Mediterr. J. Math. (2023) 20:312
https://doi.org/10.1007/s00009-023-02501-3
1660-5446/23/060001-14
published online October 7, 2023

Mediterranean Journal of Mathematics

Maps Preserving the ∂-Spectrum of Product or Triple Product of Operators

Hassane Benbouziane, Aukacha Daoudi,
Mustapha Ech-Chérif El Kettani and Ismail El Khchin

Abstract. Let X be an infinite-dimensional Banach space, and $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. A map Δ, from $\mathcal{B}(X)$ into a closed subsets of $\mathbb{C}$ is said to be ∂-spectrum if $\partial(\sigma(T)) \subseteq \Delta(T) \subseteq \sigma(T)$ for all $T \in \mathcal{B}(X)$. Here, $\sigma(T)$ is spectrum of T and $\partial(\sigma(T))$ the boundary of $\sigma(T)$. In this paper, we determine the forms of all surjective maps ϕ from $\mathcal{B}(X)$ into itself that satisfy either $\Delta(\phi(T)\phi(S)) = \Delta(TS)$ for all $T, S \in \mathcal{B}(X)$ or $\Delta(\phi(T)\phi(S)\phi(T)) = \Delta(TST)$ for all $T, S \in \mathcal{B}(X)$.

Mathematics Subject Classification. Primary 47B49; Secondary 47B48, 46H05.

Keywords. Nonlinear preservers problem, ∂-spectrum, Product, Triple product of operators.

Contents

1. Introduction and Preliminaries

Let X be an infinite-dimensional Banach space with dual space X^* and $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. The identity operator on X (resp X^*) will denote by I (resp I_{X^*}).

For $x \in X\backslash\{0\}$ and $f \in X^*\backslash\{0\}$, we denote by $x \otimes f$ the bounded linear rank one operator defined by $(x \otimes f)y = f(y)x$ for all $y \in X$. Note that every operator on X of rank one can be written as $x \otimes f$ for some $x \in X\backslash\{0\}$ and

This file has been typeset with the option `draft` to illustrate that feature and its purpose.

$x \in X^*\backslash\{0\}$. Note that $\sigma(x \otimes f) = \{0, f(x)\}$, and that $x \otimes f$ is a nilpotent operator if and only if $f(x) = 0$. The set of all rank one operators, the set of all nilpotent operators and the ideal of all finite rank operators in $\mathcal{B}(X)$ are denoted by $\mathcal{F}_1(X)$, $\mathcal{N}_1(X)$ and $\mathcal{F}(X)$, respectively.

For $T \in \mathcal{B}(X)$, we denote by $R(T)$, $N(T)$, T^*, $\sigma(T)$, $\sigma_l(T)$, $\sigma_r(T)$, $\sigma_{sur}(T)$, $\sigma_{ap}(T)$ and $\partial(\sigma(T))$, the the range, the kernel, the adjoint, the spectrum, the left spectrum, the right spectrum, the surjectivity spectrum, the approximate point spectrum and the boundary of spectrum of T, respectively. The hyper-range of $T \in \mathcal{B}(X)$ is defined by $T^\infty(X) := \bigcap_{n\in\mathbb{N}} T(X^n)$.

Consider the map $\Delta : \mathcal{B}(X) \to \{\text{closed set of } \mathbb{C}\}$ with $\Delta(.)$ is any one of the spectral functions $\sigma(.)$, $\sigma_l(.)$, $\sigma_r(.)$, $\sigma_{ap}(.)$ or $\sigma_{sur}(.)$. It is known that these spectral functions satisfy

$$\partial(\sigma(T)) \subseteq \Delta(T) \subseteq \sigma(T) \textit{ for all } T \in \mathcal{B}(X).$$

In general, a map $\Delta : \mathcal{B}(X) \to \{\text{closed subsets of } \mathbb{C}\}$ is said to be ∂-spectrum if

$$\partial(\sigma(T)) \subseteq \Delta(T) \subseteq \sigma(T), \tag{1.1}$$

for all $T \in \mathcal{B}(X)$; see [15, Def 3.2]. Note that if Δ is ∂-spectrum, then for every $T \in \mathcal{B}(X)$, $\Delta(T)$ is non-empty and

$$\Delta(T) \textit{ is countable } \iff \sigma(T) \textit{ is countable},$$

and in this case, we have $\Delta(T) = \sigma(T)$. In particular,

$$\Delta(x \otimes f) = \sigma(x \otimes f) = \{0; f(x)\}, \textit{ for all } x \in X \textit{ and } f \in X^*.$$

An operator $T \in \mathcal{B}(X)$ is called Kato operator (or semi-regular) if $R(T)$ is closed and $N(T) \subseteq T^\infty(X)$, the set $\sigma_K(T) = \{\lambda \in \mathbb{C}/T - \lambda I$ is not kato operator$\}$ is called the Kato spectrum. Mbekhta and Ouahab [16] proved that

$$\partial(\sigma(T)) \subseteq \sigma_K(T) \subseteq \sigma(T),$$

for all $T \in \mathcal{B}(X)$, and thus $\sigma_K(.)$ is ∂-spectrum.

Note that there are other spectra satisfying property (1.1), namely the generalized spectrum $\sigma_g(T)$ of operator T, the Saphar spectrum $\sigma_{rr}(T)$. For more information about these spectra, we refer the reader to [13,19].

