Mediterr. J. Math. (2023) 20:320
https://doi.org/10.1007/s00009-023-02524-w
1660-5446/23/060001-13
published online October 24, 2023

Mediterranean Journal of Mathematics

Concircular Hypersurfaces and Concircular Helices in Space Forms

Pascual Lucas and José Antonio Ortega-Yagües

Abstract. In this paper, we find a full description of concircular hypersurfaces in space forms as a special family of ruled hypersurfaces. We also characterize concircular helices in 3-dimensional space forms by means of a differential equation involving the concircular factor and their curvature and torsion, and we show that the concircular helices are precisely the geodesics of the concircular surfaces.

Mathematics Subject Classification. Primary 53A04, 53A05.

Keywords. Generalized helix, slant helix, rectifying curve, concircular helix, generalized cylinder, helix surface, conical surface, concircular surface.

1. Introduction

Generalized helices, slant helices, and rectifying curves are well-known examples of curves satisfying a certain condition with respect to a special vector field. Generalized helices are defined by the property that their tangents make a constant angle with a fixed direction. Slant helices are defined by the property that their principal normals make a constant angle with a constant vector field, [5], and rectifying curves are defined as the curves whose position vector is orthogonal to its principal normal vector field (i.e., the position vector lies in the rectifying plane), [1]. Moreover, these curves are characterized as the geodesics of some special ruled surfaces: generalized helices in cylinders, slant helices in helix surfaces, [8], and rectifying curves in conical surfaces, [2]. Motivated by these examples of curves and surfaces, the authors in [9] have extended the above conditions, and have introduced the notion of concircular submanifold in the Euclidean space $\mathbb{R}^n$. In particular, they characterize concircular helices in $\mathbb{R}^3$ by means of a differential equation involving their curvature and torsion. Moreover, they also find a full description of concircular surfaces in $\mathbb{R}^3$ as a special family of ruled surfaces and characterize the concirculares helices in $\mathbb{R}^3$ as the geodesics of these surfaces.

In this paper, we generalize the results obtained in [9] to space forms of nonzero constant curvature. Recall that a vector field $V \in \mathfrak{X}(M)$ on a Riemannian manifold M, with Levi–Civita connection ∇, is said to be *concircular* if $\nabla V = \mu I$, where $\mu \in \mathcal{C}^\infty(M)$ is a differentiable function called the *concircular factor*, [3,4,10]. We denote by $\mathrm{Con}(M)$ the set of concircular vector fields of M. The following definition extends the one given in [9]:

Definition 1. Let $\mathbb{M}^n(C)$ be an n-dimensional space form of constant curvature C. A submanifold $M^m \subset \mathbb{M}^n(C)$ is said to be a *concircular submanifold* if there exists a concircular vector field $V \in \mathrm{Con}(\mathbb{M}^n(C))$ (called the axis of M^m) such that $\langle \mathbf{n}, V\rangle$ is a constant function along M^m, $\mathbf{n}$ being any unit vector field in the first normal space of M^m.

In the particular case of a hypersurface, M^{n-1} is said to be a concircular hypersurface (with axis V) if $\langle N, V\rangle$ is a constant function along M^{n-1}, N being a unit normal vector field. Another very interesting case appears when $m = 1$: a (non-geodesic) unit speed curve γ in $\mathbb{M}^n(C)$ is said to be a *concircular helix* (with axis V) if $\langle N_\gamma, V\rangle$ is a constant function along γ, N_γ being the principal normal vector field of γ.

This paper is organized as follows: In Sect. 2, we characterize concircular vector fields in $\mathbb{M}^n(C)$, see Theorem 1. In Sect. 3, we present several properties of concircular hypersurfaces in $\mathbb{M}^n(C)$, see Propositions 4 and 5, and we finish this section with the characterization of all concircular hypersurfaces in $\mathbb{M}^n(C)$, see Theorem 6. Section 4 contains a characterization of all concircular helices in $\mathbb{M}^3(C)$, see Proposition 9 and Theorem 10. Finally, Sect. 5 contains the characterization of geodesics curves of concircular surfaces, see Proposition 11, and this characterization is used to show that concircular helices in $\mathbb{M}^3(C)$ can be described as the geodesics of the concircular surfaces, see Theorem 12.

2. Concircular Vector Fields in Space Forms

Let $\mathbb{M}^n(C)$ denote the n-dimensional space form of nonzero constant curvature C. Then $\mathbb{M}^n(C)$ stands for a sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ or a hyperbolic space $\mathbb{H}^n \subset \mathbb{R}^{n+1}_1$ according to $C > 0$ or $C < 0$, respectively. Put $C = \varepsilon/R^2$, with $\varepsilon = (-1)^\nu$, where $\nu \in \{0, 1\}$ is the index of the ambient space $\mathbb{R}^{n+1}_\nu$ that contains $\mathbb{M}^n(C)$. $\mathbb{M}^n(C)$ can be described as follows:

$$\mathbb{M}^n(C) = \{p = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_\nu \mid \langle p, p\rangle = 1/C\},$$

where as usual $\mathbb{R}^{n+1}_\nu$ is the space $\mathbb{R}^{n+1}$ endowed with the flat metric

$$\langle , \rangle = \varepsilon\, \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_{n+1}^2,$$

$(x_1, x_2, \ldots, x_{n+1})$ being the usual rectangular coordinates of $\mathbb{R}^{n+1}$.

