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The Null Boundary Controllability for a Fourth-Order Parabolic Equation with Samarskii–Ionkin-Type Boundary Conditions

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Abstract. This paper presents a moment method approach to solve the null boundary controllability problem for a fourth-order parabolic equation subject to Samarskii–Ionkin-type boundary conditions. The problem is solved in two stages. First, we demonstrate that the eigenfunctions of the system, which are not self-adjoint under these boundary conditions, form a Riesz basis in L_2 space. Using Fourier's method, we construct a biorthonormal system of functions to express the series solution. In the second stage, we use these spectral results to show that the system is null boundary controllable for a specific class of initial data. Our approach extends the existing literature on null boundary controllability of parabolic equations and provides new insights into the properties of systems subject to Samarskii–Ionkin-type boundary conditions.

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1. Introduction

Null boundary controllability for parabolic equations has been the subject of much research in recent years. A considerable amount of attention has been paid to the null boundary controllability problem with classical boundary conditions, as evidenced by numerous publications (see, for example, [1–8]). However, the literature on null boundary controllability of fourth-order linear parabolic equations remains limited. Some progress has been made in this direction. For instance, Guo [9] converted the control problem to two well-posed problems to solve the null boundary controllability problem for a fourth-order parabolic equation. Yu [10] used the method based on Lebeau–Rabbino inequality to solve the null interior controllability problem for a fourth-order parabolic equation with Dirichlet boundary conditions. Zhou [11] derived observability inequalities for a one-dimensional linear fourth-order parabolic equation with potential and obtained null controllability results for the one-dimensional fourth-order semi-linear equation. Recently, Guerrero and Kassab [12] obtained null controllability results for higher dimensional fourth-order parabolic equations.

In this paper, we focus on the null controllability problem for a fourth-order parabolic equation with Samarskii–Ionkin-type boundary conditions. To the best of our knowledge, this problem has not been studied before in the literature. The eigenfunctions of the auxiliary spectral problem of the system under these boundary conditions do not form a basis in L_2. Therefore, we first find associated eigenfunctions and complete the eigenfunctions of the system using the method developed by Ionkin in [13]. We also prove the completeness of the system using general theory. Then, using these spectral properties, we solve the null boundary controllability problem by reducing it to moment problems using the Riesz basis. Our main contribution is to provide a complete solution to this previously unstudied problem.

This paper is structured as follows: in Sect. 2, we introduce the problem and present some initial results. Section 3 is devoted to solving the auxiliary spectral problem, where we obtain the Fourier series representation of the solution of the adjoint system, find associated eigenfunctions and determine a biorthonormal system. Moreover, we prove that this biorthonormal system forms a Riesz basis in L_2. In Sect. 4, we show the existence and uniqueness of the solution of the adjoint system. Finally, in Sect. 5, we address the null boundary controllability problem, where we first determine the initial data class that guarantees the null boundary controllability of the system. Then, we show how the problem can be reduced to moment problems using the Riesz basis. Section 6 concludes with a discussion of conclusions and future work.

2. Problem Formulation

In this work, we are concerned with the null boundary controllability of the following system:

$$u_t + u_{xxxx} + cu = 0, \quad \text{in } D \tag{1a}$$

$$u(0,t) - u(1,t) = 0, \text{ in } [0,T] \tag{1b}$$

$$u_x(0,t) = v(t), \text{ in } [0,T] \tag{1c}$$

$$u_{xx}(0,t) - u_{xx}(1,t) = 0, \text{ in } [0,T] \tag{1d}$$

$$u_{xxx}(0,t) = 0, \text{ in } [0,T] \tag{1e}$$

$$u(x,0) = u^0(x), \quad \text{in } \Omega. \tag{1f}$$

Here, $D = \Omega \times [0,T]$, $\Omega = [0,1]$, $u^0(x) \in L_2(\Omega)$, $v(t) \in L_2[0,T]$, and c is any positive real number. We will later show in the paper that system (1) is not always controllable. Therefore, we aim to prove that the system is controllable for a certain initial data class, which we will specify in Remark 1 after solving the spectral problem of the adjoint system. For now, we denote this class of initial data by $\mathcal{F}_1$. With this in mind, we can define the null controllability of the system (1) as follows.

Definition 1. System (1) is null boundary controllable at time $T > 0$ from the class $\mathcal{F}_1$ if for every initial condition $u^0 \in \mathcal{F}_1$ there exists a control $v(t) \in [0, T]$ such that $u(x, T) = 0, \quad x \in \Omega$.

As you will see in Sect. 4, to establish the existence of a solution for the adjoint system, we need to impose conditions on the φ_0, and we define the following class:

$$\begin{aligned}\mathcal{F}_2 = \big\{&\varphi^0(x) \in C^8(\Omega) \mid \varphi^0(1) = \varphi^0_{xx}(1) = \varphi^0_{xxxx}(1) = 0, \varphi^0_x(0) = \varphi^0_x(1),\\ &\varphi^0_{xxx}(0) = \varphi^0_{xxx}(1), \varphi^0_{xxxxx}(0) = \varphi^0_{xxxxx}(1), \varphi^0_{xxxxxxx}(0) = \varphi^0_{xxxxxxx}(1).\big\}\end{aligned} \tag{2}$$

Next, we will present a lemma that plays a crucial role in the proof of the main results.

