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Two Spheres Uniquely Determine Infrabimonogenic Functions

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Abstract. In this paper we consider the problem of characterizing the sets of uniqueness for the solutions of the sandwich equation $\partial_{\underline{x}}^3 f \partial_{\underline{x}} = 0$, where $\partial_{\underline{x}}$ stands for the Dirac operator in $\mathbb{R}^m$. These solutions are referred to as infrabimonogenic functions and can be viewed as a non-commutative version of biharmonic functions. Our main result states that a pair of distinct spheres is a set of uniqueness for infrabimonogenic functions in a convex domain of an odd-dimensional space.

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1. Introduction

Let f be a real-valued function defined in a domain Ω of the Euclidean space $\mathbb{R}^m$. If f is $2k$-times continuously differentiable and

$$\Delta^k f(\underline{x}) = 0, \text{ for all } \underline{x} \in \Omega,$$

then f is called polyharmonic of degree k. Here and in the sequel Δ^k denotes the k-th iteration of the Laplace operator Δ.

Polyharmonic functions play an important role in pure and applied mathematics. In particular, for $k = 2$, biharmonic functions are specially important in elasticity theory.

The real Clifford algebra, $\mathbb{R}_{0,m}$ is generated by the orthonormal basis vectors $e_1, e_2, \ldots e_m$ of the Euclidean space $\mathbb{R}^m$; with the relations

$$e_i^2 = -1, \, e_i e_j = -e_j e_i, \, i, j = 1, 2, \ldots m, \, i < j.$$

We will consider functions defined on subsets of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$. An $\mathbb{R}_{0,m}$-valued function f is called left monogenic (right monogenic) in Ω if $\partial_{\underline{x}} f = 0$ ($f \partial_{\underline{x}} = 0$) in Ω, where $\partial_{\underline{x}}$ stands for the so-called Dirac operator

$$\partial_{\underline{x}} = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \cdots + \partial_{x_m} e_m.$$

It should be noticed that $\partial_{\underline{x}}^2 = \partial_{\underline{x}} \partial_{\underline{x}} = -\Delta$.

Functions that are both left and right monogenic are called two-sided monogenic (see for example [2,7,8]). More generally, an $\mathbb{R}_{0,m}$-valued function f in $C^k(\Omega)$ is called left polymonogenic of order k, or simply k-monogenic (left) if $\partial_{\underline{x}}^k f = 0$ in Ω. With regard to this generalization references may be made to [3,4,15,18].

As a natural consequence of the non-commutativity of the Clifford product, a new class of functions arises, which can be seen as a non-commutative version of harmonic functions, namely the solutions of the sandwich equation $\partial_{\underline{x}} f \partial_{\underline{x}} = 0$. Such functions are termed inframonogenic (see [16,17]) and represent a refinement of the much more recognized biharmonic ones. Interesting connections have been found in [12–14], between them and the solutions of the Lamé–Navier system in linear elasticity theory. Infrapolymonogenic functions, introduced in [1], are the solutions of the generalized sandwich equation $\partial_{\underline{x}}^{2k-1} f \partial_{\underline{x}} = 0$, with $k \in \mathbb{N}$.

Infrabimonogenic functions are, in particular, the $\mathbb{R}_{0,m}$-valued solutions of the fourth-order partial differential equation $\partial_{\underline{x}}^3 f \partial_{\underline{x}} = 0$ (a non-commutative version of the biharmonic equation). As easily seen, a function f is infrabimonogenic if and only if its Laplacian Δf is inframonogenic, or, equivalently, if and only if $\partial_{\underline{x}} f \partial_{\underline{x}}$ is harmonic.

In [6], Edenhoffer proved that a polyharmonic function of order k is completely determined by its values on k concentric spheres. This result was extended by Hayman and Korenblum, who proved in [9] that the concentric assumption may be removed.

As proved in [10], one sphere represents a set of uniqueness for inframonogenic functions in odd-dimensional spaces. However, in even-dimensional spaces, it is possible to construct non-zero inframonogenic functions whose restrictions vanish on a sphere. More recently in [11] has been proved that k distinct concentric spheres is a set of uniqueness for infrapolymonogenic functions in odd dimensional spaces. The main result of this article ensures that, in the case of infrabimonogenic functions, the above concentric requirement may be removed in convex domains. Indeed, we prove that two distinct spheres in a convex domain uniquely determine an infrabimonogenic function. Incidentally, a characterization of sets of uniqueness for a generalized Lamé–Navier equation has been derived.

