{24}$$

Then it follows from Lemma 4.3 that $k = 0$ and $J_\rho(\bar{u}_0) = c_\rho > 0$, that is $\bar{u}_0$ is a nontrivial critical point of J_ρ. By Theorem 3.1 again, we get that $c_\rho = \inf_{N_\rho} J_\rho$.

In order to prove Theorem 1.3, we first prove a crucial lemma for the relation between c_ρ and c_0 as $\rho \to 0^+$.

Lemma 5.1. *Let $0 \leq \rho < \frac{\tilde{\rho}^+}{\kappa}$. Assume that (H) and (F1)–(F5) hold. If u_ρ and u_0 are the ground state solutions of J_ρ and J_0, then we have that*

$$\lim_{\rho \to 0^+} c_\rho = c_0.$$

Proof. Let $u_0 \in N_0$ be a ground state solution of J_0. By Corollary 3.4, there exist $t' > 0$ and $v' \in X^-$ such that $t'u_0 + v' \in N_\rho$. Then it follows from Lemma 3.5 that

$$\begin{aligned} c_0 = J_0(u_0) \geq J_0(t'u_0 + v') &= J_\rho(t'u_0 + v') + \frac{1}{2}\int_{\mathbb{V}} \frac{\rho}{(|x|^2 + 1)}|t'u_0 + v'|^2 \,\mathrm{d}\mu \\ &\geq c_\rho + \frac{1}{2}\int_{\mathbb{V}} \frac{\rho}{(|x|^2 + 1)}|t'u_0 + v'|^2 \,\mathrm{d}\mu, \end{aligned}$$

which implies that

$$c_0 \geq c_\rho. \tag{25}$$

Let $u_\rho \in N_\rho$ be a ground state solution of J_ρ. Similarly, there exist $t > 0$ and $v \in X^-$ such that $tu_\rho + v \in N_0$, and hence

$$\begin{aligned} c_\rho = J_\rho(u_\rho) \geq J_\rho(tu_\rho + v) &= J_0(tu_\rho + v) - \frac{1}{2}\int_{\mathbb{V}} \frac{\rho}{(|x|^2 + 1)}|tu_\rho + v|^2 \,\mathrm{d}\mu \\ &\geq c_0 - \frac{1}{2}\int_{\mathbb{V}} \frac{\rho}{(|x|^2 + 1)}|tu_\rho + v|^2 \,\mathrm{d}\mu. \end{aligned} \tag{26}$$

We claim that as $\rho \to 0^+$,

$$\int_{\mathbb{V}} \frac{\rho}{(|x|^2 + 1)}|tu_\rho + v|^2 \,\mathrm{d}\mu \to 0. \tag{27}$$

Then the result of this lemma follows from (25)–(27).

Now we prove (27). In fact, by Theorem 3.1 and Lemma 4.1, we have that $\{u_\rho\}$ is bounded if $\rho \to 0^+$. Take any sequence $\rho_n \to 0^+$ such that $\rho_n \leq \rho < \frac{\tilde{\rho}^+}{\kappa}$ and let $u_n = u_{\rho_n}$.

We first prove that there is a sequence $\{y_n\} \subset \mathbb{V}$ such that

$$|u_n^+(y_n)| > 0.$$

Otherwise by Lemma 2.7, we get that $u_n^+ \to 0$ in $\ell^t(\mathbb{V})$ for $t > 2$. Since $u_n \in N_{\rho_n}$, by (2) and the Hölder inequality, we have that

$$\|u_n^+\|^2 = \int_{\mathbb{V}} \frac{\rho_n}{(|x|^2+1)} u_n u_n^+ \,\mathrm{d}\mu + \int_{\mathbb{V}} f(x, u_n) u_n^+ \,\mathrm{d}\mu \to 0, \quad n \to +\infty.$$

Hence $\limsup_{n\to\infty} J_{\rho_n}(u_n) \leq 0$. However, for sufficiently small $r > 0$,

$$J_{\rho_n}(u_n) \geq J_{\rho_n}\Big(\frac{r}{\|u_n^+\|} u_n^+\Big) \geq \inf_{n\in\mathbb{N}} \inf_{u\in X^+:\|u\|=r} J_{\rho_n}(u) > 0.$$

This yields a contradiction.