Lemma 1. *The system* (1) *is null controllable in time* $T > 0$ *if and only if for any* $u^0 \in \mathcal{F}_1$ *there exists* $v(t) \in L_2[0, T]$ *such that*

$$\int_0^1 u^0(x)\varphi(x,0)\mathrm{d}x + \int_0^T v(t)\varphi_{xx}(0,t)\mathrm{d}t = 0 \tag{3}$$

holds for any $\varphi^0 \in \mathcal{F}_2$*, where* $\varphi(x, t)$ *is a classical solution of the backward adjoint problem given by*

$$\varphi_t - \varphi_{xxxx} - c\varphi = 0, \text{ in } D \tag{4a}$$
$$\varphi(1,t) = 0, \text{ in } [0,T] \tag{4b}$$
$$\varphi_x(0,t) - \varphi_x(1,t) = 0, \text{ in } [0,T] \tag{4c}$$
$$\varphi_{xx}(1,t) = 0, \text{ in } [0,T] \tag{4d}$$
$$\varphi_{xxx}(0,t) - \varphi_{xxx}(1,t) = 0, \text{ in } [0,T] \tag{4e}$$
$$\varphi(x,T) = \varphi^0(x), \text{ in } \Omega. \tag{4f}$$

Proof. Let v be arbitrary in $L_2[0, T]$, and let φ be the solution of (4). If we multiply (1a) by φ and integrate the obtained result on D using the integration by parts, we get

$$\begin{aligned}0 &= \int_0^T\int_0^1 (u_t + u_{xxxx} + cu)\varphi\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int_0^1 u(-\varphi_t + \varphi_{xxxx} + c\varphi)\mathrm{d}x\mathrm{d}t\\ &\quad + \int_0^1 u\varphi\mid_0^T \mathrm{d}x + \int_0^T [\varphi u_{xxx} - \varphi_x u_{xx} + \varphi_{xx}u_x - u\varphi_{xxx}]\mid_0^1 \mathrm{d}t.\end{aligned}$$

Using the given initial condition and boundary conditions, we have

$$\int_0^1 u(x,T)\varphi^0(x)\mathrm{d}x - \int_0^1 u^0(x)\varphi(x,0)\mathrm{d}x - \int_0^T v(t)\varphi_{xx}(0,t)\mathrm{d}t = 0. \tag{5}$$

If Eq. (3) holds, it follows that $\int_0^1 u(x,T)\varphi^0(x)\mathrm{d}x = 0$ for all $\varphi^0(x) \in \mathcal{F}_2$ and $u(x,T) = 0$. Consequently, system (1) is null-controllable. On the contrary, suppose that $v(t)$ is a control for system (1). Then, $u(x,T) = 0$, and substituting this into (5), we conclude that (3) holds. □

Lemma 1 states that system (1) is null boundary controllable if and only if Eq. (3) holds. Therefore, it is necessary to find a solution of the system (4) to determine the null boundary controllability of the system. In the following sections, we will solve the auxiliary spectral problem of system (4).

3. Auxiliary Spectral Problem

To find the solution of system (4), we will apply the method of separation of variables by letting $\varphi(x,t) = X(x)T(t)$. With the help of this expression, we obtain

$$\begin{cases} X''''(x) = (\lambda - c)X, & \text{in } \Omega \\ X(1) = 0, & \text{in } [0,T] \\ X_x(0) - X_x(1) = 0, & \text{in } [0,T] \\ X_{xx}(1) = 0, & \text{in } [0,T] \\ X_{xxx}(0) - X_{xxx}(1) = 0, & \text{in } [0,T]. \end{cases} \tag{6}$$

This boundary value problem is a non-self-adjoint and it has the following conjugate problem:

$$\begin{cases} Y''''(x) = (\lambda - c)Y, & \text{in } \Omega \\ Y(1) = Y(0), & \text{in } [0,T] \\ Y_x(0) = 0, & \text{in } [0,T] \\ Y_{xx}(1) = Y_{xx}(0), & \text{in } [0,T] \\ Y_{xxx}(0) = 0, & \text{in } [0,T]. \end{cases} \tag{7}$$

Spectral problem (6) has the eigenvalues

$$\lambda_0 = c \text{ and } \lambda_n = (2n\pi)^4 + c$$

and the eigenfunctions

$$\hat{X}_0(x) = 1 - x \text{ and } \hat{X}_n(x) = \sin(2n\pi x), \quad n = 1, 2, \ldots$$

The system of eigenfunctions is not a complete system in $L_2(\Omega)$ and therefore, they do not form a basis in L_2. To complete the eigenfunctions, we use the associated functions of the spectral problem (6). Using similar approaches given in [13], let us define the associated functions $\tilde{X}_n(x)$ for λ_n corresponding to $\hat{X}_n(x)$, $n = 1, 2, \ldots$ Let $\tilde{X}_n(x)$ be a solution of the following problem:

$$\begin{cases} \tilde{X}_n''''(x) - (\lambda_n - c)\tilde{X}_n(x) = -P_n\hat{X}_n(x), & \\ \tilde{X}(1) = 0, & \text{in } [0,T] \\ \tilde{X}_x(0) - \tilde{X}_x(1) = 0, & \text{in } [0,T] \\ \tilde{X}_{xx}(1) = 0, & \text{in } [0,T] \\ \tilde{X}_{xxx}(0) - \tilde{X}_{xxx}(1) = 0, & \text{in } [0,T], \end{cases} \tag{8}$$

where $P_n \neq 0$ is an arbitrary constant and $n = 1, 2, \ldots$ For $P_n = 4(2n\pi)^3$ with $n = 1, \ldots,$ we find $\tilde{X}_n(x) = (1-x)\cos(2n\pi x)$. We can rewrite the eigenfunctions and associated functions of the auxiliary spectral problem as follows.