The problem of characterizing maps on matrices or operators that preserve certain functions, subsets and relations has attracted the attention of many mathematicians in the last decade; see for example [1,4–6,8,11,12,14, 15,20,22].

In [6], Cui and Hou showed that if Δ is a ∂-spectrum, and ϕ is a linear map from a semisimple Banach algebra $\mathcal{A}$ onto another one $\mathcal{B}$ such that

$$\Delta(\Phi(T)) \subset \Delta(T), \quad T \in \mathcal{A},$$

then ϕ is idempotent preserving and $\phi(I) = I$.

In the last decades, many authors, investigated maps preserving a certain property of the product or triple product without assuming linearity or additivity. We refer the interested reader to [2–4,9,10,17,18,23].

In [18], Molnár described maps preserving the spectrum of product of operators. In particular, he showed that a surjective map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ (H is an infinite-dimensional complex Hilbert space) satisfies

$$\sigma(\phi(T)\phi(S)) = \sigma(TS), \quad T, S \in \mathcal{B}(H),$$

if and only if there exists an operator invertible $A \in \mathcal{B}(H)$ such that either $\phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\phi(T) = -ATA^{-1}$ for all $T \in \mathcal{B}(H)$. He obtained, in the same paper, a similar result by considering the surjectivity spectrum and the point spectrum instead the usual spectrum.

Instead of the usual product, certain authors investigated maps preserving spectra of triple product of operators or matrices; See for instance [7,9,23]. In [23], Zhang and Hou gave the form of maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfying

$$\sigma_\pi(\phi(T)\phi(S)\phi(T)) = \sigma_\pi(TST), \quad T, S \in \mathcal{B}(H),$$

where $\sigma_\pi(T) := \{\lambda \in \sigma(T)/ \mid \lambda \mid = r(T)\}$ is the peripheral spectrum of $T \in \mathcal{B}(H)$.

Our objective of this paper is to study nonlinear maps preserving any part of the spectrum, which contains the boundary of the spectrum (∂-spectrum), of the product of operator. We thus obtain Molnár's main results in the case of an arbitrary Banach space using a different approach. We also obtain the form of any surjective map preserving the ∂-spectrum of the triple product of operators. Our proofs are inspired by those of the main results of the papers [2,3].

Throughout this paper, let Δ be a ∂-spectrum map.

2. Maps Preserving the ∂-Spectrum of Product of Operators

The following theorem is our main result in this section which characterizes nonlinear maps preserving a ∂-spectrum $\Delta(.)$ of the product of operators.

Theorem 2.1. *Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map satisfying*

$$\Delta(\phi(T)\phi(S)) = \Delta(TS) \text{ for all } T,\ S \in \mathcal{B}(X), \tag{2.1}$$

then there exists a scalar $\alpha = \pm 1$ and either there is a bounded invertible operator $A : X \to X$ such that

$$\phi(T) = \alpha ATA^{-1} \text{ for all } T \in \mathcal{B}(X), \tag{2.2}$$

or there is a bounded invertible operator $C : X^* \to X$ such that

$$\phi(T) = \alpha CT^*C^{-1} \text{ for all } T \in \mathcal{B}(X). \tag{2.3}$$

In the last case, X is automatically reflexive.

Before embarking on the proof, we need several lemmas. The first one was proved in [20].

Lemma 2.2. [20, Theorem 3.3] *Let $\phi : \mathcal{F}(X) \to \mathcal{F}(X)$ be a bijective linear map preserving rank one operators in both directions. Then, either there are linear bijective mappings $A : X \to X$ or $B : X^* \to X^*$ such that*

$$\phi(x \otimes f) = Ax \otimes Bf \ (x \in X, f \in X^*), \tag{2.4}$$

or, there are linear bijective mappings $C : X^* \to X$ and $D : X \to X^*$ such that

$$\phi(x \otimes f) = Cf \otimes Dx \ (x \in X, f \in X^*). \tag{2.5}$$

For the remaining lemmas and the proofs of the main theorems, we will need the following notation—

$$\Delta^*(T) = \begin{cases} \Delta(T) \setminus \{0\} & \text{if } \Delta(T) \neq \{0\} \\ \{0\} & \text{if } \Delta(T) = \{0\}, \end{cases} \tag{2.6}$$

where Δ is ∂-spectrum and $T \in \mathcal{B}(X)$. In particular, we have $\Delta^*(x \otimes f) = \{f(x)\}$ for all $x \in X$ and $f \in X^*$.

The next lemma gives necessary and sufficient conditions for two operators to be equal in term of ∂-spectrum.

Lemma 2.3. *Let $T,\ S\ \in \mathcal{B}(X)$. Then the following statements are equivalent.*

1. *$T = S$.*
2. *$\Delta(TR) = \Delta(SR)$ for all $R \in \mathcal{F}_1(X)$.*
3. *$\Delta^*(TR) = \Delta^*(SR)$ for all $R \in \mathcal{F}_1(X)$.*

Proof. We only need to prove that the implication (3) $\Rightarrow$ (1) holds.