Let us write ∇^0 and $\overline{\nabla}$ to denote the Levi–Civita connections of $\mathbb{R}^{n+1}_\nu$ and $\mathbb{M}^n(C)$, respectively. If $\phi : \mathbb{M}^n(C) \to \mathbb{R}^{n+1}_\nu$ denotes the usual isometric immersion (the position vector), then the Gauss formula is

$$\nabla^0_X Y = \overline{\nabla}_X Y - C\,\langle X, Y\rangle\, \phi, \tag{1}$$

for any vector fields X and Y tangent to $\mathbb{M}^n(C)$.

Given a point $p \in \mathbb{M}^n(C)$ and a unit vector $w \in T_p\mathbb{M}^n(C)$, the exponential map $\exp_p$ is given by

$$\exp_p(tw) = f\Big(\frac{t}{R}\Big)p + Rg\Big(\frac{t}{R}\Big)w, \tag{2}$$

where functions f and g are given by $f(t) = \cos t$ and $g(t) = \sin t$ when $C > 0$, or $f(t) = \cosh t$ and $g(t) = \sinh t$ when $C < 0$. Note that $f^2 + \varepsilon g^2 = 1$, $f' = -\varepsilon g$ and $g' = f$.

The following result characterizes the concircular vector fields.

Theorem 1. *A vector field $V \in \mathfrak{X}(\mathbb{M}^n(C))$ is concircular if and only if V is the tangential part of a constant vector field p_0 in $\mathbb{R}^{n+1}_\nu$. Moreover, if μ is the concircular factor of V, then $V = p_0 + \mu\phi$, where $\mu = -C\langle p_0, \phi\rangle$.*

Proof. The curvature tensor of $\mathbb{M}^n(C)$ is given by

$$R_{XY}Z = \overline{\nabla}_{[X,Y]}Z - \overline{\nabla}_X(\overline{\nabla}_Y Z) + \overline{\nabla}_Y(\overline{\nabla}_X Z).$$

Then, if V is a concircular vector field with concircular factor μ, we have

$$R_{XV}V = V(\mu)X - X(\mu)V. \tag{3}$$

On the other hand, since $\mathbb{M}^n(C)$ is a space of constant curvature C, its curvature tensor is given by

$$R_{XV}V = C\{\langle V, X\rangle V - \langle V, V\rangle X\}. \tag{4}$$

By assuming that X and V are two linearly independent vector fields, from (3) and (4), we get $-C\langle V, X\rangle = X(\mu)$ and $-C\langle V, V\rangle = V(\mu)$, and therefore

$$-CV = \nabla\mu. \tag{5}$$

Take the vector field $\psi = V - \mu\phi$, then

$$\nabla^0_X\psi = \overline{\nabla}_X V - C\langle X, V\rangle\phi - X(\mu)\phi - \mu X.$$

From here and again (5) we get ψ is constant, and so there exists a constant vector field $p_0 \in \mathbb{R}^{n+1}_\nu$ such that

$$p_0 = V - \mu\phi, \quad \text{with } \mu = -C\langle p_0, \phi\rangle. \tag{6}$$

Conversely, let $V = \{p_0\}^\top$ be the tangential part of a constant vector in $\mathbb{R}^{n+1}_\nu$. Then we have (6), and by derivating there, we get $0 = \overline{\nabla}_X V - C\langle X, V\rangle\phi - X(\mu)\phi - \mu X$, where X is any tangent vector field in $\mathbb{M}^n(C)$. Hence $\overline{\nabla}_X V = \mu X$ for any X, so that V is a concircular vector field with concircular factor μ. □

As a consequence of (5), we have the following result:

Corollary 2. *In a space form $\mathbb{M}^n(C)$ of nonzero curvature C, the concircular factor is a nonconstant function.*

Proposition 3. *The set $\mathrm{Con}(\mathbb{M}^n(C))$ of all concircular vector fields of $\mathbb{M}^n(C)$ is a real vector space of dimension $n + 1$.*

Proof. It is sufficient to show that each concircular vector field V is determined by one, and only one, constant vector p_0. Let us suppose that V is determined by two constant vectors p_0 and q_0, that is, $V = p_0 + \mu_1\phi$ and $V = q_0 + \mu_2\phi$, for certain differentiable functions μ_1 and μ_2. Then $p_0 - q_0 = (\mu_2 - \mu_1)\phi$, and so $\mu_1 = \mu_2$ and $p_0 = q_0$. □

3. Concircular Hypersurfaces

To begin with, let us show some examples of concircular hypersurfaces in $\mathbb{M}^n(C)$.