$$X_0(x) = 1 - x,\ X_{2n-1} = (1-x)\cos(2n\pi x),\ \text{and}\ X_{2n} = \sin(2n\pi x) \quad (9)$$

for $n = 1, 2, \ldots$ Similarly, solving problem (7), we obtain the eigenvalues

$$\lambda_0 = c \text{ and } \lambda_n = (2n\pi)^4 + c, \quad n = 1, 2, \ldots$$

and the eigenfunctions $\hat{Y}_0(x) = 2$ and $\hat{Y}_n(x) = S_n \cos(2n\pi x), \quad n = 1, 2, \ldots.$ To obtain biorthonormal system, we choose $S_n = 4$ for $n = 1, 2, \ldots.$ Similarly, we find the associated functions of the conjugate problem (7) as follows:

$$Y_0(x) = 2,\ Y_{2n-1} = 4\cos(2n\pi x), Y_{2n} = 4x\sin(2n\pi x), \quad (10)$$

for $n = 1, 2, \ldots$

Lemma 2. *The system of functions given in* (9) *and* (10) *are biorthonormal on* Ω, *namely, for all* $i, j = 0, 1, \ldots$

$$(X_i, Y_j) = \int_0^1 X_i(x)Y_j(x)\mathrm{d}x = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Proof. The proof is trivial. □

Lemma 3. *The systems of functions given in* (9) *and in* (10) *are complete in* $L_2(\Omega)$.

Proof. We will give the proof only for the system of functions in (9), but note that the same argument applies to the system of functions in (10). Suppose $f(x)$ is a function in $L_2(\Omega)$ that is orthogonal to the functions in the system (9), i.e., $(f, X_m) = 0$ for $m = 0, 1, 2, \ldots.$ Then we have

$$\int_0^1 f(x)(1-x)\mathrm{d}x = 0, \int_0^1 f(x)(1-x)\cos(2n\pi x)\mathrm{d}x = 0,\ \text{and}$$
$$\int_0^1 f(x)\sin(2n\pi x)\mathrm{d}x = 0.$$

Since the functions $\{1, \sqrt{2}\cos(2n\pi x), \sqrt{2}\sin(2n\pi x)\}_{n=1}^{\infty}$ form an orthonormal basis for $L_2(\Omega)$, we can represent $f(x)$ as a series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2n\pi x) \quad (11)$$

which converges in $L_2(\Omega)$. Since $f(x)$ is orthogonal to the functions in (9), we obtain

$$\begin{aligned} 0 &= \int_0^1 f(x)(1-x)\cos(2k\pi x)\mathrm{d}x \\ &= \int_0^1 \frac{a_0}{2}(1-x)\cos(2k\pi x)\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_0^1 (1-x)\cos(2n\pi x)\cos(2k\pi x) = \frac{a_k}{4} \end{aligned}$$

$k = 0, 1, 2, \ldots$ Substituting $a_k = 0$ for $k = 0, 1, 2, \ldots$ in (11), we obtain $f(x) \equiv 0$. Therefore, the system of functions in (9) is complete in $L_2(0, 1)$. □

To verify that the systems of functions (9) and (10) are Riesz bases in $L_2(\Omega)$, we will employ the following lemma.

Lemma 4. (Theorem 2.1 in [14]) *For a sequence $(f_k)_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$, the following conditions are equivalent:*
(1) $(f_k)_{k=1}^{\infty}$ *forms a Riesz basis for $\mathcal{H}$.*
(2) $(f_k)_{k=1}^{\infty}$ *is complete in $\mathcal{H}$ and it has a complete biorthogonal sequence $(g_k)_{k=1}^{\infty}$ so that*

$$\sum_{k=1}^{\infty} |(f, f_k)|^2 < \infty \text{ and } \sum_{k=1}^{\infty} |(f, g_k)|^2 < \infty$$

for every $f \in \mathcal{H}$.

Lemma 5. *The systems of functions (9) and (10) are Riesz bases in $L_2(\Omega)$.*

Proof. Since the systems of functions given in (9) and (10) are complete in $L_2(\Omega)$, and

$$\begin{aligned}\sum_{k=0}^{\infty} |\, (f, X_k)\, |^2 &= \left(\int_0^1 (1-x) f(x) \mathrm{d}x\right)^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(\int_0^1 f(x)(1-x)\sqrt{2}\cos(2k\pi x)\mathrm{d}x\right)^2 \\ &\quad + \frac{1}{2}\sum_{k=1}^{\infty}\left(\int_0^1 f(x)\sqrt{2}\sin(2k\pi x)\mathrm{d}x\right)^2 \\ &\le \frac{1}{2}\|f\|^2_{L_2(\Omega)} < \infty,\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty} |\, (f, Y_k)\, |^2 &= 4\left(\int_0^1 f(x)\mathrm{d}x\right)^2 + 8\sum_{k=1}^{\infty}\left(\int_0^1 f(x)\sqrt{2}\cos(2k\pi x)\mathrm{d}x\right)^2 \\ &\quad + 8\sum_{k=1}^{\infty}\left(\int_0^1 f(x)x\sqrt{2}\sin(2k\pi x)\mathrm{d}x\right)^2 \\ &\le 12\|f\|^2_{L_2(\Omega)} + 8\|xf\|^2_{L_2(\Omega)} < \infty,\end{aligned}$$

are convergent for each $f(x) \in L_2(\Omega)$ using classical Bessel's inequality. Thus, from Lemma 4, we obtain the systems of functions (9) and (10) form Riesz bases in $L_2(\Omega)$. □

Then, the solution of (4) can be represented by the sum of the series as follows:

$$\begin{aligned}\varphi(x,t) = \beta_0(1-x)e^{-\lambda_0(T-t)} + \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)}[\beta_{2n-1}(1-x)\cos(2n\pi x) \\ + (\beta_{2n} + 4(2n\pi)^3(T-t)\beta_{2n-1})\sin(2n\pi x)]\end{aligned} \tag{12}$$

where $\beta_n = (\varphi^0(x), Y_n(x))$ for $n = 0, 1, 2, \ldots$

4. Existence and Uniqueness of the Solution

The existence and uniqueness for a fractional partial differential equation with this type of boundary conditions were demonstrated in [15]. In this section, we will show the existence and uniqueness of (4). To achieve this, we need to begin by proving the following lemma.