2. Auxiliary Results

In this section, some definitions and basic properties of a Clifford algebra will be recalled. Besides, we will provide some auxiliary results before stating the main theorems. The elements of $\mathbb{R}_{0,m}$ will be described in the form $a = \sum_A a_A e_A$, where as indices the elements A of the set containing the ordered subsets of $\{1, 2, \ldots, m\}$ will be used, with the empty subset corresponding to the index 0. An arbitrary element $a \in \mathbb{R}_{0,m}$ may be written in a unique way as

$$a = [a]_0 + [a]_1 + \cdots + [a]_m, \tag{1}$$

where $[\,]_p$ denotes the projection of $\mathbb{R}_{0,m}$ onto the subspace $\mathbb{R}_{0,m}^{(p)}$ of p-vectors defined by

$$\mathbb{R}_{0,m}^{(p)} = \operatorname{span}_{\mathbb{R}}(e_A : \ |A| = p).$$

We will make repeated use of the operator $\Psi : \mathbb{R}_{0,m} \mapsto \mathbb{R}_{0,m}$ given by

$$\Psi(a) = \sum_{j=1}^{m} e_j a e_j,$$

for $a \in \mathbb{R}_{0,m}$.

The operator Ψ keeps the subspace $\mathbb{R}_{0,m}^{(p)}$ invariant and, moreover, we have (see [16])

$$\Psi(Y^p) = (-1)^{p+1}(m-2p)Y^p, \tag{2}$$

for a p-vector Y^p.

When restricting to odd dimension m, the operator Ψ becomes a bijection and its inverse is given by

$$\Psi^{-1}(a) = \sum_{p=0}^{m} \frac{(-1)^{p+1}}{m-2p}[a]_p.$$

We will also deal with $\mathbb{R}_{0,m}$-valued homogeneous polynomials of degree k given by

$$\mathcal{P}_k = \sum_{|\mathbf{k}|=k} a_{\mathbf{k}} \underline{x}^{\mathbf{k}},$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_m)$ denotes a multiindex, $|\mathbf{k}| = k_1 + k_2 + \cdots + k_m$ and $\underline{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$.

Lemma 1. *Let f be a two-sided 3-monogenic function in a convex set Ω, and $\underline{y} \in \Omega$. Then, there exists a unique harmonic function h, and a unique two-sided monogenic function ϕ, such that*

$$f(\underline{x}) = h(\underline{x}) + |\underline{x} - \underline{y}|^2 \phi(\underline{x}).$$

Proof. The Almansi-type decomposition obtained in [15, Theorem 2.1] enables us to infer the following representation (uniquely determined)

$$f(\underline{x}) = f_1'(\underline{x}) + \underline{x} f_2'(\underline{x}) + \underline{x}^2 f_3'(\underline{x}), \tag{3}$$

with $f_1'(\underline{x})$, $f_2'(\underline{x})$ and $f_3'(\underline{x})$ monogenic in Ω. Moreover from [14, Proposition 2] it follows that $\underline{x} f_2'(\underline{x})$ is harmonic, then also $f_1(\underline{x}) = f_1'(\underline{x}) + \underline{x} f_2'(\underline{x})$. Hence (3) can be rewritten as

$$f(\underline{x}) = f_1(\underline{x}) + |\underline{x}|^2 f_2(\underline{x}), \tag{4}$$

where f_1, and f_2 are harmonic, and left monogenic, respectively.

It will be proved that f_2 is also right monogenic. If we apply (on the left) the Dirac operator $\partial_{\underline{x}}$ to both sides of (4), we get

$$\partial_{\underline{x}} f(\underline{x}) = \partial_{\underline{x}} f_1(\underline{x}) + 2\underline{x} f_2(\underline{x}), \tag{5}$$

where $\partial_{\underline{x}} f_1$ and f_2 are both left monogenic functions.

Since, by assumption $\partial_{\underline{x}} f$ is bimonogenic (harmonic), the uniqueness of the above representation is guaranteed by [15, Theorem 2.1].

On the other hand, by assumption we have that $\partial_{\underline{x}} f$ is also inframonogenic and so, we can apply [5, Corollary 2.5] to yield the alternative (but unique!) representation

$$\partial_{\underline{x}} f(\underline{x}) = f_1^*(\underline{x}) + \underline{x} f_2^*(\underline{x}), \tag{6}$$

where f_1^* and f_2^* are left and two-sided monogenic, respectively.