Example 1. A *totally umbilical hypersurface* $Q^{n-1}(c)$ in $\mathbb{M}^n(C)$ can be obtained as the intersection $\mathbb{M}^n(C) \cap H(p_0)$, where $H(p_0)$ is a hyperplane in $\mathbb{R}^{n+1}_\nu$ orthogonal to a constant vector $p_0 \in \mathbb{R}^{n+1}_\nu$. If N denotes the unit vector field normal to $Q^{n-1}(c)$ in $\mathbb{M}^n(C)$, then $\langle N, V\rangle = \langle N, p_0\rangle$, where $V = p_0 - C\langle p_0, \phi\rangle\,\phi$. By derivating here, we get $X\langle N, p_0\rangle = \langle -AX, p_0\rangle = 0$, for any vector field X tangent to $Q^{n-1}(c)$, where A denotes the shape operator associated to N. Hence $\langle N, V\rangle$ is constant and so $Q^{n-1}(c)$ is a concircular hypersurface. We will say a totally umbilical hypersurface is a *trivial concircular hypersurface.*

Example 2. A *conical hypersurface* M in $\mathbb{M}^n(C)$ (with vertex at $p_0 \in \mathbb{M}^n(C)$) can be described as follows: Let P^{n-2} be an $(n-2)$-dimensional hypersurface in the unit hypersphere $S^{n-1}(1)$ of the tangent space $T_{p_0}\mathbb{M}^n(C)$. For $\epsilon > 0$ sufficiently small, the map $\Psi : P^{n-2} \times (-\epsilon, \epsilon) \to \mathbb{M}^n(C)$ given by

$$\Psi(v, t) = \exp_{p_0}(tv) = f\Big(\frac{t}{R}\Big)p_0 + Rg\Big(\frac{t}{R}\Big)v,$$

defines an immersion. The image $M = \Psi(P^{n-2} \times (-\epsilon, \epsilon))$ is said to be a conical hypersurface in $\mathbb{M}^n(C)$ (see [6,7] in the case $n = 3$). We can identify in a natural way P^{n-2} with $P^{n-2} \times \{0\}$ and $P^{n-2} \times (-\epsilon, \epsilon)$ with M, and then the unit vector field normal to M in $\mathbb{M}^n(C)$ is given without loss of generality by $N(v, t) = \eta(v)$, η being the unit vector field normal to P^{n-2} in $S^{n-1}(1)$. Hence, $\langle N, V\rangle = 0$, for $V = p_0 - C\langle p_0, \phi\rangle\,\phi$, showing that M is a concircular hypersurface.

Before addressing the characterization of the concircular hypersurfaces, we will present a couple of results.

Proposition 4. *Given a hypersurface $M \subset \mathbb{M}^n(C)$, then there exists a concircular vector field parallel to its normal vector field along M if and only if M is a totally umbilical hypersurface in $\mathbb{M}^n(C)$.*

Proof. Suppose there exists a concircular vector field V such that $V|_M = \lambda N$, for a nonzero differentiable function λ. Then we get $\mu X = X(\lambda)N - \lambda AX$, for any vector field X tangent to M, A being the shape operator associated to N. From that equation, we have that λ is a nonzero constant and $AX = -(\mu/\lambda)X$. Now, from (5), we get $X(\mu) = 0$ for any tangent vector field X, and so M is a totally umbilical hypersurface.

Conversely, if M is a totally umbilical hypersurface, then there exists a constant m such that $\overline{\nabla}_X N = -AX = -m\,X$, and then the vector field $N+m\phi$ is constant along M. This shows that N is collinear with a concircular vector field along M. □

Let $M \subset \mathbb{M}^n(C)$ be a nontrivial concircular hypersurface with axis V, and write $\langle V, N\rangle = \lambda$, λ being a constant. By decomposing V in its tangential and normal components, we have

$$V|_M = \alpha\, T + \lambda\, N, \tag{7}$$

where T is a unit vector field tangent to M and $\alpha \neq 0$ (otherwise, M would be a trivial concircular hypersurface).

Proposition 5. *The integral curves of T are geodesics in $\mathbb{M}^n(C)$.*

Proof. By derivating (7) and using the Gauss and Weingarten equations, we obtain

$$\mu X = X(\alpha)T + \alpha\nabla_X T + \alpha\sigma(X,T) - \lambda AX, \tag{8}$$

where ∇ and σ denote the Levi–Civita connection and the second fundamental form of M. From here, we get $\sigma(X,T) = 0$, for any tangent vector field X, or equivalently

$$AT = 0. \tag{9}$$

By putting $X = T$ in (8) and using (9), we obtain

$$\nabla_T T = 0. \tag{10}$$

From here and (9), we deduce the result. □

In what follows, we will characterize nontrivial concircular hypersurfaces in $\mathbb{M}^n(C)$. Let $Q^{n-1}(c) = H(p_0)\cap\mathbb{M}^n(C)$ be a totally umbilical hypersurface of constant curvature c, where $H(p_0)$ is a hyperplane in $\mathbb{R}^{n+1}_\nu$ orthogonal to a unit vector $p_0 \in \mathbb{R}^{n+1}_\nu$. Let P^{n-2} be a hypersurface of $Q^{n-1}(c)$, and denote by η_1 and η_2 the unit vector fields normal to P^{n-2} in $Q^{n-1}(c)$ and normal to $Q^{n-1}(c)$ in $\mathbb{M}^n(C)$, respectively. For a real number a, define the unit vector field $W_a(p) = \cos(a)\;\eta_1(p) + \sin(a)\;\eta_2(p)$, where $p \in P^{n-2}$. The map $\Psi_a : P^{n-2} \times I_0 \to \mathbb{M}^n(C)$ given by

$$\Psi_a(p,z) = \exp_p(zW_a(p)) = f\Big(\frac{z}{R}\Big)p + Rg\Big(\frac{z}{R}\Big)W_a(p),$$