Assume that $\Delta^*(TR) = \Delta^*(SR)$ for all $R \in \mathcal{F}_1(X)$ and let $x \in X$ and $f \in X^*$. We have

$$\{f(Tx)\} = \Delta^*(T(x \otimes f)) = \Delta^*(S(x \otimes f)) = \{f(Sx)\}$$

and thus $Tx = Sx$. By the arbitrariness of x, clearly $T = S$.

□

The following lemma gives a characterization of rank one operators in term of $\partial-$ spectrum.

Lemma 2.4. *Let $R \in \mathcal{B}(X) \setminus \{0\}$. The following statements are equivalent:*

1. *R has rank one.*
2. *$\Delta^*(TR)$ is a singleton for all operator $T \in \mathcal{B}(X)$.*

Proof. Note that, for any $T \in \mathcal{B}(X)$, if $\Delta^*(T)$ is a singleton, then $\Delta^*(T) = \sigma_\pi(T)$. Thus, this Lemma is an immediate consequence of [17, Lemma 2.1].

□

Lemma 2.5. *Let $T, S \in \mathcal{B}(X)$. Then for every $R \in \mathcal{F}_1(X)$, we have*

$$\Delta^*((T + S)R) = \Delta^*(TR) + \Delta^*(SR).$$

Proof. Let $R \in \mathcal{F}_1(X)$ such that $R = x \otimes f$ where $x \in X,\ f \in X^*$. Note that $TR = Tx \otimes f$ and $SR = Sx \otimes f$. Then

$$\begin{aligned}\Delta^*((T+S)R) &= \{f((T+S)x)\}\\ &= \{f(Tx) + f(Sx)\}\\ &= \{f(Tx)\} + \{f(Sx)\}\\ &= \Delta^*(TR) + \Delta^*(SR).\end{aligned}$$

□

Proof of Theorem 2. 1.. We will prove the theorem in five steps—

Step 1. ϕ is injective, ϕ^{-1} also satisfies (2.1) and $\phi(0) = 0$.

Let $A, B \in \mathcal{B}(X)$ such that $\Phi(A) = \Phi(B)$. For every $R \in \mathcal{F}_1(X)$, we have

$$\Delta(AR) = \Delta(\phi(A)\phi(R)) = \Delta(\phi(B)\phi(R)) = \Delta(BR).$$

By Lemma 2.3, we get $A = B$ which proves that ϕ is injective. It follows that ϕ is a bijection and it is obvious that ϕ^{-1} satisfies (2.1).

For any $T \in \mathcal{B}(X)$, we have

$$\Delta^*(0\phi(T) = \{0\} = \Delta^*(0T) = \Delta^*(\phi(0)\phi(T)).$$

Since ϕ is surjective, lemma 2.3 implies that $\phi(0) = 0$.

Step 2. ϕ preserves rank one operators in both directions.

Let $R \in \mathcal{F}_1(X)$ such that $R = x \otimes f$ where $x \in X$ and $f \in X^*$. For every $T \in \mathcal{B}(X)$ there exists $S \in \mathcal{B}(X)$ such that $T = \phi(S)$. We have

$$\begin{aligned}\Delta^*(\phi(R)T) &= \Delta^*(\phi(R)\phi(S))\\ &= \Delta^*(RS)\\ &= \{f(Sx)\}.\end{aligned}$$

Then $\Delta^*(\phi(R)T)$ has one element for all $T \in \mathcal{B}(X)$. By Lemma 2.4, we see that $\phi(R) \in \mathcal{F}_1(X)$. In addition, since ϕ is bijective and ϕ^{-1} satisfies (2.1), ϕ preserves rank one operators in the both direction.

Note also that ϕ preserves non-nilpotent rank one operators in both directions. Indeed, let $R = x \otimes f$ where $x \in X$ and $f \in X^*$ with $f(x) \neq 0$. Then $\phi(R) = y \otimes g$ where $y \in X$ and $g \in X^*$. On the other hand, we have

$$\begin{aligned}\{f^2(x)\} &= \Delta^*(f(x)x \otimes f)\\ &= \Delta^*(R^2)\\ &= \Delta^*(\phi(R)^2)\\ &= \Delta^*(g(y)y \otimes g)\\ &= \{g^2(y)\}.\end{aligned}$$

Since $f(x) \neq 0$, then $g(y) \neq 0$. It follows that $\phi(R) \in \mathcal{F}_1(X)\backslash\mathcal{N}_1(X)$. The reverse direction is obvious since ϕ is bijective.

Step 3. ϕ is a linear map preserving finite rank operators in both directions.

Let $T, S \in \mathcal{B}(X)$. For every $R \in \mathcal{F}_1(X)$ we have

$$\begin{aligned}\Delta^*(\phi(T+S)\phi(R)) &= \Delta^*((T+S)R)\\ &= \Delta^*(TR + SR)\end{aligned}$$

$$\begin{aligned} &= \Delta^*(TR) + \Delta^*(SR) \\ &= \Delta^*(\phi(T)\phi(R)) + \Delta^*(\phi(S)\phi(R)) \\ &= \Delta^*((\phi(T) + \phi(S))\phi(R)). \end{aligned}$$