The uniqueness of (5) and (6) yields $2f_2 = f_2^*$ and hence f_2 is two-sided monogenic.

Then, due to the translation invariance of monogenic functions, $f(\underline{x}+\underline{y})$ is a two-sided 3-monogenic function as well, and

$$f(\underline{x}+\underline{y}) = f_1^*(\underline{x}) + |\underline{x}|^2 f_2^*(\underline{x}),$$

with $\partial_{\underline{x}}^2 f_1^* = 0$, and $\partial_{\underline{x}} f_2^* = f_2^* \partial_{\underline{x}} = 0$.

Hence

$$f(\underline{x}) = f_1^*(\underline{x}-\underline{y}) + |\underline{x}-\underline{y}|^2 f_2^*(\underline{x}-\underline{y}).$$

The proof is completed by defining $h(\underline{x}) = f_1^*(\underline{x}-\underline{y})$, and $\phi(\underline{x}) = f_2^*(\underline{x}-\underline{y})$. □

Lemma 2. *Suppose that* Ω *is a star-like domain with center* 0*, if* f *is two-sided 5-monogenic then* f *admits in* Ω *the unique representation*

$$f(\underline{x}) = h(\underline{x}) + |\underline{x}|^4 \psi(\underline{x}),$$

where h *is 4-monogenic and* ψ *is two-sided monogenic in* Ω*.*

Proof. By similar arguments to those given in the proof of Lemma 1. It follows from the Almansi-type decomposition [15, Theorem 2.1] and [14, Proposition 2] that

$$f = h + |\underline{x}|^4 \psi, \tag{7}$$

$$f = h^* + |\underline{x}|^4 \psi^*, \tag{8}$$

with $\partial_{\underline{x}} \psi = \psi^* \partial_{\underline{x}} = 0$ and $\partial_{\underline{x}}^4 h = \partial_{\underline{x}}^4 h^* = 0$. By (7) and (8) we have

$$\partial_{\underline{x}}^4 \big(|\underline{x}|^4 (\psi^* - \psi) \big) = 0,$$

then

$$\partial_{\underline{x}}^4 \big((|\underline{x}|^2 - r_1^2)(|\underline{x}|^2 - r_2^2)(\psi^* - \psi) \big) = 0,$$

where r_1 and r_2 have been chosen such that $\overline{B(0,r_1)} \cup \overline{B(0,r_2)} \subset \Omega$. Hence, $\psi^* - \psi \equiv 0$ in Ω, since two distinct spheres is a set of uniqueness for biharmonic functions. □

Lemma 3. *Suppose that g is two-sided monogenic in a convex Ω, then*

$$\partial_{\underline{x}}^4(|\underline{x}|^4 g) = \partial_{\underline{x}}^4\big(|\underline{x}|^2|\underline{x}-\underline{y}|^2\phi\big),$$

where ϕ is two-sided monogenic, and $\underline{y} \in \Omega$.

Proof. We have that $|\underline{x}|^4 g = |\underline{x}|^2\big(|\underline{x}|^2 g\big)$, and $\partial_{\underline{x}}^3\big(|\underline{x}|^2 g\big) = \big(|\underline{x}|^2 g\big)\partial_{\underline{x}}^3 = 0$. Indeed, from Lemma 1, there exist unique functions h and ϕ, such that

$$|\underline{x}|^2 g = h(\underline{x}) + |\underline{x}-\underline{y}|^2\phi(\underline{x}),$$

and by direct calculations we obtain the desired result. □

Lemma 4. *Let $\mathcal{P}_k$ be a two-sided monogenic homogeneous polynomial of degree k in the odd-dimensional space $\mathbb{R}^m$, and let*

$$Q := |\underline{x}|^2|\underline{x}-\underline{y}|^2\mathcal{P}_k.$$

Then,

$$\partial_{\underline{x}}^3 Q\partial_{\underline{x}} = (-16+8\alpha_k)\Psi(\mathcal{P}_k) - 4\partial_{\underline{x}}\big(\underline{y}\Psi(\mathcal{P}_k)\big).$$