defines an immersion, for an enough small interval I_0 around the origin. Let M denote the ruled hypersurface in $\mathbb{M}^n(C)$ given by $\Psi_a(P^{n-2}\times I_0)$. We can identify, in a natural way, P^{n-2} with $\Psi_a(P^{n-2}\times\{0\})$ and $P^{n-2}\times I_0$ with M. Without loss of generality, we can assume that the unit vector field normal to M in $\mathbb{M}^n(C)$ is given by

$$N(p,z) = -\sin(a)\;\eta_1(p) + \cos(a)\;\eta_2(p).$$

Since $P^{n-2} \subset H(p_0)\cap\mathbb{M}^n(C)$, it is not difficult to see that $p_0 \in \text{span}\{\eta_2(p), \phi(p)\}$, for any $p \in P^{n-2}$, so we can write $p_0 = A\eta_2 + B\phi$, for two differentiable functions A and B. By taking derivative here, and using that p_0 is constant, we get that A and B are also constants. Let us consider the concircular vector field in $\mathbb{M}^n(C)$ given by $V = p_0 - C\,\langle p_0, \phi\rangle\,\phi$. It is not difficult to see that $\langle V, N\rangle = \langle p_0, N\rangle = A\cos(a)$ is constant, and so M is a concircular hypersurface.

Note that a hypersurface M in $\mathbb{M}^n(C)$ is concircular if and only if there exists a point $p_0 \in \mathbb{R}^{n+1}_\nu$ such that $\langle N, p_0\rangle$ is constant, N being the unit normal vector field of M in $\mathbb{M}^n(C)$.

The main result of this section is the following. We will show that every nontrivial concircular hypersurface in $\mathbb{M}^n(C)$ can be obtained by the construction described above.

Theorem 6. *Let $M \subset \mathbb{M}^n(C)$ be a nontrivial concircular hypersurface with axis V. Then there exists a hypersurface P^{n-2} in a totally umbilical hypersurface $Q^{n-1}(c) \subset \mathbb{M}^n(C)$, such that M can be locally described by*

$$\Psi_a(p, z) = \exp_p(zW_a(p)) = f\Big(\frac{z}{R}\Big)p + Rg\Big(\frac{z}{R}\Big)W_a(p), \tag{11}$$

where $a \in \mathbb{R}$ and $(p, z) \in P^{n-2} \times I_0$, I_0 being an interval around the origin.

Proof. Let us suppose that the axis V of M is given by $V = p_0 - C\langle p_0, \phi\rangle\,\phi$, for a constant vector $p_0 \in \mathbb{R}^{n+1}_\nu$, and assume $\langle V, N\rangle = \lambda$. Pick a point q in M and let $Q^{n-1}(c) = H(p_0) \cap \mathbb{M}^n(C)$ be a totally umbilical hypersurface containing q, $H(p_0)$ being a hyperplane in $\mathbb{R}^{n+1}_\nu$ orthogonal to p_0. Since M is a nontrivial concircular hypersurface, then there is an $(n-2)$-dimensional submanifold $P^{n-2} \subset Q^{n-1}(c) \cap M$ with $q \in P^{n-2}$.

Let T be the unit vector field tangent to M which is collinear with the tangential component of V. From Proposition 5 we deduce that there exists a neighborhood $U(q)$ of q in M given by

$$U(q) = \{f\left(\tfrac{z}{R}\right)p + R\,g\left(\tfrac{z}{R}\right)T(p) \mid p \in U_1(q),\ z \in (-\varepsilon_2, \varepsilon_2)\},$$

where $U_1(q)$ is an neighborhood of q in P^{n-2}. Since T is orthogonal to P^{n-2}, but tangent to M, then there exist a differentiable function $a \in \mathcal{C}^\infty(P^{n-2})$ such that

$$T(p) = \cos(a(p))\,\eta_1(p) + \sin(a(p))\,\eta_2(p),$$

and, up to the sign, the unit normal vector field N along P^{n-2} is given by

$$N(p) = -\sin(a(p))\,\eta_1(p) + \cos(a(p))\,\eta_2(p).$$

Then we have $\lambda = \langle V, N\rangle(p) = \langle p_0, N\rangle(p) = A\cos(a(p))$ is constant and so a is a constant function. Hence the open set $U(q)$ can be rewritten as in (11). □

In Example 2, we have seen that the conical hypersurfaces are concircular hypersurfaces associated to the constant value $\lambda = 0$. Now, we will prove that these hypersurfaces are the only ones that satisfy this property.

Proposition 7. *Let $M \subset \mathbb{M}^n(C)$ be a concircular hypersurface, with axis V, such that $\langle V, N\rangle = 0$, N being the unit normal vector field. Then M is a conical hypersurface.*

Proof. From Theorem 6, we know that M can be locally described by

$$\Psi_a(p,z) = f\Big(\frac{z}{R}\Big)p + Rg\Big(\frac{z}{R}\Big)W_a(p), \quad W_a(p) = \cos(a)\,\eta_1(p) + \sin(a)\,\eta_2(p),$$

where $(p,z) \in P^{n-2} \times I_0$, for certain $P^{n-2} \subset Q^{n-1}(c) \subset \mathbb{M}^n(C)$, $Q^{n-1}(c) = H(p_0) \cap \mathbb{M}^n(C)$ and $I_0 \subset \mathbb{R}$. Suppose that the interval I_0 is the largest possible interval.