Using Lemma 2.3, we conclude that $\phi(T+S) = \phi(T) + \phi(S)$. Let $\lambda \in \mathbb{C}$ and $T \in \mathcal{B}(X)$, let us prove that $\phi(\lambda T) = \lambda\phi(T)$.
If $\lambda = 0$, Then $\phi(0T) = \phi(0) = 0 = 0\phi(T)$.
If $\lambda \neq 0$, let $R \in \mathcal{F}_1(X)$,

$$\begin{aligned} \Delta(\phi(\lambda T)\phi(R)) &= \Delta(\lambda TR) \\ &= \sigma(\lambda TR) \\ &= \lambda\sigma(TR) \\ &= \lambda\Delta(TR) \\ &= \lambda\Delta(\phi(T)\phi(R)) \\ &= \lambda\sigma(\phi(T)\phi(R)) \\ &= \sigma(\lambda\phi(T)\phi(R)) \\ &= \Delta(\lambda\phi(T)\phi(R)). \end{aligned}$$

By Lemma 2.3 and the surjectivity of ϕ, we conclude that $\phi(\lambda T) = \lambda\phi(T)$, which proves that ϕ is linear. Finally, since every $T \in \mathcal{F}(X)$ is a finite linear combination of rank one operators and ϕ preserves $\mathcal{F}_1(X)$ in both directions, by linearity we find that ϕ preserves finite rank operators in both directions.

Step 4. $\phi(I) = \alpha I$ where $\alpha^2 = 1$.
Suppose, by the way of contradiction, that $\Phi(I)$ and I are linearly independent, then there exists a nonzero vector $x \in X$ such that $\phi(I)x$ and x are linearly independent. Let $f \in X^*$ such that $f(x) = 1$ and $f(\phi(I)x) = 0$. For $R = x \otimes f \in \mathcal{F}_1(X)\backslash\mathcal{N}_1(X)$, by step 2 and surjectivity of ϕ, there is $T \in \mathcal{F}_1(X)\backslash\mathcal{N}_1(X)$ such that $\phi(T) = R$ and $T = y \otimes g$ where $y \in$ X and $g \in X^*$ with $g(y) \neq 0$. Therefore,

$$\begin{aligned} \{g(y)\} &= \Delta^*(T) \\ &= \Delta^*(\phi(I)\phi(T)) \\ &= \Delta^*(\phi(I)R) \\ &= \Delta^*(\phi(I)x \otimes f) \\ &= \{f(\phi(I)x)\}. \end{aligned}$$

Thus $g(y) = 0$. This contradiction shows that $\Phi(I) = \alpha I$ for some non-zero scalar $\alpha \in \mathbb{C}$.

On the other hand, we have

$$\{1\} = \Delta(I^2) = \Delta(\phi(I)^2) = \Delta(\alpha^2 I) = \{\alpha^2\}.$$

Which implies that $\alpha^2 = 1$; as desired.

Step 5. ϕ has the desired form.
Let $\varphi := \alpha^{-1}\phi$, note that φ satisfies (2.1) and $\varphi(I) = I$. It follows, from Step 3 and Lemma 2.2, that ϕ takes either form (2.4) or form (2.5).

Suppose that (2.4) holds. Let $x \in X$ and $f \in X^*$, and note that

$$\{f(x)\} = \Delta^*(x \otimes f)$$

$$= \Delta^*(\varphi(I)\varphi(x \otimes f))$$
$$= \Delta^*(Ax \otimes Bf)$$
$$= \{Bf(Ax)\}.$$

Then

$$Bf(Ax) = f(x).$$

Now, we are ready to prove that A is bounded and $B = (A^*)^{-1}$.
Let $(x_n)_n$ be a sequence of elements of X and $x, y \in A$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$. We show that $Ax = y$.
For every $f \in X^*$, we have

$$Bf(y) = Bf(\lim_{n\to\infty} Ax_n) = \lim_{n\to\infty} Bf(Ax_n) = \lim_{n\to\infty} f(x_n) = f(x) = Bf(Ax).$$

Since B is bijective, then $f(Ax) = f(y)$ for all $x \in X$ and all $f \in X^*$. By Hahn–Banach theorem, we conclude that $Ax = y$. The closed graph theorem shows that A is bounded. Moreover,

$$\begin{aligned} Bf(Ax) = f(x) &\Rightarrow f(x) = A^*Bf(x) \ for \ all \ f \in X^* \\ &\Rightarrow A^*B = I_{X*} \\ &\Rightarrow B = (A^*)^{-1}. \end{aligned}$$

Therefore, for every $x \in X$ and $f \in X^*$ we have

$$\varphi(x \otimes f) = Ax \otimes (A^*)^{-1}f = A(x \otimes f)A^{-1}.$$

Then

$$\varphi(R) = ARA^{-1} \ for \ all \ R \in \mathcal{F}_1(X).$$

Now, let $T \in \mathcal{B}(X)$ and $R \in \mathcal{F}_1(X)$, we have

$$\begin{aligned} \Delta(\varphi(T)\varphi(R)) &= \Delta(TR) \\ &= \Delta(ATRA^{-1}) \\ &= \Delta(ATA^{-1}ARA^{-1}) \\ &= \Delta(ATA^{-1}\varphi(R)). \end{aligned}$$

Since φ is surjective, by Lemma 2.3, we conclude that

$$\varphi(T) = ATA^{-1} \ for \ all \ T \in \mathcal{B}(X).$$

Finally, $\phi(T) = \alpha ATA^{-1}$ with $\alpha^2 = 1$ for all $T \in \mathcal{B}(X)$.