Proof. First, we consider

$$\begin{aligned}\partial_{\underline{x}}(\underline{x}P_k) &= \sum_{j=1}^{m} e_j\partial_{x_j}(\underline{x}P_k) = -mP_k + \sum_{j=1}^{m} e_j\underline{x}\partial_{x_j}P_k\\ &= -mP_k + \sum_{j=1}^{m}(-2x_j - \underline{x}e_j)\partial_{x_j}P_k\\ &= -mP_k - \underline{x}(\partial_{\underline{x}}P_k) - 2\sum_{j=1}^{m} x_j\partial_{x_j}P_k\\ &= -mP_k - 2kP_k = -(m+2k)P_k,\end{aligned}$$

and for simplicity, in what follows we use the notation $\alpha_k = -(m+2k)$. Therefore $\partial_{\underline{x}}(\underline{x}\mathcal{P}_k) = \alpha_k\mathcal{P}_k$, and we have the following chain of identities:

$$\begin{aligned}\partial_{\underline{x}}Q &= \big(2\underline{x}|\underline{x}-\underline{y}|^2 + 2(\underline{x}-\underline{y})|\underline{x}|^2\big)\mathcal{P}_k\\ &= 2\underline{x}|\underline{x}-\underline{y}|^2\mathcal{P}_k + 2(\underline{x}-\underline{y})|\underline{x}|^2\mathcal{P}_k.\end{aligned}$$

$$\begin{aligned}\partial_{\underline{x}}^2 Q &= 4(\underline{x}-\underline{y})\underline{x}\mathcal{P}_k + 2|\underline{x}-\underline{y}|^2\alpha_k\mathcal{P}_k + 4\underline{x}(\underline{x}-\underline{y})\mathcal{P}_k + 2|\underline{x}|^2\alpha_k\mathcal{P}_k - 2|\underline{x}|^2\partial_{\underline{x}}(\underline{y}\mathcal{P}_k)\\ &= -8|\underline{x}|^2\mathcal{P}_k - 4\underline{y}\underline{x}\mathcal{P}_k + 2\alpha_k|\underline{x}-\underline{y}|^2\mathcal{P}_k - 4\underline{x}\underline{y}\mathcal{P}_k + 2\alpha_k|\underline{x}|^2\mathcal{P}_k - 2|\underline{x}|^2\partial_{\underline{x}}(\underline{y}\mathcal{P}_k).\end{aligned}$$

$$\begin{aligned}\partial_{\underline{x}}^3 Q = \quad & -16\underline{x}\mathcal{P}_k - 4\partial_{\underline{x}}(\underline{y}\underline{x}\mathcal{P}_k) + 4\alpha_k(\underline{x}-\underline{y})\mathcal{P}_k - 4\partial_{\underline{x}}(\underline{x}\underline{y}\mathcal{P}_k)\\ & +4\alpha_k\underline{x}\mathcal{P}_k - 4\underline{x}\partial_{\underline{x}}(\underline{y}\mathcal{P}_k).\end{aligned}$$

Then,

$$\begin{aligned}\partial_{\underline{x}}^3 Q\partial_{\underline{x}} &= -16\Psi(\mathcal{P}_k) - 4\partial_{\underline{x}}\big(\underline{y}\Psi(\mathcal{P}_k)\big) + 8\alpha_k\Psi(\mathcal{P}_k) - 4\partial_{\underline{x}}(\Psi(\underline{y}\mathcal{P}_k)) - 4\Psi(\partial_{\underline{x}}(\underline{y}\mathcal{P}_k))\\ &= (-16+8\alpha_k)\Psi(\mathcal{P}_k) - 4\partial_{\underline{x}}\big(\underline{y}\Psi(\mathcal{P}_k)\big),\end{aligned}$$

where use has been made of [14, Lemma 2.1]. □

3. Main results

We state and prove our main results in this section. In the sequel, it will be assumed that one is working in an odd-dimensional Euclidean space $\mathbb{R}^m$. The standard notations $B(0,r)$ and $\overline{B(0,r)}$ for the open and closed ball with radius r centered at the origin will be used, respectively. In a similar manner, the sphere with radius r centered at the origin is denoted by $\partial B(0,r)$.