The point p_0 can be written as $p_0 = A\eta_2 + B\phi$, for certain constants A and B related by $Ak - B = 0$, where $k = \sqrt{|c - C|}$. From the equality $\langle V, N\rangle = A\cos(a)$, we get $\cos(a) = 0$ (since A cannot vanish) and then $W_a(p) = \eta_2(p)$. Take $z_0 \in \mathbb{R}$ such that $f(z_0/R) = kRg(z_0/R)$, and define a differentiable function $\varphi : P^{n-2} \to \mathbb{R}^{n+1}_\nu$ by $\varphi(p) = \Psi_a(p, z_0)$. A straightforward computation yields

$$d\varphi_p(v) = \Big[f\Big(\frac{z_0}{R}\Big) - kRg\Big(\frac{z_0}{R}\Big)\Big]\, v = 0,$$

for any $p \in P^{n-2}$ and $v \in T_pP^{n-2}$. Hence, φ is a constant $q_0 \in \mathbb{R}^{n+1}_\nu$ and this shows that M is a conical hypersurface with vertex at q_0. □

4. Concircular Helices in $\mathbb{M}^3(C)$

We begin this section by showing a couple of examples of concircular helices in $\mathbb{M}^3(C)$. Let $\gamma : I \to \mathbb{M}^3(C)$ be an arclength parametrized curve satisfying the following Frenet–Serret equations:

$$\begin{aligned} T'_\gamma(s) &= \nabla^0_{T_\gamma} T_\gamma(s) = -C\gamma(s) + \kappa_\gamma(s)\, N_\gamma(s), \\ N'_\gamma(s) &= \nabla^0_{T_\gamma} N_\gamma(s) = -\kappa_\gamma(s)\, T_\gamma(s) + \tau_\gamma(s)\, B_\gamma(s), \\ B'_\gamma(s) &= \nabla^0_{T_\gamma} B_\gamma(s) = -\tau_\gamma(s)\, N_\gamma(s), \end{aligned} \tag{12}$$

where ∇^0 denotes the Levi–Civita connection on $\mathbb{R}^4_\nu$. As usual, κ_γ and τ_γ are called the curvature and torsion of γ.

Example 3. *Planar curves*, i.e., curves γ with zero torsion. These curves live in a surface $M^2(C)$ totally geodesic in $\mathbb{M}^3(C)$. That means there is a constant vector p_0 in $\mathbb{R}^4_\nu$ such that $M^2(C) = \mathbb{M}^3(C) \cap H_0(p_0)$, where $H_0(p_0)$ is the hyperplane through the origin orthogonal to p_0. Since $\langle N_\gamma, V\rangle = 0$, for $V = p_0 - C\,\langle p_0, \phi\rangle\,\phi$, γ is a concircular helix.

Example 4. *Rectifying curves*, i.e., curves for which there exists a point p_0 in $\mathbb{M}^3(C)$ such that the geodesics connecting p_0 with $\gamma(s)$ are orthogonal to the principal normal geodesic starting from $\gamma(s)$, see [6,7]. The nonplanar rectifying curves are characterized by the condition $\langle N_\gamma, p_0\rangle = 0$. Then γ is a concircular helix since $\langle N_\gamma, V\rangle = 0$ for the concircular vector field $V = p_0 - C\,\langle p_0, \phi\rangle\,\phi$.

In the following result, we show that the restriction to a curve γ of a concircular vector field V is a vector field along γ satisfying a property similar to that of concircular vector fields. For this reason, such a vector field will be called a *concircular vector field along a curve.*

Proposition 8. *Let $\gamma : I \to \mathbb{M}^n(C)$ be a differentiable curve and consider a vector field $\mathbf{v}$ along γ. Then $\mathbf{v}$ is the restriction to γ of a concircular vector field V on $\mathbb{M}^n(C)$ if and only if $\frac{\overline{D}\mathbf{v}}{dt} = \omega\, T_\gamma$, where $\omega : I \to \mathbb{R}$ is a differentiable function with $\omega' = -C\,\langle \mathbf{v}, T_\gamma\rangle$, $\overline{D}$ being the covariant derivative along γ.*

Proof. Let us assume that V is an extension of $\mathbf{v}$ to $\mathbb{M}^n(C)$ which is a concircular vector field. Then there exists a constant vector $p_0 \in \mathbb{R}^{n+1}_\nu$ such that $p_0 = V + C\,\langle p_0, \phi\rangle\,\phi$. If X is a local extension of the tangent vector T_γ then

$$\frac{\overline{D}\mathbf{v}}{dt} = \overline{\nabla}_X V|_\gamma = (\mu X)|_\gamma = \omega T_\gamma,$$

where the differentiable function $\omega : I \to \mathbb{R}$ is the restriction of μ along γ. On the other hand, from Theorem 1 we have $\mu = -C\,\langle p_0, \phi\rangle$, and then $\omega = -C\,\langle p_0, \gamma\rangle$ and $\omega' = -C\,\langle \mathbf{v}, T_\gamma\rangle$.