Now, suppose that φ takes the second form (2.5). Let $f \in X^*$ and $x \in X$, we have

$$\begin{aligned} \{f(x)\} &= \Delta^*(x \otimes f) \\ &= \Delta^*(\varphi(I)\varphi(x \otimes f)) \\ &= \Delta^*(Cf \otimes Dx) \\ &= \{Dx(Cf)\}. \end{aligned}$$

This shows that

$$Dx(Cf) = f(x).$$

Consequently, by similar reasoning as the first case and using the closed graph theorem, separately for each of C and D, we conclude that these operators are bounded. Therefore, both $C^* : X^* \to X^{**}$ and $D^* : X^{**} \to X^*$ are invertible. Moreover, if $\jmath$ is the canonical embedding of X in X^{**}, then we have $D^* \circ \jmath \circ C = I_{X^*}$ which implies that $\jmath \circ C = (D^*)^{-1}$. Since both C and $(D^*)^{-1}$ are surjective, $\jmath$ is also surjective and hence X is reflexive. By identifying X with X^{**}, we conclude that $D^* \circ C = I_{X^*}$ and $C^* \circ D = I$. Which implies that $D = (C^*)^{-1} = (C^{-1})^*$ and

$$\varphi(R) = CR^*C^{-1} \; for \; all \; R \in \mathcal{F}_1(X).$$

Let $T \in \mathcal{B}(X)$. Then for every $f \in X^*$ and $x \in X$, we obtain

$$\begin{aligned} \Delta^*(T(x \otimes f)) &= \Delta^*(\varphi(T)\varphi(x \otimes f)) \\ &= \Delta^*(\varphi(T)(C(x \otimes f)^*C^{-1}) \\ &= \Delta^*(\varphi(T)(Cf \otimes (C^{-1})^*x)). \end{aligned}$$

By identifying X with X^{**}, we get $((C^{-1})^*x)(\varphi(T)(Cf)) = f(Tx)$. Thus, for every $f \in X^*$ and $x \in X$, we have $C^{-1}\varphi(T)Cf(x) = T^*f(x)$. Therefore,

$$\varphi(T) = CT^*C^{-1} \; for \; all \; T \in \mathcal{B}(X).$$

Finally, we get $\phi(T) = \alpha CT^*C^{-1}$ with $\alpha^2 = 1$, and the proof is thus complete. □

3. Surjective Maps Preserving the ∂-Spectrum of Triple Product of Operators

In this section, we will study maps (without assuming linearity or additivity) preserving the ∂-spectrum of triple product of operators. We begin this section with the following identity principal which will be used frequently in the proof of Theorem 3.4.

Lemma 3.1. *If $T, \; S \; \in \mathcal{B}(X)$, then the following statements are equivalent.*

1. $T = S$.
2. $\Delta(RTR) = \Delta(RSR)$ *for all* $R \in \mathcal{F}_1(X) \setminus \mathcal{N}_1(X)$.
3. $\Delta^*(RTR) = \Delta^*(RSR)$ *for all* $R \in \mathcal{F}_1(X) \setminus \mathcal{N}_1(X)$.

Proof. We only need to prove that ($3 \implies 1$). Assume that $\Delta^*(RTR) = \Delta^*(RSR)$ for all $R \in \mathcal{F}_1(X) \setminus \mathcal{N}_1(X)$.
Let $R = x \otimes f$, where $x \in X \backslash \{0\}$ and $f \in X^*$. If $f(x) \neq 0$, note that $RTR = f(Tx)x \otimes f$ and $RSR = f(Sx)x \otimes f$. Then

$$\begin{aligned} \Delta^*(RTR) = \Delta^*(RSR) &\Rightarrow \Delta^*(f(Tx)x \otimes f) = \Delta^*(f(Sx)x \otimes f). \\ &\Rightarrow \{f(Tx)f(x)\} = \{f(Sx)f(x)\} \\ &\Rightarrow f(Tx) = f(Sx). \end{aligned}$$

Now, if $f(x) = 0$, let $g \in X^*$ such that $g(x) \neq 0$. Then by the first case

$$(f + g)(Tx) = (f + g)(Sx) \; and \; g(Tx) = g(Sx).$$

Then

$$\begin{aligned} f(Tx) + g(Sx) &= f(Tx) + g(Sx) \\ &= (f+g)(Tx) \\ &= (f+g)(Sx) \\ &= f(Sx) + g(Sx). \end{aligned}$$

Which implies that $f(Tx)) = f(Sx)$ in this case too. Therefore, by Hahn-Banach theorem, $Tx = Sx$ for all $x \in X$, which proves that $T = S$. □

The second result provides necessary and sufficient condition for an operator to be rank 1 in term of ∂-spectrum.

Lemma 3.2. *Let $R \in \mathcal{B}(X) \setminus \{0\}$, the following statements are equivalent.*

1. *R has rank one.*
2. *$\Delta^*(TRT)$ is a singleton for all operator $T \in \mathcal{B}(X)$.*

Proof. This is an immediate consequence of [23, Lemma 2.2]. □

The next lemma will be used to show that if a surjective map ϕ from $\mathcal{B}(X)$ into itself preserves the ∂-spectrum of triple product of operators, then it's automatically additive.