Theorem 1. *Let* $\overline{B(\underline{y},r)} \subset \Omega$, $A \in \mathbb{R}$, $A \neq \pm 1$, *and* $[u]_k \in C^2(\Omega)$. *If*

$$A \neq (-1)^{k+1}\frac{2j+m}{m-2k}, \quad \forall j \in (\mathbb{N} \cup \{0\}), \tag{9}$$

then the problem

$$\begin{aligned} A\partial_{\underline{x}}([u]_k)\partial_{\underline{x}} + \partial_{\underline{x}}^2([u]_k) \quad &= 0 \quad in \quad \Omega, \\ [u]_k = 0 \quad in \quad &\partial B(\underline{y},r), \end{aligned} \tag{10}$$

has only the trivial solution $[u]_k \equiv 0$.

Proof. Suppose that $[u]_k$ is a solution of the above problem. Let $R \in \mathbb{R}$, such that $\overline{B(\underline{y},r)} \subset \overline{B(\underline{y},R)} \subset \Omega$. It follows from (10) that

$$A[u]_k\partial_{\underline{x}}^3 + \partial_{\underline{x}}^3[u]_k = 0,$$

$$[u]_k\partial_{\underline{x}}^3 + A\partial_{\underline{x}}^3[u]_k = 0,$$

and so, we have

$$(1-A^2)[u]_k\partial_{\underline{x}}^3 = 0,$$
$$(1-A^2)\partial_{\underline{x}}^3[u]_k = 0.$$

Hence $[u]_k\partial_{\underline{x}}^3 = \partial_{\underline{x}}^3[u]_k = 0$, since $A \neq \pm 1$.

Next, by Lemma 1, we have

$$[u]_k = h + |\underline{x}-\underline{y}|^2\phi, \ \underline{x} \in B(\underline{y},R), \tag{11}$$

where h and ϕ are harmonic and two-sided monogenic, respectively.

Now, we introduce the auxiliary function $G = r^2\phi$ which is obviously harmonic. Then, (11) can be rewritten as

$$[u]_k = h + G + (|\underline{x}-\underline{y}|^2 - r^2)\phi.$$

Since

$$[u]_k = 0 \quad \text{in} \quad \partial B(\underline{y},r),$$

thus

$$h + G = 0 \quad \text{in} \quad \partial B(\underline{y},r).$$

The harmonicity of $h+G$ yields $h \equiv -G$ in $B(\underline{y},r)$, and so in $B(\underline{y},R)$.

Therefore,

$$[u]_k = (|\underline{x}-\underline{y}|^2 - r^2)[\phi]_k, \quad \text{in} \quad B(\underline{y},R).$$

Since $\phi = [\phi]_k$ is two-sided monogenic it can be expanded into the converging Taylor series in $B(\underline{y}, r)$

$$[\phi]_k = \sum_{j=0}^{\infty} [P_j(\underline{x} - \underline{y})]_k,$$

with $[P_j]_k$ being two-sided monogenic as well.

Consequently,

$$[u]_k = \sum_{j=0}^{\infty} (|\underline{x} - \underline{y}|^2 - r^2)[P_j(\underline{x} - \underline{y})]_k,$$

and

$$\partial_{\underline{x}}^2([u]_k) = \sum_{j=0}^{\infty} \partial_{\underline{x}}^2\{(|\underline{x} - \underline{y}|^2 - r^2)[P_j(\underline{x} - \underline{y})]_k\} = \sum_{j=0}^{\infty} 2\alpha_j [P_j(\underline{x} - \underline{y})]_k.$$

On the other hand,

$$\partial_{\underline{x}}([u]_k)\partial_{\underline{x}} = \sum_{j=0}^{\infty} 2\Psi([P_j(\underline{x} - \underline{y})]_k) = \sum_{j=0}^{\infty} (-1)^{k+1} 2(m - 2k)[P_j(\underline{x} - \underline{y})]_k,$$

using [14, Lemma 2.1]. Summarizing, we have

$$A\partial_{\underline{x}}([u]_k)\partial_{\underline{x}} + \partial_{\underline{x}}^2([u]_k) = \sum_{j=0}^{\infty} \{(-1)^{k+1} 2A(m - 2k) + 2\alpha_j\}[P_j(\underline{x} - \underline{y})]_k = 0.$$

Under the assumption (9), the last identity yields

$$[P_j(\underline{x} - \underline{y})]_k = 0, \quad \forall \quad j \in \{0, 1, 2, \ldots\},$$

and hence $[u]_k \equiv 0$. □

A simple corollary is the following.