To prove the converse, let us consider the vector field Y along γ given by $Y = \mathbf{v} - \omega\gamma$. By derivating here with the Euclidean derivative, we obtain that Y is a constant vector $p_0 \in \mathbb{R}^{n+1}_\nu$ along γ. By defining $V = p_0 - C\,\langle p_0, \phi\rangle\,\phi$ and bearing Theorem 1 in mind, we deduce $\mathbf{v}$ is the restriction to γ of a concircular vector field. □

For simplicity, and since $\omega(s) = \mu(\gamma(s))$, in what follows we will use $\mu(s)$ instead of $\omega(s)$. In the following, we will characterize the concircular helices in $\mathbb{M}^3(C)$. Note that γ is a concircular helix if and only if there exists a point $p_0 \in \mathbb{R}^4_\nu$ such that $\langle N_\gamma, p_0\rangle = \lambda$ constant.

Since the case $\lambda = 0$ reduces to planar or rectifying curves, we will exclude this case from our study. A concircular helix γ (with axis V) in $\mathbb{M}^3(C)$ is said to be *proper* if γ is a nonplanar curve with $\lambda \neq 0$, λ being the constant function $\langle N_\gamma, V\rangle$.

Let $\gamma(s) \subset \mathbb{M}^3(C) \subset \mathbb{R}^4_\nu$ be an arclength parametrized concircular helix, and suppose it is a proper one. From Proposition 8, we can write

$$\mathbf{v}(s) = V(\gamma(s)) = t(s)\,T_\gamma(s) + \lambda\,N_\gamma(s) + z(s)\,B_\gamma(s), \tag{13}$$

for certain differentiable functions t and z. To simplify the writing, we will eliminate the s parameter. By derivating in (13), we get

$$\mu\,T_\gamma = (t' - \lambda\kappa_\gamma)\,T_\gamma + (t\kappa_\gamma - z\tau_\gamma)\,N_\gamma + (z' + \lambda\tau_\gamma)\,B_\gamma,$$

and then

$$t' - \lambda\kappa_\gamma = \mu, \quad t\kappa_\gamma - z\tau_\gamma = 0, \quad z' + \lambda\tau_\gamma = 0. \tag{14}$$

Note that $z \neq 0$ since γ is a proper concircular helix. Now we distinguish two cases, according to t/z (called the *rectifying slope* of γ) is a constant or not.

Case 1: the rectifying slope t/z of γ is a constant function. Then the Lancret curvature $\rho = \tau_\gamma/\kappa_\gamma$ is constant, and using the first and third equations of (14), we get

$$\kappa_\gamma = \frac{-1}{\lambda(1+\rho^2)}\mu \qquad \text{and} \qquad \tau_\gamma = \rho\kappa_\gamma, \tag{15}$$

where the function μ satisfies

$$\mu'' + C\frac{\rho^2}{1+\rho^2}\mu = 0. \tag{16}$$

Now we will show that Eqs. (15) and (16) characterize proper concircular helices with constant rectifying slope. Let γ be a curve satisfying these two equations. Define a function z by

$$z = -\frac{\mu'}{C\rho},$$

and consider the vector field $\mathbf{v} = z\, D_\gamma + \lambda N_\gamma$, where $D_\gamma = \rho T_\gamma + B_\gamma$ is the modified Darboux vector and $\lambda = -1/(m(1+\rho^2))$. Then

$$\overline{\nabla}_{T_\gamma}\mathbf{v} = \frac{\rho}{1+\rho^2}\mu\,(\rho T_\gamma + B_\gamma) - \lambda m\mu T_\gamma + \lambda\rho m\mu B_\gamma = \mu T_\gamma.$$

Since $\mu' = -C\,\langle \mathbf{v}, T_\gamma\rangle$, from Proposition 8 we get γ is a concircular helix.

Hence, we have shown the following result:

Proposition 9. *Let γ be an arclength parametrized nonplanar curve in $\mathbb{M}^3(C)$. Then γ is a proper concircular helix with constant rectifying slope if and only if its curvature and torsion are given by $\kappa_\gamma = m\mu$ and $\tau_\gamma = \rho\kappa_\gamma$, where m and ρ are nonzero constants and the function μ satisfies (16).*

Case 2: the rectifying slope t/z is a nonconstant function. From the first equation of (14), bearing in mind that $\mu' = -C\,\langle \mathbf{v}, T_\gamma\rangle$, we have

$$\mu'' + C\mu = -C\lambda\kappa_\gamma, \tag{17}$$

and then the second and third equations of (14) lead to

$$\left(\frac{\mu'}{\rho}\right)' = C\lambda\tau_\gamma. \tag{18}$$

As before, we will show that equations (17) and (18) characterize concircular helices (when ρ is a nonconstant function). Let γ be an arclength parametrized curve satisfying (17) and (18), for a constant λ and a differentiable function μ. Define the functions

$$t = -\frac{\mu'}{C} \qquad \text{and} \qquad z = \frac{t}{\rho}, \tag{19}$$

and consider the vector field $\mathbf{v}$ along γ given by

$$\mathbf{v} = zD_\gamma + \lambda N_\gamma = t\,T_\gamma + \lambda\,N_\gamma + z\,B_\gamma. \tag{20}$$

From (17), we get $t' - \lambda\kappa_\gamma = \mu$, and from (18), we obtain $z' = -\lambda\tau_\gamma$. Then by derivating in (20), we have

$$\overline{\nabla}_{T_\gamma}\mathbf{v} = (t' - \lambda\kappa_\gamma)T_\gamma + (t\kappa_\gamma - z\tau_\gamma)N_\gamma + (\lambda\tau_\gamma + z')B_\gamma = \mu\,T_\gamma.$$

Therefore, since $\mu' = -C\,\langle \mathbf{v}, T_\gamma\rangle$, from Proposition 8, we deduce γ is a concircular helix.