Lemma 6. *Assuming that the function $\varphi^0(x) \in \mathcal{F}_2$. Then, the following series are convergent:*

$$\sum_{n=1}^{\infty} n^7 \mid \beta_{2n-1} \mid, \quad \sum_{n=1}^{\infty} n^4 \mid \beta_{2n} \mid .$$

Proof. Let $\varphi^0(x)$ satisfy the assumptions of the Lemma 6. From (12), it can be observe that $\beta_{2n-1} = (\varphi^0(x), Y_{2n-1}(x)) = \int_0^1 \varphi^0(x) 4\cos(2n\pi x)\mathrm{d}x$. Applying integration by parts eight times and utilizing the assumptions of the Lemma, we obtain

$$\beta_{2n-1} = \int_0^1 \varphi^0(x) 4\cos(2n\pi x)\mathrm{d}x = \frac{4}{(2n\pi)^8} \int_0^1 (\varphi^0(x))_{xxxxxxxx} \cos(2n\pi x)\mathrm{d}x.$$

Using this equation, one can get

$$\sum_{n=1}^{\infty} n^7 \mid \beta_{2n-1} \mid = \sum_{n=1}^{\infty} \frac{4}{(2\pi)^8 n} \left| \int_0^1 (\varphi^0(x))_{xxxxxxxx} \cos(2n\pi x)\mathrm{d}x \right| .$$

By utilizing the Cauchy–Schwartz and Bessel inequalities, we arrive at:

$$\sum_{n=1}^{\infty} n^7 \mid \beta_{2n-1} \mid \leq \frac{1}{(2\pi)^7 \sqrt{3}} \|(\varphi^0(x))_{xxxxxxxx}\|_{L_2(\Omega)}.$$

Similarly,

$$\beta_{2n} = \int_0^1 \varphi^0(x) x \sin(2n\pi x)\mathrm{d}x = \frac{4}{(2n\pi)^5} \int_0^1 \left[x\varphi^0_{xxxxx} + 5\varphi^0_{xxxx}\right] \cos(2n\pi x)\mathrm{d}x$$

and

$$\sum_{n=1}^{\infty} n^4 \mid \beta_{2n} \mid = \sum_{n=1}^{\infty} \frac{4}{(2\pi)^5 n} \mid \int_0^1 \left[x\varphi^0_{xxxxx} + 5\varphi^0_{xxxx}\right] \cos(2n\pi x) \mid .$$

Applying Cauchy–Schwartz and Bessel inequalities again shows that

$$\sum_{n=1}^{\infty} n^4 \mid \beta_{2n} \mid \leq \frac{1}{(2\pi)^4 \sqrt{3}} \|x\varphi^0_{xxxxx} + 5\varphi^0_{xxxx}\|_{L_2(\Omega)}.$$

□

Lemma 7. *Suppose $\varphi^0(x)$ satisfies the conditions stated in Lemma (6). Then, the system (4) has a unique solution $\varphi(x,t) \in C^{8,1}(D)$ as described by the form (12).*

Proof. Given the basis $\{X_n(x)\}_{n \geq 0}$ in the Hilbert space $L_2(\Omega)$, we express $\varphi(x,t)$ using Eq. (12). To establish that the solution for system (4) is valid, we need to confirm the continuity of both the first partial derivative of $\varphi(x,t)$ with respect to t and the fourth partial derivative with respect to x. In addition, this solution should adhere to (4a) within D. Furthermore, it is essential for the function defined in equation (12) and its spatial derivatives up to the third order, as well as its first partial derivative with respect to time, to exhibit continuity at the boundary points. To this end, we need to show that the following series converge uniformly for $t \geq 0$.

$$\begin{aligned}\varphi_t(x,t) \sim \beta_0\lambda_0(1-x)e^{-\lambda_0(T-t)} + \sum_{n=1}^{\infty}\lambda_n e^{-\lambda_n(T-t)}[\beta_{2n-1}(1-x)\cos(2n\pi x) \\ + (\beta_{2n} + 4(2n\pi)^3(T-t)\beta_{2n-1})\sin(2n\pi x)] \\ - \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)}4(2n\pi)^3\beta_{2n-1}\sin(2n\pi x),\end{aligned} \tag{13}$$

and

$$\begin{aligned}\varphi_{xxxx}(x,t) \sim \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)}\beta_{2n-1}\left[(2n\pi)^4(1-x)\cos(2n\pi x) - 4(2n\pi)^3\sin(2n\pi x)\right] \\ + \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)}(2n\pi)^4\left[\beta_{2n} + 4(2n\pi)^3(T-t)\beta_{2n-1}\right]\sin(2n\pi x).\end{aligned} \tag{14}$$

Due to the convergence of the following majorant series from Weierstrass M-test and Lemma 6, the function $\varphi(x,t)$ becomes continuous over D.