Then passing to a subsequence if necessary, there exists $u \in X$ with $u^+(0) \neq 0$ such that

$$u_n(x+y_n) \rightharpoonup u(x), \quad \text{in } X, \quad \text{and} \quad u_n(x+y_n) \to u(x), \quad \text{pointwise in } \mathbb{V}. \tag{28}$$

Denote $\tilde{u}_n(x) = u_n(x+y_n)$, let $t_n\tilde{u}_n + \tilde{v}_n \in N_0$ with $t_n > 0$, $\tilde{v}_n(x) = v_n(x+y_n) \in X^-$. By (3), we have that

$$\begin{aligned} \|\tilde{u}_n^+\|^2 &= \|\tilde{u}_n^- + \frac{\tilde{v}_n}{t_n}\|^2 + \frac{1}{t_n^2}\int_{\mathbb{V}} f(x, t_n\tilde{u}_n + \tilde{v}_n)(t_n\tilde{u}_n + \tilde{v}_n)\,\mathrm{d}\mu \\ &\geq \|\tilde{u}_n^- + \frac{\tilde{v}_n}{t_n}\|^2 + 2\int_{\mathbb{V}} \frac{F(x, t_n(\tilde{u}_n + \frac{\tilde{v}_n}{t_n}))}{t_n^2}\,\mathrm{d}\mu, \end{aligned} \tag{29}$$

which means that $\|\tilde{u}_n^- + \frac{\tilde{v}_n}{t_n}\|$ is bounded. We may assume that $\tilde{u}_n^-(x) + \frac{\tilde{v}_n(x)}{t_n} \to v(x)$ pointwise in $\mathbb{V}$ for some $v \in X^-$. If $t_n \to +\infty$, then $|t_n\tilde{u}_n + \bar{v}_n| = t_n|\tilde{u}_n^+ + (\tilde{u}_n^- + \frac{\tilde{v}_n}{t_n})| \to +\infty$ since $u^+(x) + v(x) \neq 0$. By the Fatou lemma and (F4), we obtain that

$$\int_{\mathbb{V}} \frac{F(x, t_n(\tilde{u}_n + \frac{\tilde{v}_n}{t_n}))}{t_n^2|\tilde{u}_n + \frac{\tilde{v}_n}{t_n}|^2} |\tilde{u}_n + \frac{\tilde{v}_n}{t_n}|^2\,\mathrm{d}\mu \to +\infty,$$

which contradicts (29). Therefore $\{t_n\}$ is bounded. As a consequence, $\|t_n\tilde{u}_n^+\|$ and $\|t_n\tilde{u}_n^- + \tilde{v}_n\|$ are bounded. Then by the Hardy inequality (5),

$$\frac{1}{2}\int_{\mathbb{V}} \frac{\rho_n}{(|x|^2+1)} |t_n\tilde{u}_n + \tilde{v}_n|^2\,\mathrm{d}\mu \to 0 \text{ as } n \to +\infty.$$

□

Proof of Theorem 1.3. Let $\{u_n\}$ be a sequence of ground state solutions of J_{ρ_n}. By similar arguments as in Lemma 5.1, we can find a sequence $\{x_n\} \subset \mathbb{V}$

such that $|u_n^+(x_n)| > 0$. Passing to a subsequence if necessary, there exists $u \in X$ with $u^+(0) \neq 0$ such that

$$u_n(x+x_n) \rightharpoonup u(x), \quad \text{in } X, \quad \text{and} \quad u_n(x+x_n) \to u(x), \quad \text{pointwise in } \mathbb{V}. \tag{30}$$

For $(x,u) \in \mathbb{V} \times \mathbb{R}$, we define

$$G(x,u) = \frac{1}{2} f(x,u)u - F(x,u).$$

Note that $\rho_n \to 0$ as $n \to \infty$. Hence for any $\phi \in X$,

$$\begin{aligned} &\langle J_0'(u_n(x+x_n)), \phi\rangle \\ &= \langle J_{\rho_n}'(u_n(x)), \phi(x-x_n)\rangle + \int_{\mathbb{V}} \frac{\rho_n}{(|x|^2+1)} u_n(x)\phi(x-x_n)\,\mathrm{d}\mu \to 0. \end{aligned}$$

Similar to the proof of (ii) in Lemma 3.2, we get that $\langle J_0'(u_n(x+x_n)), \phi\rangle \to \langle J_0'(u), \phi\rangle$. Then u is a nontrivial critical point of J_0, and hence $u \in \mathcal{N}_0$. By Lemma 5.1 and the Fatou lemma, we have that

$$\begin{aligned} c_0 &= \liminf_{n\to\infty} J_{\rho_n}(u_n) = \liminf_{n\to\infty} (J_{\rho_n}(u_n) - \frac{1}{2}\langle J_{\rho_n}'(u_n), u_n\rangle) \\ &= \liminf_{n\to\infty} \int_{\mathbb{V}} G(x,u_n)\,\mathrm{d}\mu = \liminf_{n\to\infty} \int_{\mathbb{V}} G(x, u_n(x+x_n))\,\mathrm{d}\mu \\ &\geq \int_{\mathbb{V}} G(x,u)\,\mathrm{d}\mu = J_0(u) \geq c_0. \end{aligned} \tag{31}$$

This implies that u is a ground state solution of J_0.