In conclusion, putting cases 1 and 2 together, we have proved the following result. Note that equations (17) and (18) are equivalent to (15) and (16) in the case ρ constant.

Theorem 10. *Let γ be an arclength parametrized nonplanar curve in $\mathbb{M}^3(C)$. Then γ is a proper concircular helix if and only if Eqs. (17) and (18) are satisfied, for a constant $\lambda \in \mathbb{R}$ and a differentiable function μ. Moreover, the* *axis V of γ is the extension of the vector field $\mathbf{v}$ given in (20), t and z being the differentiable functions given in (19).*

5. Geodesics of Concircular Surfaces

Let M be a nontrivial concircular surface in $\mathbb{M}^3(C)$ with axis V, and let us consider $\gamma(s)$ an arclength parametrized geodesic of M. From Theorem 6, we can assume that γ is locally written as $X(t(s), z(s))$, where X is the parametrization (11). Here, the submanifold P^{n-2} of Theorem 6 is a curve $\delta(t)$. Since the principal normal vector field N_γ of γ is collinear with the unit normal vector field N of M in $\mathbb{M}^3(C)$, then $\langle N_\gamma, V|_\gamma\rangle$ is constant, and then γ is a concircular helix in $\mathbb{M}^3(C)$. The goal of this section is to prove the converse.

First, we are going to obtain the equations of geodesics in a concircular surface M. Let $\gamma(s) = X\big(t(s), z(s)\big)$ be an arclength parametrized geodesic of M, with $\kappa_\gamma > 0$. Then $T_\gamma(s) = t'(s)X_t(t(s), z(s)) + z'(s)X_z(t(s), z(s))$ and so there exists a differentiable function θ such that

$$t'(s)\sqrt{E(t(s), z(s))} = \sin\theta(s), \tag{21}$$

$$z'(s) = \cos\theta(s). \tag{22}$$

Hence $T_\gamma(s) = \sin\theta(s)\, T_\delta(t(s)) + \cos\theta(s)\, X_z(t(s), z(s))$, and by taking derivative here, we get

$$\begin{aligned} -C\gamma(s) + \kappa_\gamma(s)\, N_\gamma(s) = {} & \theta'(s)\big(\cos\theta(s)T_\delta(t(s)) - \sin\theta(s)X_z(t(s), z(s))\big) \\ & + \sin\theta(s)t'(s)\big(k\eta(t(s)) - C\delta(t(s)) + \kappa_\delta(t(s))N_\delta(t(s))\big) \\ & + \cos\theta(s)\big(t'(s)X_{tz}(t(s), z(s)) + z'(s)X_{zz}(t(s), z(s))\big). \end{aligned} \tag{23}$$

Bearing in mind that $\{T_\delta,\ X_z,\ N,\ \frac{1}{R}\gamma\}$ is an orthonormal frame of $\mathbb{R}^4_\nu$ along γ, we have

$$\begin{aligned} \delta(t(s)) &= f\big(\tfrac{z(s)}{R}\big)\gamma(s) - Rg\big(\tfrac{z(s)}{R}\big)X_z(t(s), z(s)), \\ N_\delta(t(s)) &= -\sin(a)N(t(s), z(s)) + \cos(a)\Big(f\big(\tfrac{z(s)}{R}\big)X_z(t(s), z(s)) + \tfrac{\varepsilon}{R}g\big(\tfrac{z(s)}{R}\big)\gamma(s)\Big), \\ \eta(t(s)) &= \cos(a)N(t(s), z(s)) + \sin(a)\Big(f\big(\tfrac{z(s)}{R}\big)X_z(t(s), z(s)) + \tfrac{\varepsilon}{R}g\big(\tfrac{z(s)}{R}\big)\gamma(s)\Big), \\ X_{tz}(t(s), z(s)) &= -\Big(\tfrac{\varepsilon}{R}g\big(\tfrac{z(s)}{R}\big) + f\big(\tfrac{z(s)}{R}\big)\big(\sin(a)k + \cos(a)\kappa_\delta(t(s))\big)\Big)T_\delta(t(s)), \\ X_{zz}(t(s), z(s)) &= -C\gamma(s). \end{aligned}$$

From these equations, jointly with (23) and the fact that γ is a geodesic in M (and so $N_\gamma(s) = N(\gamma(s))$; the case $N_\gamma(s) = -N(\gamma(s))$ is similar), we deduce

$$\theta'(s) = t'(s)\Big(CRg\big(\tfrac{z(s)}{R}\big) + f\big(\tfrac{z(s)}{R}\big)\big(\sin(a)k + \cos(a)\kappa_\delta(t(s))\big)\Big), \tag{24}$$

$$\kappa_\gamma(s) = \sin\theta(s)t'(s)\big(k\cos(a) - \sin(a)\kappa_\delta(t(s))\big). \tag{25}$$