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n e^{\lambda_n T}\left[\mid \beta_{2n-1}\mid + \mid \beta_{2n}\mid + 4(2n\pi)^3 T \mid \beta_{2n-1}\mid\right] \\ + \sum_{n=1}^{\infty} e^{\lambda_n T}4(2n\pi)^3 \mid \beta_{2n-1}\mid\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} e^{\lambda_n T}\mid \beta_{2n-1}\mid\left[(2n\pi)^4 + 4(2n\pi)^3\right] \\ + \sum_{n=1}^{\infty} e^{\lambda_n T}(2n\pi)^4\left[\mid \beta_{2n}\mid + 4(2n\pi)^3 T\mid \beta_{2n-1}\mid\right].\end{aligned}$$

The function in Eq. (12) and its first, second, and third partial derivatives with respect to spatial variable and first partial derivative with respect to time must be continuous at boundary points. Namely, the series in Eq. (12) must be continuous at $t = T$

$$\varphi(x,T) = \beta_0(1-x) + \sum_{n=1}^{\infty}[\beta_{2n-1}(1-x)\cos(2n\pi x) + \beta_{2n}\sin(2n\pi x)]$$

and the following functions must be continuous at boundary points $x = 0$ and $x = 1$:

$$\begin{aligned}\varphi_x(x,t) &\sim -\beta_0 e^{-\lambda_0(T-t)} \\ &\quad + \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)} \left[\beta_{2n-1}\left(-\cos(2n\pi x) - (1-x)2n\pi\sin(2n\pi x)\right)\right] \\ &\quad + \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)} 2n\pi \left[\left(\beta_{2n} + 4(2n\pi)^3(T-t)\beta_{2n-1}\right)\cos(2n\pi x)\right],\end{aligned}$$

$$\begin{aligned}&\varphi_{xx}(x,t) \\ &\quad \sim \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)} \left[\beta_{2n-1}\left(2(2n\pi)\sin(2n\pi x) - (1-x)(2n\pi)^2\cos(2n\pi x)\right)\right] \\ &\quad - \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)} (2n\pi)^2 \left[\left(\beta_{2n} + 4(2n\pi)^3(T-t)\beta_{2n-1}\right)\sin(2n\pi x)\right],\end{aligned}$$

$$\begin{aligned}&\varphi_{xxx}(x,t) \\ &\quad \sim \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)} \left[\beta_{2n-1}\left(3(2n\pi)^2\cos(2n\pi x) + (1-x)(2n\pi)^3\sin(2n\pi x)\right)\right] \\ &\quad - \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)} (2n\pi)^3 \left[\left(\beta_{2n} + 4(2n\pi)^3(T-t)\beta_{2n-1}\right)\cos(2n\pi x)\right].\end{aligned}$$

Applying the Weierstrass M-test and referring to Lemma 6, it becomes evident that the subsequent majorant series are uniformly convergent.

$$\begin{aligned}&\sum_{n=1}^{\infty} \mid \beta_{2n-1} \mid + \mid \beta_{2n} \mid, \\ &\sum_{n=1}^{\infty} \mid \beta_{2n-1} \mid + \sum_{n=1}^{\infty} 2n\pi \left[\mid \beta_{2n} \mid + 4(2n\pi)^3 T \mid \beta_{2n-1} \mid\right], \\ &\sum_{n=1}^{\infty} (2n\pi)^2 \mid \beta_{2n-1} \mid \\ &\sum_{n=1}^{\infty} \mid \beta_{2n-1} \mid 3(2n\pi)^2 + \sum_{n=1}^{\infty} (2n\pi)^3 \left[\mid \beta_{2n} \mid + 4(2n\pi)^3 T \mid \beta_{2n-1} \mid\right]\end{aligned}$$

Therefore, these series are continuous at the boundary points. Finally, we obtain a function $\varphi(x,t) \in C^{8,1}(D)$ which is a classical solution of system (4) given by the biorthonormal series in Eq. (12). To establish the uniqueness of the solution, consider φ_1 and φ_2 as two solutions of the problem (4) within the domain D. Here, $\beta_n = (\varphi_1(x,T), Y_n(x))$ for $n = 0,1,2,\ldots$, and $\gamma_n = (\varphi_2(x,T), Y_n(x))$ for $n = 0,1,2,\ldots$. Let $\varphi = \varphi_1 - \varphi_2$. This function φ satisfies (4). By taking the inner product of both sides of the last equation with the biorthogonal function $Y_m(x)$, where $m = 0,1,2,\ldots$, we derive $\beta_n = \gamma_n$ for all $n = 0,1,\ldots$. Consequently, this implies that the solution is unique. □

5. Main Result

Before presenting the main theorem, we provide the following remark, which helps us identify uncontrollable situations.

Remark 1. System (1) is not always controllable. To verify that, we first represent $u^0 \in L_2(\Omega)$ using the biorthonormal series as follows:

$$u^0(x) = 2\eta_0 + \sum_{n=1}^{\infty} \eta_{2n-1} 4\cos(2n\pi x) + \eta_{2n} 4x \sin(2n\pi x),$$

where $\eta_n = (u^0(x), X_n(x))$. Next, we take the following solution of (4) with the initial data $\sin(2n\pi x)$ for arbitrary fixed positive integer n:

$$\varphi_n(x,t) = \sin(2n\pi x)e^{-\lambda_n(T-t)}.$$

From (3), we obtain

$$\int_0^1 \sin(2n\pi x)u^0(x)\mathrm{d}x = 0,$$

which is equivalent to $\eta_{2n} = 0$ for $n = 1, \ldots$. Similarly, it can be seen that if $\eta_0 = 0$, then system is also not controllable. Therefore, the null controllability of the system is possible over the following initial data class:

$$\mathcal{F}_1 = \{u^0(x) \in L_2(\Omega) \mid \eta_{2n} = (u^0(x), X_{2n}(x)) = 0, n = 0, 1, \ldots\}.$$

Next, we state the main theorem of the paper.