Let us denote $w_n(x) = u_n(x+x_n)$ and observe that

$$\begin{aligned} \int_{\mathbb{V}} G(x,w_n) - G(x,w_n-u)\,\mathrm{d}\mu &= \int_{\mathbb{V}} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} G(x, w_n - u + tu)\,\mathrm{d}t\,\mathrm{d}\mu \\ &= \int_0^1 \int_{\mathbb{V}} g(x, w_n - u + tu)u\,\mathrm{d}\mu\,\mathrm{d}t, \end{aligned}$$

where $g(x,s) = \frac{\partial}{\partial s} G(x,s)$ for $s \in \mathbb{R}$ and $x \in \mathbb{V}$. Since $\{w_n - u + tu\}$ is bounded in X, by (F6) and (2), we can prove that the family $\{g(x, w_n - u + tu)u\}$ is uniformly summable and tight over $\mathbb{V}$. In addition, note that $g(x, w_n - u + tu)u \to g(tu)u$ pointwise in $\mathbb{V}$, then by the Vitali convergence theorem, we get that $g(x,tu)u$ is summable and

$$\int_{\mathbb{V}} g(x, w_n - u + tu)u\,\mathrm{d}\mu \to \int_{\mathbb{V}} g(x,tu)u\,\mathrm{d}\mu, \quad n \to +\infty.$$

Then we have that

$$\int_{\mathbb{V}} G(x,w_n) - G(x,w_n-u)\,\mathrm{d}\mu \to \int_0^1 \int_{\mathbb{V}} g(x,tu)u\,\mathrm{d}\mu\,\mathrm{d}t = \int_{\mathbb{V}} G(x,u)\,\mathrm{d}\mu, \quad n \to +\infty.$$

Combined with (31), we get that $\lim_{n\to\infty} \int_{\mathbb{V}} G(x, w_n - u)\,\mathrm{d}\mu = 0$. By (F6), we have that $w_n \to u$ in $\ell^q(\mathbb{V})$ with $2 < q \leq p$. Note that $\{w_n\}$ is bounded in $\ell^2(\mathbb{V})$, and hence in $\ell^\infty(\mathbb{V})$. By interpolation inequality, for $p < t < +\infty$,

$$\|w_n - u\|_t^t \leq \|w_n - u\|_p^p \|w_n - u\|_\infty^{t-p} \to 0.$$

Hence, one has that $w_n \to u$ in $\ell^t(\mathbb{V})$ for $2 < t < +\infty$. Combined with $\rho_n \to 0$ as $n \to \infty$., we get that

$$\begin{aligned}\|w_n^+ - u^+\|^2 &= \langle J'_{\rho_n}(u_n), (w_n^+ - u^+)(x - x_n)\rangle - (u^+, w_n^+ - u^+)\\ &\quad + \int_{\mathbb{V}} \frac{\rho_n}{(|x|^2+1)} u_n(w_n^+ - u^+)(x - x_n)\,\mathrm{d}\mu\\ &\quad + \int_{\mathbb{V}} f(x, w_n)(w_n^+ - u^+)\,\mathrm{d}\mu\\ &\to 0,\\ \|w_n^- - u^-\|^2 &= \langle J'_{\rho_n}(u_n), (w_n^- - u^-)(x - x_n)\rangle - (u^-, w_n^- - u^-)\\ &\quad - \int_{\mathbb{V}} \frac{\rho_n}{(|x|^2+1)} u_n(w_n^- - u^-)(x - x_n)\,\mathrm{d}\mu\\ &\quad - \int_{\mathbb{V}} f(x, w_n)(w_n^- - u^-)\,\mathrm{d}\mu\\ &\to 0.\end{aligned}$$

Therefore, $\|w_n - u\|^2 = \|w_n^+ - u^+\|^2 + \|w_n^- - u^-\|^2 \to 0$, as $n \to +\infty$, which means that $w_n \to u$ in X. □

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