Corollary 1. *Let $\Omega \subset \mathbb{R}^3$, and suppose that $A \in \mathbb{N}$ is even. Then a sphere $\partial B(\underline{y}, r)$ is a set of uniqueness for the vector solutions of the system*

$$A\partial_{\underline{x}}[u]_1\partial_{\underline{x}} + \partial_{\underline{x}}^2[u]_1 = 0.$$

Proof. It follows directly from Theorem 1. Indeed, since in this case $m = 3$ and $k = 1$, we have

$$A \neq \frac{2j + 3}{3 - 2} = 2j + 3, \quad j \in \{0, 1, 2, \ldots\},$$

for any even number A. □

Remark 1. When u is a bivector, i.e. $u = [u]_2$, an analogous statement holds, since

$$A \neq \frac{-(2j + 3)}{3 - 4} = 2j + 3.$$

It is important to mention that Corollary 1 and Remark 1 can be generalized in case that m is odd and A is even. We are now in a position to state and prove our main result.

Theorem 2. *Let Ω be a convex domain containing two distinct balls B_1 and B_2, such that $\overline{B_1 \cup B_2} \subset \Omega$. If f is infrabimonogenic in Ω and $f|_{\partial B_1} = f|_{\partial B_2} = 0$, then $f = 0$ identically in Ω. In other words, two distinct spheres is a set of uniqueness for infrabimonogenic functions.*

Proof. Without loss of generality, assume that one of the balls is centered at 0. The another center will be denoted by $\underline{y}$. Since $\partial_{\underline{x}}^3 f \partial_{\underline{x}} = 0$, it follows that $\partial_{\underline{x}}^5 f = f\partial_{\underline{x}}^5 = 0$. From Lemma 2, there exist h^* and ψ, such that

$$f = h^* + |\underline{x}|^4 \psi,$$

where h^* is 4-monogenic and ψ is two-sided monogenic.

Then

$$\partial_{\underline{x}}^4 f = \partial_{\underline{x}}^4 (|\underline{x}|^4 \psi),$$

and by Lemma 3, there exists a two-sided monogenic function ϕ, such that

$$\partial_{\underline{x}}^4 (|\underline{x}|^4 \psi) = \partial_{\underline{x}}^4 (|\underline{x}|^2 |\underline{x} - \underline{y}|^2 \phi).$$

Therefore,

$$\partial_{\underline{x}}^4 f = \partial_{\underline{x}}^4 (|\underline{x}|^2 |\underline{x} - \underline{y}|^2 \phi),$$

and

$$\partial_{\underline{x}}^4 (f - |\underline{x}|^2 |\underline{x} - \underline{y}|^2 \phi) = 0.$$

Now let r and R be the radius of the corresponding balls centered at 0 and $\underline{y}$.

Notice that

$$\partial_{\underline{x}}^4 (|\underline{x}|^2 |\underline{x} - \underline{y}|^2 \phi) = \partial_{\underline{x}}^4 \big((|\underline{x}|^2 - r^2)(|\underline{x} - \underline{y}|^2 - R^2)\phi \big),$$

and hence

$$\partial_{\underline{x}}^4 \big(f - (|\underline{x}|^2 - r^2)(|\underline{x} - \underline{y}|^2 - R^2)\phi \big) = 0.$$

Since two distinct spheres are a set of uniqueness for biharmonic functions, one has then

$$f \equiv (|\underline{x}|^2 - r^2)(|\underline{x} - \underline{y}|^2 - R^2)\phi \quad \text{in } \Omega.$$

On the other hand, direct calculations give

$$0 = \partial_{\underline{x}}^3 \{ (|\underline{x}|^2 - r^2)(|\underline{x} - \underline{y}|^2 - R^2)\phi \} \partial_{\underline{x}} = \partial_{\underline{x}}^3 (|\underline{x}|^2 |\underline{x} - \underline{y}|^2 \phi) \partial_{\underline{x}},$$

for $\underline{x} \in B(0, \delta) \subset B(0, r)$.