Theorem 1. *The system* (1) *is null controllable in time* $T > 0$ *from the class* $\mathcal{F}_1$ *if and only if for any* $u^0 \in \mathcal{F}_1$ *with Fourier expansion*

$$u^0(x) = \sum_{n=1}^{\infty} \eta_{2n-1} 4\cos(2n\pi x),$$

there exists a function $f \in L^2(0,T)$ *such that*

$$\int_0^T f(t)e^{-\lambda_n t}\mathrm{d}t = \frac{\eta_{2n-1}e^{-\lambda_n T}}{(2n\pi)^2}, \qquad n = 1, 2, \ldots, \tag{15}$$

where $\lambda_n = (2n\pi)^4 + c$, $n \geq 0$ *and* c *is any positive number.*

Proof. We observe that $v(t)$ is a control for system (1) if and only if Eq. (3) holds. We obtain the solution of the backward adjoint problem of the system (1) using (4), which can be represented by (12). Substituting the values of $\varphi(x,t)$ and $u^0(x)$ into (3), we get the equation

$$\sum_{n=1}^{\infty} e^{-\lambda_n T}\beta_{2n-1}\eta_{2n-1} = \int_0^T v(t) \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)}(2n\pi)^2\beta_{2n-1}\mathrm{d}t. \tag{16}$$

Since systems of functions (9) and (10) form a biorthonormal system of functions on $L_2(\Omega)$, the equation (3) holds if and only if it is verified by $\varphi_m^0 = X_m(x)$, $m = 0, 1, 2, \ldots$. If $\varphi_m^0 = X_m(x)$, then $\beta_n = (X_m, Y_n) = \delta_{m,n}, n, m = 0, 1, 2, \ldots$ and

$$\int_0^T v(t)e^{-\lambda_n(T-t)}\mathrm{d}t = \frac{\eta_{2n-1}e^{-\lambda_n T}}{(2n\pi)^2},$$

for $n = 1, 2, \ldots$. After replacing $T - t$ by t in the last integrals and choosing $v(T - t) = f(t)$, we complete the proof. □

We seek to find $f(t)$ that satisfies (15) to determine control $v(t)$. This is a moment problem in $L_2[0, T]$ with respect to the family $\Lambda = \{e^{-\lambda_n t}\}_{n \geq 0}$. According to Theorem (1), controllability holds only if the moment problem (15) is solvable. To solve this moment problem, the general theory developed by Fattorini and Russell in [16] can be applied. Assuming that $\{\Psi_m\}_{m \geq 0}$ is a family of functions biorthogonal to the set Λ in $L_2[0, T]$ can be constructed, such that

$$\int_0^T e^{-\lambda_n t} \Psi_m(t) \mathrm{d}t = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

for all $m, n = 0, 1, 2 \ldots$, then moment problems (15) have solutions by setting

$$f(t) = \sum_{m=1}^{\infty} \frac{\eta_{2m-1} e^{-\lambda_m T}}{(2m\pi)^2} \Psi_m(t).$$

Muntz's Theorem shows that the biorthogonal sequence $\{\Psi_m\}_{m \geq 0}$ exists since

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \sum_{n=0}^{\infty} \frac{1}{(2n\pi)^4 + c} < \infty, \tag{17}$$

holds. The general estimations of $\|\Psi_m\|_{L_2(0,\infty)}$ were calculated by Fattoroni and Russell in [2]. They demonstrated that if the λ_n are real and satisfy the following asymptotic relationship

$$\lambda_n = K(n + \alpha)^\zeta + o(n^{\zeta - 1}) \quad (n \to \infty),$$

where $K > 0, \zeta > 1$ and α is real, then there exists constants $\hat{K}, K_\zeta$ such that

$$\|\Psi_n(t)\|_{L_2(0,\infty)} \leq \hat{K} \exp\left[(K_\zeta + o(1)) \lambda_n^{1/\zeta}\right] \quad (n \geq 1),$$

where $o(1)$ indicates a term tending to zero as n goes to infinity. The computation of the constant K_ζ is given in [16]. To relate the interval $[0, \infty]$ with the finite interval $[0, T]$, they used results given in [17].

Since $\lambda_n = (2n\pi)^4 + c$, using these results it can be seen that

$$\|\Psi_m(t)\|_{L_2[0,T]} \leq K e^{m\rho} \text{ for } m \geq 0$$

where K and ρ some positive constants. Let us determine the value of ρ for the case addressed in this article. Fattorini and Russell [2] shown that

$$\|\Psi_m(t)\|_{L_2(0,\infty)} = \left(\frac{\lambda_m}{2}\right)^{\frac{1}{2}} \frac{\prod_{k=0}^{\infty} \left(1 + \frac{\lambda_m}{\lambda_k}\right)}{\prod_{\substack{k=0 \\ k \neq m}}^{\infty} \left(1 - \frac{\lambda_m}{\lambda_k}\right)} \tag{18}$$

for all $m \geq 0$.

Since $\lambda_n = (2n\pi)^4 + c$, from (18), we obtain

$$\|\Psi_m(t)\|_{L_2(0,\infty)} = \sqrt{2\lambda_m} \prod_{\substack{k=0 \\ k\neq m}}^{\infty} \frac{k^4 + m^4 + 2s}{\mid k^4 - m^4 \mid}. \tag{19}$$

For the sake of simplicity in computations, we have chosen $c = (2\pi)^4 s$, where s is any positive number. Now, we prove the following Lemma to estimate the infinite product which appears in (19).