On the other hand, it is easy to see that $B_\gamma(s) = \cos\theta(s)T_\delta(t(s)) - \sin\theta(s)$ $X_z(t(s), z(s))$ and by taking derivative here, we have

$$\tau_\gamma(s) = \cos\theta(s)t'(s)\big(-\cos(a)k + \sin(a)\kappa_\delta(t(s))\big). \tag{26}$$

Hence, we have shown the following result:

Proposition 11. *Let M be a nontrivial concircular surface in $\mathbb{M}^3(C)$, locally parametrized by (11). An arclength parametrized curve $\gamma(s) = X\big(t(s), z(s)\big)$, with $\kappa_\gamma > 0$, is a geodesic if and only if there is a differentiable function $\theta(s)$ such that* Eqs. (21), (22) *and* (24) *are satisfied. Moreover, the curvature and torsion of γ are given by* (25) *and* (26)*, respectively.*

We finish this section with the following characterization of concircular helices in $\mathbb{M}^3(C)$.

Theorem 12. *Let $\gamma(s)$ be an arclength parametrized curve in $\mathbb{M}^3(C)$, $\kappa_\gamma > 0$. Then γ is a proper concircular helix if and only if γ is (congruent to) a geodesic of a proper concircular surface.*

Proof. We need only prove the direct implication. Let $\gamma(s)$ be a arclength parametrized proper concircular helix in $\mathbb{M}^3(C)$ with axis $V = p_0 - C\,\langle p_0, \phi\rangle\,\phi$, such that $\langle N_\gamma, V\rangle = \lambda$ is constant along γ. Let M be the ruled surface with base curve γ and director curve D_γ (the unit Darboux vector field of γ), which can be parametrized as follows:

$$X(s,z) = f\Big(\frac{z}{R}\Big)\gamma(s) + R\,g\Big(\frac{z}{R}\Big)\,\left(\frac{\rho(s)}{\sqrt{1+\rho(s)^2}}T_\gamma(s) + \frac{1}{\sqrt{1+\rho(s)^2}}B_\gamma(s)\right). \tag{27}$$

Since $X_s \in \text{span}\{T_\gamma, B_\gamma\}$ and $X_z \in \text{span}\{\gamma,\, T_\gamma,\, B_\gamma\}$, we obtain that the unit normal vector field N is collinear with the principal normal vector field N_γ. From here, we conclude that M is a concircular surface in $\mathbb{M}^3(C)$ with axis V, and that γ is a geodesic in M. This concludes the proof. □

Acknowledgements

This research is part of the grant PID2021-124157NB-I00, funded by MCIN/ AEI/ 10.13039/ 501100011033/ "ERDF A way of making Europe". Also supported by "Ayudas a proyectos para el desarrollo de investigación científica y técnica por grupos competitivos", included in the "Programa Regional de Fomento de la Investigación Científica y Técnica (Plan de Actuación 2022)" of the Fundación Séneca-Agencia de Ciencia y Tecnología de la Región de Murcia, Ref. 21899/PI/22.

Author contributions Both authors have collaborated equally in the writing of the manuscript.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors declare no competing interests.

References

[1] Chen, B.Y.: When does the position vector of a space curve always lie in its rectifying plane? Am. Math. Mon. **110**, 147–152 (2003)

[2] Chen, B.Y.: Rectifying curves and geodesics on a cone in the Euclidean 3-space. Tamkang J. Math. **48**, 209–214 (2017)

[3] Fialkow, A.: Conformals geodesics. Trans. Am. Math. Soc. **45**(3), 443–473 (1939)

[4] Kim, I.B.: Special concircular vector fields in Riemannian manifolds. Hirosima Math. J. **12**, 77–91 (1982)

[5] Izumiya, S., Takeuchi, N.: New special curves and developable surfaces. Turk. J. Math. **28**, 153–163 (2004)

[6] Lucas, P., Ortega-Yagües, J.A.: Rectifying curves in the three-dimensional sphere. J. Math. Anal. Appl. **421**, 1855–1868 (2015)

[7] Lucas, P., Ortega-Yagües, J.A.: Rectifying curves in the three-dimensional hyperbolic space. Mediterr. J. Math. **13**, 2199–2214 (2016)

[8] Lucas, P., Ortega-Yagües, J.A.: Slant helices in the Euclidean 3-space revisited. Bull. Belg. Math. Soc. Simon Stevin **23**, 133–150 (2016)

[9] Lucas, P., Ortega-Yagües, J.A.: Concircular helices and concircular surfaces in Euclidean 3-space $\mathbb{R}^3$. Hacet. J. Math. Stat. **52**, 995–1005 (2023)

[10] Yano, K.: Concircular geometry I, concircular transformations. Proc. Imp. Acad. Tokyo **16**, 195–200 (1940)

Pascual Lucas and José Antonio Ortega-Yagües
Departamento de Matemáticas
Universidad de Murcia, Campus de Espinardo
30100 Murcia
Spain
e-mail: plucas@um.es

José Antonio Ortega-Yagües
e-mail: yagues1974@hotmail.com

Received: May 26, 2023.
Revised: August 24, 2023.
Accepted: September 26, 2023.