Lemma 3.3. *Let $T, S \in \mathcal{B}(X)$. Then, for every $R \in \mathcal{F}_1(X)$, we have*

$$\Delta^*(R(T+S)R) = \Delta^*(RTR) + \Delta^*(RSR).$$

Proof. Let $R \in \mathcal{F}_1(X)$ such that $R = x \otimes f$ where $x \in X,\ f \in X^*$. Then $R(T+S)R = f((T+S)x)x \otimes f$, and

$$\begin{aligned} \Delta^*(R(T+S)R) &= \{f(x)f((T+S)x)\} \\ &= \{f(x)f(Tx) + f(x)f(Sx)\} \\ &= \{f(x)f(Tx)\} + \{f(x)f(Sx)\} \\ &= \Delta^*(RTR) + \Delta^*(RSR). \end{aligned}$$

□

Now, we are in a position to give the second main results in this paper.

Theorem 3.4. *Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map satisfying*

$$\Delta(\phi(T)\phi(S)\phi(T)) = \Delta(TST) \text{ for all } T,\ S\ \in \mathcal{B}(X). \tag{3.1}$$

Then there exists a scalar $\alpha \in \mathbb{C}$ with $\alpha^3 = 1$ and either there is a bounded invertible operator $A : X \to X$ such that

$$\phi(T) = \alpha ATA^{-1} \text{ for all } T \in \mathcal{B}(X), \tag{3.2}$$

or there is a bounded invertible operator $C : X^* \to X$ such that

$$\phi(T) = \alpha CT^*C^{-1} \text{ for all } T \in \mathcal{B}(X). \tag{3.3}$$

In the last case, X is automatically reflexive.

Proof. We break down the proof into several steps—

Step 1. ϕ is injective and $\phi(0) = 0$.

Let $A, B \in \mathcal{B}(X)$ such that $\phi(A) = \phi(B)$. Then for every $R \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$, we have

$$\Delta(RAR) = \Delta(\phi(R)\phi(A)\phi(R)) = \Delta(\phi(R)\phi(B)\phi(R)) = \Delta(RBR).$$

By Lemma 3.1, we establish that $A = B$ which proves that ϕ is injective. Consequently ϕ is a bijection and it is easy to show that ϕ^{-1} satisfies the Eq. 3.1.

Let us prove that $\phi(0) = 0$. For every $T \in \mathcal{B}(X)$, we have

$$\Delta^*(\phi(T)0\phi(T)) = \{0\} = \Delta^*(T0T) = \Delta^*(\phi(T)\phi(0)\phi(T)).$$

By Lemma 3.1, we see that $\phi(0) = 0$.

Step 2. ϕ preserves rank one operators in both directions.

Let $R \in \mathcal{F}_1(X)$, then $Card(\Delta^*(TRT)) = 1$ for all $T \in \mathcal{B}(X)$. Which implies that $Card(\Delta^*(\phi(T)\phi(R)\phi(T)) = 1$ for all $T \in \mathcal{B}(X)$. Its follows, by surjectivity of ϕ and Lemma 3.2, that $\phi(R) \in \mathcal{F}_1(X)$. Since ϕ is a bijection and ϕ^{-1} satisfies (3.1), ϕ preserves $\mathcal{F}_1(X)$ in both directions.

Now, we show also that ϕ preserves non-nilpotent rank one operators in both directions. Indeed, let $R = x \otimes f$ where $x \in X$ and $f \in X^*$ with $f(x) \neq 0$. Then $\phi(R) = y \otimes g$ where $y \in X$ and $g \in X^*$. We have

$$\begin{aligned}
\{f^3(x)\} &= \Delta^*(f^2(x)x \otimes f) \\
&= \Delta^*(R^3) \\
&= \Delta^*(\phi(R)^3) \\
&= \Delta^*(g^2(y)y \otimes g) \\
&= \{g^3(y)\}.
\end{aligned}$$

Since $f(x) \neq 0$, then $g(y) \neq 0$. It follows that $\phi(R) \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$ and ϕ preserves $\mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$.

Step 3. ϕ is a linear map.

Let $T, S \in \mathcal{B}(X)$ and $R \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$. We have

$$\begin{aligned}
\Delta^*(\phi(R)\phi(T+S)\phi(R)) &= \Delta^*(R(T+S)R) \\
&= \Delta^*(RTR) + \Delta^*(RSR) \\
&= \Delta^*(\phi(R)\phi(T)\phi(R)) + \Delta^*(\phi(R)\phi(S)\phi(R)) \\
&= \Delta^*(\phi(R)(\phi(T) + \phi(S))\phi(R)).
\end{aligned}$$

Since $R \in \mathcal{F}_1(X)$ is arbitrary and ϕ is surjective, by Lemma 3.1, we get $\phi(T+S) = \phi(T) + \phi(S)$. Let $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$. For every $R \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$, we have

$$\begin{aligned}
\Delta(\phi(R)\phi(\lambda T)\phi(R)) &= \Delta(\lambda RTR) \\
&= \sigma(\lambda RTR) \\
&= \lambda\sigma(RTR) \\
&= \lambda\Delta(RTR) \\
&= \lambda\Delta(\phi(R)\phi(T)\phi(R))
\end{aligned}$$

$$
\begin{aligned}
&= \lambda\sigma(\phi(R)\phi(T)\phi(R)) \\
&= \sigma(\lambda\phi(R)\phi(T)\phi(R)) \\
&= \Delta(\phi(R)\lambda\phi(T)\phi(R)).
\end{aligned}
$$

By Lemma 3.1 and the surjectivity of ϕ, we get $\phi(\lambda T) = \lambda\phi(T)$, which proves that ϕ is a linear map.