Lemma 8. *There exist two positive constants M and ρ such that for any $m \geq 0$*

$$\prod_{\substack{k=0 \\ k\neq m}}^{\infty} \frac{k^4 + m^4 + 2s}{\mid k^4 - m^4 \mid} \leq M e^{\rho m}.$$

Proof. To prove this, it is necessary to consider two distinct cases: namely, when $m \geq 1$ and when $m = 0$. In the first case, we assume that $m \geq 1$. Note that

$$\prod_{\substack{k=0 \\ k\neq m}}^{\infty} \frac{k^4 + m^4 + 2s}{\mid k^4 - m^4 \mid} \leq \exp\left[\sum_{\substack{k=0 \\ k\neq m}}^{\infty} \ln\left(1 + \frac{2m^4 + 2s}{\mid k^4 - m^4 \mid}\right)\right].$$

Then,

$$\begin{aligned}
\sum_{\substack{k=0 \\ k\neq m}}^{\infty} \ln\left(1 + \frac{2m^4 + 2s}{\mid k^4 - m^4 \mid}\right) \leq & \int_0^m \ln\left(1 + \frac{2m^4 + 2s}{m^4 - x^4}\right) \mathrm{d}x \\
& + \int_m^{2m} \ln\left(1 + \frac{2m^4 + 2s}{x^4 - m^4}\right) \mathrm{d}x \\
& + \int_{2m}^{\infty} \ln\left(1 + \frac{2m^4 + 2s}{x^4 - m^4}\right) \mathrm{d}x \\
& = m\Bigg[\int_0^1 \ln\left(1 + \frac{c_1}{1 - x^4}\right) \mathrm{d}x \\
& + \int_1^2 \ln\left(1 + \frac{c_1}{x^4 - 1}\right) \mathrm{d}x + \int_2^{\infty} \ln\left(1 + \frac{c_1}{x^4 - 1}\right) \mathrm{d}x\Bigg] \\
& = m(I_1 + I_2 + I_3)
\end{aligned},$$

where $c_1 = 2 + \frac{2s}{m^4}$. Using integration by parts, these integrals can be calculated as follows.

$$\begin{aligned}
I_1 &= \int_0^1 \ln\left(1 + \frac{c_1}{1 - x^4}\right) \mathrm{d}x = \int_0^1 \ln\left(1 + \frac{c_1}{(1 - x)(1 + x)(1 + x^2)}\right) \mathrm{d}x \\
&\leq \int_0^1 \ln\left(1 + \frac{c_1}{(1 - x)}\right) \mathrm{d}x = (1 - x) \ln\left(1 + \frac{c_1}{(1 - x)}\right) \Big|_0^1 + \int_0^1 \frac{c_1}{1 + c_1 - x} \mathrm{d}x \\
&= \ln \frac{(1 + c_1)^{c_1 + 1}}{c_1^{c_1}}
\end{aligned}$$

$$I_2 = \int_1^2 \ln\left(1+\frac{c_1}{x^4-1}\right)\mathrm{d}x \le \int_1^2 \ln\left(1+\frac{c_1}{x^2-1}\right)\mathrm{d}x \le \int_1^2 \ln\left(1+\frac{c_1}{(x-1)^2}\right)\mathrm{d}x$$
$$= (x-1)\ln\left(1+\frac{c_1}{(x-1)^2}\right)|_1^2 + \int_1^2 \frac{2c_1}{(x-1)^2+c_1}$$
$$= \ln(1+c_1) + 2\sqrt{c_1}\arctan\left(\frac{1}{\sqrt{c_1}}\right)$$

For the third one, since $\ln(1+x) < x$ for all $x > 0$ we have

$$I_3 = \int_2^\infty \ln\left(1+\frac{c_1}{x^4-1}\right)\mathrm{d}x \le \int_2^\infty \ln\left(1+\frac{c_1}{x^2-1}\right)\mathrm{d}x \le \int_2^\infty \ln\left(1+\frac{c_1}{(x-1)^2}\right)\mathrm{d}x$$
$$\le \int_2^\infty \left(\frac{c_1}{(x-1)^2}\right)\mathrm{d}x = c_1.$$

So, for the case $m \ge 1$, it is seen that

$$\prod_{\substack{k=0\\k\ne m}}^{\infty} \frac{k^4+m^4+2s}{\mid k^4-m^4\mid} \le Me^{\rho m},$$

where $\rho = \ln\left(\frac{(1+c_1)^{c_1+2}}{c_1^{c_1}}\right) + 2\sqrt{c_1}\arctan(\frac{1}{\sqrt{c_1}}) + c_1$ and $M = 1$.

For the second case, we assume that $m = 0$. Using equation (18), we obtain

$$\|\Psi_m(t)\|_{L_2(0,\infty)} = \sqrt{2\lambda_0}\prod_{k=1}^{\infty}\frac{k^4+2s}{k^4}$$

and

$$\prod_{k=1}^{\infty}\frac{k^4+2s}{k^4} = \exp\left[\sum_{k=1}^{\infty}\ln\left(1+\frac{2s}{k^4}\right)\right].$$

Since

$$\sum_{k=1}^{\infty}\ln\left(1+\frac{2s}{k^4}\right) \le \int_1^\infty \ln\left(1+\frac{2s}{x^4}\right)\mathrm{d}x \le \int_1^\infty \frac{2s}{x^4}\mathrm{d}x = \frac{2s}{3},$$

, we get

$$\prod_{k=1}^{\infty}\frac{k^4+2s}{k^4} \le M.$$

where $M = \sqrt{2\lambda_0}e^{\frac{2s}{3}}$. □

By means of Lemma 8, we deduce that for all $m \ge 0$

$$\|\Psi_m(t)\|_{L_2(0,\infty)} = \sqrt{2\lambda_m}\prod_{\substack{k=0\\k\ne m}}^{\infty}\frac{k^4+m^4+2s}{\mid k^4-m^4\mid}$$
$$\le Ke^{\rho m}.$$

where $K = M\sqrt{2\lambda_m}$. Fattoroni and Russell [2] proved that if the exponential moment problem is solvable for $T = \infty$, then it is solvable for every time $T > 0$. Now, we can estimate the norm of $\Psi_m(t)$ in $L_2[0, T]$ using the result given in [17].