Next, we expand the two-sided monogenic function ϕ in $B(0, \delta) \subset \overline{B(0, r)} \subset \Omega$, into the converging Taylor series

$$\phi = \sum_{j=0}^{\infty} P_j.$$

Accordingly, we have

$$
\begin{aligned}
0 &= \sum_{j=0}^{\infty}\{(8\alpha_j - 16)\Psi(P_j) - 4\partial_{\underline{x}}(\underline{y}\Psi(P_j))\},\\
0 &= 8\sum_{j=0}^{\infty}\alpha_j\Psi(P_j) - 16\sum_{j=0}^{\infty}\Psi(P_j) - 4\partial_{\underline{x}}\left(\underline{y}\sum_{j=0}^{\infty}\Psi(P_j)\right),\\
0 &= 8\partial_{\underline{x}}\left(\underline{x}\sum_{j=0}^{\infty}\Psi(P_j)\right) - 16\sum_{j=0}^{\infty}\Psi(P_j) - 4\partial_{\underline{x}}\left(\underline{y}\sum_{j=0}^{\infty}\Psi(P_j)\right),\\
0 &= 8\partial_{\underline{x}}(\underline{x}\Psi(\phi)) - 16\Psi(\phi) - 4\partial_{\underline{x}}(\underline{y}\Psi(\phi)),
\end{aligned}
$$

where use has been made of Lemma 4, the bijectivity of Ψ, and the fact that $\partial_{\underline{x}}(\underline{x}P_j) = \alpha_j P_j$.

Moreover, define $G = 8\partial_{\underline{x}}(\underline{x}\Psi(\phi)) - 16\Psi(\phi) - 4\partial_{\underline{x}}(\underline{y}\Psi(\phi))$. Of course, since G represents a real analytic function, which vanishes in the open set $B(0,\delta) \subset \Omega$, it follows that $G \equiv 0$ in the whole domain Ω.

Then, we indeed have in Ω

$$2\partial_{\underline{x}}(\underline{x}\Psi(\phi)) - 4\Psi(\phi) - \partial_{\underline{x}}(\underline{y}\Psi(\phi)) = 0,$$

from this identity and [14, Lemma 2.1] we have

$$\partial_{\underline{x}}^2[(|\underline{x} - \underline{y}/2|^2 - \varepsilon^2)\Psi(\phi)] = 4\Psi(\phi), \tag{12}$$

where ε was chosen such that $B(\underline{y}/2,\varepsilon) \subset \Omega$. Similarly,

$$\partial_{\underline{x}}[(|\underline{x} - \underline{y}/2|^2 - \varepsilon^2)\phi]\partial_{\underline{x}} = 2\Psi(\phi),$$

and so

$$\partial_{\underline{x}}[(|\underline{x} - \underline{y}/2|^2 - \varepsilon^2)[\phi]_k]\partial_{\underline{x}} = 2\Psi([\phi]_k) = 2(-1)^{k+1}(m-2k)[\phi]_k \tag{13}$$

after taking the k-vector part.

Applying Ψ^{-1} to (12), yields

$$\partial_{\underline{x}}^2[(|\underline{x} - \underline{y}/2|^2 - \varepsilon^2)\phi] = 4\phi,$$

thus

$$\partial_{\underline{x}}^2[(|\underline{x} - \underline{y}/2|^2 - \varepsilon^2)[\phi]_k] = 4[\phi]_k. \tag{14}$$

Let us denote $[\omega]_k = (|\underline{x} - \underline{y}/2|^2 - \varepsilon^2)[\phi]_k$. It follows from (13) and (14) that

$$(-1)^{k+1}(m-2k)\partial_{\underline{x}}^2([\omega]_k) - 2\partial_{\underline{x}}([\omega]_k)\partial_{\underline{x}} = 0,$$

or equivalently

$$\partial_{\underline{x}}^2([\omega]_k) + \frac{2(-1)^k}{m-2k}\partial_{\underline{x}}([\omega]_k)\partial_{\underline{x}} = 0.$$

On the other hand,

$$[\omega]_k = 0 \quad \text{in} \quad \partial B(\underline{y}/2,\varepsilon),$$

which is obvious from the definition of $[\omega]_k$.

Thus by Theorem 1, $[\omega]_k \equiv 0$ in Ω for every k. Then $\phi \equiv 0$, and finally $f \equiv 0$ in Ω, as desired. $\square$

4. Concluding Remark

In the context of the search for possible generalizations we can ask whether Theorem 2 remains valid for infrapolymonogenic functions of arbitrary order k if instead of two distinct spheres we consider k of them. This conclusion cannot be reached using the inductive method similar to that used in the proof of [9, Theorem 4], because it is based on a direct application of the maximum principle for harmonic functions. The above question will inspire further analysis and researches.

Author contributions All authors equally contributed to the results established in the paper.

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Declarations

Conflict of Interest The authors declare no competing interests.

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