By linearity, we conclude that ϕ preserves finite rank operators in both directions.

Step 4. $\phi(I) = \alpha I$ for some scalar $\alpha \in \mathbb{C}$ such that $\alpha^3 = 1$.
Suppose that $\phi(I)$ and I are linearly independent. Let $x \in X \setminus \{0\}$ such that $\phi(I)x$ and x are linearly independent and let $f \in X^*$ such that $f(x) = 1$ and $f(\phi(I)x) = 0$. For $R = x \otimes f$, step 2 and surjectivity of ϕ ensures the existence of a non-nilpotent kank one operator $T = y \otimes g$, where $y \in$ X and $g \in X^*$ with $g(y) \neq 0$ such that $\phi(T) = R$. Therefore,

$$
\begin{aligned}
\{g^2(y)\} &= \Delta^*(T^2) \\
&= \Delta^*(\phi(T)\phi(I)\phi(T)) \\
&= \Delta^*(R\phi(I)R) \\
&= \Delta^*((x \otimes f)\phi(I)(x \otimes f)) \\
&= \{f(\phi(I)x)f(x)\} \\
&= \{0\}.
\end{aligned}
$$

Thus $g(y) = 0$. This contradiction shows that $\phi(I) = \alpha I$ where $\alpha \in \mathbb{C}^*$.
On the other hand, we have

$$
\{1\} = \Delta(I) = \Delta(\phi(I)^3) = \Delta(\alpha^3 I) = \{\alpha^3\}.
$$

This implies that $\alpha^3 = 1$, as desired.

Step 5. ϕ has the desired form.

Let $\varphi := \alpha^{-1}\phi$, and note that φ satisfies the Eq. (3.1) and $\varphi(I) = I$. It follows, from Step 3 and Lemma 2.2, that φ takes either the form (2.4) or the form (2.5).

Suppose (2.4) holds. Let $x \in X$ and $f \in X^*$, we have

$$
\begin{aligned}
\{f(x)\} &= \Delta^*(x \otimes f) \\
&= \Delta^*(I(x \otimes f)I) \\
&= \Delta^*(\varphi(I)\varphi(x \otimes f)\varphi(I)) \\
&= \Delta^*(Ax \otimes Bf) \\
&= \{Bf(Ax)\}.
\end{aligned}
$$

We get

$$
Bf(Ax) = f(x).
$$

By applying the closed graph theorem, we conclude that A is bounded. Moreover, $Bf(Ax) = f(x)$ implies that $A^*B = I_{X^*}$. Thus, $B = (A^*)^{-1} = (A^{-1})^*$. So, for every $x \in X$ and $f \in X^*$, we have

$$
\varphi(x \otimes f) = Ax \otimes Bf = Ax \otimes (A^{-1})^* f = A(x \otimes f)A^{-1}.
$$

Then

$$\varphi(R) = ARA^{-1} \ for\ all\ R \in \mathcal{F}_1(X).$$

Now, let $T \in \mathcal{B}(X)$ and $R \in \mathcal{F}_1(X)$, we have

$$\begin{aligned}\Delta(\varphi(R)\varphi(T)\varphi(R)) &= \Delta(RTR)\\ &= \Delta(ARTRA^{-1})\\ &= \Delta(ARA^{-1}ATA^{-1}ARA^{-1})\\ &= \Delta(\varphi(R)ATA^{-1}\varphi(R)).\end{aligned}$$

Since φ is surjective, Lemma 3.1 implies that

$$\varphi(T) = ATA^{-1} \ for\ all\ T \in \mathcal{B}(X).$$

Consequently $\phi(T) = \alpha ATA^{-1}$ with $\alpha^3 = 1$ for all $T \in \mathcal{B}(X)$.

In a similar way, we will treat the case where φ takes the form (2.5). We get

$$Dx(Cf) = f(x).$$

Using the closed graph theorem, separately for C and D, we conclude that these operators are bounded. Therefore, both $C^* : X^* \to X^{**}$ and $D^* : X^{**} \to X^*$ are invertible.

Moreover, by a similar way as in the proof of Theorem 2.1, we show that X is reflexive and $D = (C^*)^{-1} = (C^{-1})^*$. Therefore,

$$\varphi(R) = CR^*C^{-1} \ for\ all\ R \in \mathcal{F}_1(X).$$

Let T be an arbitrary operator in $\mathcal{B}(X)$. For every $R \in \mathcal{F}_1(X)$ we have

$$\begin{aligned}\Delta(\varphi(R)\varphi(T)\varphi(R)) &= \Delta(RTR)\\ &= \Delta(R^*T^*R^*)\\ &= \Delta(CR^*T^*R^*C^{-1})\\ &= \Delta(CR^*C^{-1}CT^*C^{-1}CR^*C^{-1})\\ &= \Delta(\varphi(R)CT^*C^{-1}\varphi(R)).\end{aligned}$$

By Lemma 3.1 and surjectivity of φ, we conclude that $\varphi(T) = CT^*C^{-1}$ for all $T \in \mathcal{B}(X)$. This implies that $\phi(T) = \alpha CT^*C^{-1}$ for all $T \in \mathcal{B}(X)$ with $\alpha^3 = 1$, and finishes the proof. □

Acknowledgements

The authors wish to express their thanks to the referee for carefully reading the paper and for giving valuable suggestions.