Theorem 2. *Suppose that* $\sum_{n\geq 0}\dfrac{1}{\lambda_n}<\infty$. *The restriction operator*

$$R_T : E(\Lambda,\infty)\to E(\Lambda,T)$$
$$R_T(v)=v\mid_{[0,T]}.$$

Moreover, there exists a positive constant C *which only depends on* T *such that* $\|(R_T)^{-1}\|\leq C$. *Here,* $E(\Lambda,\infty)=\cup_{n\geq 0}E^n(\Lambda,\infty)$ *is the space generated by* $\Lambda^n:=\{e^{-\lambda_k t}\}_{0\leq k\leq n}$ *in* $L_2[0,T]$ *and* $E(\Lambda,T)$ *is the space generated by* Λ *in* $L_2[0,T]$.

Proof. The proof of theorem can be found [2] or [17]. □

In view of the above theorem, we see that if $p_n(t)=e^{-\lambda_n t}$ for $t>0$ and $n\geq 0$, then

$$R_T(p_n(t))=p_n(t)\mid_{[0,T]}.$$

Also,

$$\begin{aligned}\delta_{n,m}&=(p_n,\Psi_m(t))_{L_2(0,\infty)}=(R_T^{-1}R_Tp_n,\Psi_m(t))_{L_2(0,\infty)}\\&=(p_n,(R_T^{-1})^*\Psi_m(t))_{L_2[0,T]}.\end{aligned}$$

Therefore, the family $\{(R_T^{-1})^*\Psi_m(t)\}_{m\geq 0}$ is biorthogonal to $\{e^{-\lambda_n t}\}_{n\geq 0}$ in $L_2[0,T]$. Together with $\|(R_T^{-1})^*\|=\|R_T^{-1}\|$, this means that

$$\|\Psi_m(t))\|_{L_2[0,T]}=\|(R_T^{-1})^*\Psi_m(t)\|_{L_2[0,T]}\leq\|R_T^{-1}\|\|\Psi_m(t)\|_{L_2(0,\infty)}.$$

At the end, we can state the following corollary.

Corollary 1. *Given any* $T>0$, *suppose that there exists a sequence* $\{\Psi_m$
$(t)\}_{n\geq 0}$ *in* $L_2[0,T]$ *biorthogonal to the set* Λ *such that*

$$\|\Psi_m\|_{L_2[0,T]}\leq Ke^{m\rho},\quad \forall m\geq 0 \tag{20}$$

holds, where K *and* ρ *are some positive constants. Then, system* (1) *is null-controllable in time* T.

Proof. According to Theorem (1), system (1) is null controllable in time $T>0$ from the class $\mathcal{F}_1$ if for any $u^0\in\mathcal{F}_1$ with Fourier expansion

$$u^0(x)=\sum_{n=1}^{\infty}\eta_{2n-1}4\cos(2n\pi x)$$

there exists a function $f\in L_2[0,T]$ which holds (15). Choose

$$f(t)=\sum_{m=1}^{\infty}\frac{\eta_{2m-1}e^{-\lambda_m T}}{(2m\pi)^2}\Psi_m(t). \tag{21}$$

Since $\|\Psi_m\|_{L_2[0,T]}\leq Ke^{m\rho}$, for all $m\geq 0$, we deduce that

$$\left\|\sum_{m=1}^{\infty}\frac{\eta_{2m-1}e^{-\lambda_m T}}{(2m\pi)^2}\Psi_m\right\|_{L_2[0,T]}\leq\sum_{m=1}^{\infty}\frac{\mid\eta_{2m-1}\mid}{(2m\pi)^2}e^{-\lambda_m T}\|\Psi_m\|_{L_2[0,T]}$$
$$\leq K\sum_{m=1}^{\infty}\frac{\mid\eta_{2m-1}\mid}{(2m\pi)^2}e^{-\lambda_m T+m\rho}<\infty,$$

i.e., $f(t)$ converges in $L_2[0,T]$. Hence, (20) implies that f satisfies (15) and the proof finishes. □

6. Conclusions and Future Work

In this work, we investigated the null boundary controllability for a fourth-order parabolic equations with Samarkii–Ionkin-type conditions. Due to the absence of self-adjointness under these boundary conditions, we initially illustrate that the eigenfunctions of the system form a Riesz basis in L_2. In addition, we have established the existence and uniqueness of the adjoint problem. Furthermore, applying the moment method, we have derived necessary and sufficient conditions for null boundary controllability of the given system for some classes of initial data.

As a direction for future research, it would be valuable to explore the cost of controllability for this system. Similar to the approaches introduced in [18–20] within the existing literature, such an investigation could provide deeper insights into the controllability aspects of the system.

Author Contributions IO wrote and reviewed the main manuscript text.

Declarations

Conflict of Interest The authors declare no competing interests.

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