Author contributions The authors have no competing interests to declare that are relevant to the content of this article.

Funding No funding was received for this manuscript.

References

[1] Aupetit, B.: Spectrum-preserving linear mappings between Banach algebras or Jordan Banach algebras. J. Lond. Math. Soc. **62**, 917–924 (2000)

[2] Bourhim, A., Mashreghi, J.: Maps preserving the local spectrum of product of operators. Glasg. Math. J. **57**(3), 709–718 (2015). https://doi.org/10.1017/S0017089514000585

[3] Bourhim, A., Mashreghi, J.: Maps preserving the local spectrum of triple product of operators. Linear Multilinear Algebra. **63**(4), 765–773 (2015)

[4] Chan, J.T., Li, C.K., Sze, N.S.: Mapping preserving spectra of products of matrices. Proc. Am. Math. Soc. **135**(4), 977–986 (2007)

[5] Cui, J.-L., Hou, J.-C.: Additive maps on standard operator algebras preserving parts of the spectrum. J. Math. Anal. Appl. **282**, 266–278 (2003)

[6] Cui, J., Hou, J.: Linear maps between Banach algebras compressing certain spectral functions. Rocky Mountain J. Math. **34**(2), 565–585 (2004)

[7] Cui, J.L., Li, C.K.: Maps preserving peripheral spectrum of Jordan products of operators. Oper. Matr. **6**, 129–146 (2012)

[8] Cui, J., Li, C.K., Sze, N.S.: Unitary similarity invariant function preservers of skew products of operators. J. Math. Anal. Appl. **454**, 716–729 (2017)

[9] Hajighasemi, S., Hejazian, S.: Maps preserving some spectral domains of operator products. Linear Multilinear Algebra (2020). https://doi.org/10.1080/03081087.2020.1801567

[10] Hou, J.C., Li, C.K., Wong, N.C.: Maps preserving the spectrum of generalized Jordan product of operators. Linear Algebra Appl. **432**, 1049–1069 (2010)

[11] Jafarian, A., Sourour, A.R.: Spectrum preserving Linear maps. J. Funct. Anal. **66**, 255–261 (1986)

[12] Kaplansky, I.: Algebraic and analytic aspects of operator algebras, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 1. Providence (RI): American Mathematical Society; 1970

[13] Mbekhta, M.: Résolvant généralisé et théorie spectrale. J. Oper. Theory. **21**, 69–105 (1989)

[14] Mbekhta, M.: Linear Maps Preserving the Generalized Spectrum. Extracta Math. **22**, 45–54 (2007)

[15] Mbekhta, M.: Linear maps preserving the minimum and surjectivity modulus of operators. Oper. Matr. **4**, 511–518 (2010)

[16] Mbekhta, M., Ouahab, A.: Opérateur s-régulier dans un espace de Banach et théorie spectrale. Acta Sci. Math. (Szeged) **59**, 525–543 (1994)

[17] Miura, T., Honma, D.: A generalization of peripherally-multiplicative surjections between standard operator algebras. Cent. Eur. J. Math. **7**(3), 479–486 (2009)

[18] Molnár, L.: Some characterizations of the automorphisms of B(H) and C(X). Proc. Am. Math. Soc. **130**, 111–120 (2005)

[19] Muller, V.: Spectral theory of linear operators and spectral systems in banach algebras. Oper. Theory Adv. Appl. **139**, 2 (2007)

[20] Omladič, M., Šemrl, P.: Additive mappings preserving operators of rank one. Linear Algebra Appl. **182**, 239–256 (1993)

[21] Schmoeger, C.H.: Ein Spektralubhidungssatz. Arch. Math. **55**, 484–489 (1990)

[22] Sourour, A.R.: Invertibility preserving linear maps on L(X). Trans. Am. Math. Soc. **348**, 13–30 (1996)

[23] Zhang, W., Hou, J.: Maps preserving peripheral spectrum of Jordan semi-triple products of operators. Linear Algebra Appl. **435**, 1326–1335 (2011)

Hassane Benbouziane, Aukacha Daoudi, Mustapha Ech-Chérif El Kettani
and Ismail El Khchin
Faculty of Sciences DharMahraz Fes
University Sidi Mohammed Ben Abdellah
Fez
Morocco
e-mail: hassane.benbouziane@usmba.ac.ma;
benbhassan@yahoo.fr

Aukacha Daoudi
e-mail: daukacha@yahoo.fr

Mustapha Ech-Chérif El Kettani
e-mail: mostapha.echcherifelkettani@usmba.ac.ma

Ismail El Khchin
e-mail: ismail.elkhchin@usmba.ac.ma

Received: January 3, 2023.
Revised: April 20, 2023.
Accepted: August 24, 2023.