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On the Transfinite Symmetric Strong Diameter Two Property

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Abstract. We study transfinite analogues of the *symmetric strong diameter two property*. We investigate the stability of these properties under c_0, ℓ_∞ sums and under projective tensor products. Moreover, we characterize Banach spaces of the form $C_0(X)$, where X is a Hausdorff locally compact space, which possesses these transfinite properties via cardinal functions over X. As an application, we are able to produce a variety of examples of Banach spaces which enjoy or fail these properties.

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1. Introduction

Given an infinite-dimensional real Banach space X, its topological dual, its unit ball and its unit sphere are denoted by X^*, B_X and S_X, respectively.

Definition 1.1. A Banach space X has the *symmetric strong diameter two property* (SSD2P) if, and only if, for every $x_1^*, \dots, x_n^* \in S_{X^*}$ and $\varepsilon > 0$, there are $x_1, \dots, x_n, y \in B_X$ such that $\|y\| \geq 1-\varepsilon$, $x_i \pm y \in B_X$ and $x_i^*(x_i) \geq 1-\varepsilon$ for all $1 \leq i \leq n$.

The SSD2P was introduced in [2], but the original definition contains the additional requirement that $x_i^*(x_i \pm y) \geq 1 - \varepsilon$, which is redundant. Indeed, if we require that $x_i^*(x_i) \geq 1 - \varepsilon/2$, since $x_i \pm y \in B_X$, then $|x_i^*(y)| \leq \varepsilon/2$ and, therefore, Definition 1.1 is equivalent to the original one.

Examples of Banach spaces enjoying the SSD2P include Lindenstrauss spaces, uniform algebras, almost square Banach spaces, Banach spaces with an infinite dimensional centralizer, somewhat regular subspaces of $C_0(X)$ spaces, where X is an infinite locally compact Hausdorff space, and Müntz spaces (see [8]).

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In [3], transfinite analogs of the SSD2P were defined, but, before recalling these definitions, let us introduce some notation. Given $r \in (0,1)$, $B \subset B_X$ and $A \subset S_{X^*}$, we say that B *r-norms* A if, for every $x^* \in A$, there is $x \in B$ such that $x^*(x) \geq r$. In addition, we say that B *norms* A if it r-norms it for all $r \in (0,1)$.

Definition 1.2. [3, Definition 5.3] Let X be a Banach space and κ an infinite cardinal.

(i) X has the SSD2P$_\kappa$ if, for every set $A \subset S_{X^*}$ of cardinality $< \kappa$ and $\varepsilon > 0$, there are $B \subset B_X$, which $(1-\varepsilon)$-norms A, and $y \in B_X$ satisfying $B \pm y \subset B_X$ with $\|y\| \geq 1-\varepsilon$.
(ii) X has the 1-ASSD2P$_\kappa$ if, for every set $A \subset S_{X^*}$ of cardinality $< \kappa$, there are $B \subset S_X$, which norms A, and $y \in S_X$ satisfying $B \pm y \subset S_X$.

Here 1-A stands for 1-*norming* and *attaining*, respectively. In the following, we aim to investigate these transfinite extensions of the SSD2P and, in particular, to show differences in their behavior when compared to the regular SSD2P.

Now, let us also recall the transfinite extensions of almost squareness and the strong diameter two property.

Definition 1.3. [3, Definition 2.1] Let X be a Banach space and κ a cardinal.

(i) X is ASQ$_\kappa$ if, for every set $A \subset S_X$ of cardinality $< \kappa$ and $\varepsilon > 0$, there exists $y \in S_X$ such that $\|x \pm y\| \leq 1+\varepsilon$ holds for all $x \in A$.
(ii) X is SQ$_\kappa$ if, for every set $A \subset S_X$ of cardinality $< \kappa$, there exists $y \in S_X$ such that $\|x \pm y\| \leq 1$ holds for all $x \in A$.

It should be noted that a small change of notation is here applied. We denote (A)SQ$_\kappa$ what was written (A)SQ$_{<\kappa}$ in [3, Definition 2.1].

Definition 1.4. [5, Definitions 2.11 and 2.12] Let X be a Banach space and κ an infinite cardinal.

(i) X has the SD2P$_\kappa$ if, for every set $A \subset S_{X^*}$ of cardinality $< \kappa$ and $\varepsilon > 0$, there are $B \subset B_X$, which $(1-\varepsilon)$-norms A, and $x^* \in B_{X^*}$ satisfying $x^*(x) \geq 1-\varepsilon$ for all $x \in B$.
(ii) X has the 1-ASD2P$_\kappa$ if, for every set $A \subset S_{X^*}$ of cardinality $< \kappa$, there are $B \subset S_X$, which norms A, and $x^* \in S_{X^*}$ satisfying $x^*(x) = 1$ for all $x \in B$.

It is clear that every ASQ$_\kappa$ (SQ$_\kappa$, respectively) Banach space enjoys the SSD2P$_\kappa$ (1-ASSD2P$_\kappa$, respectively). Moreover, it was shown in [3, Proposition 5.4] that the SSD2P$_\kappa$ (1-ASSD2P$_\kappa$, respectively) implies the SD2P$_\kappa$ (1-ASD2P$_\kappa$, respectively). To sum up, the following implications hold true.

$$\begin{array}{ccccc} \mathrm{SQ}_\kappa & \Longrightarrow & \text{1-ASSD2P}_\kappa & \Longrightarrow & \text{1-ASD2P}_\kappa \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathrm{ASQ}_\kappa & \Longrightarrow & \mathrm{SSD2P}_\kappa & \Longrightarrow & \mathrm{SD2P}_\kappa \end{array}$$

1.1. Content of the Paper

In Sect. 2, we study the stability of the transfinite SSD2P with respect to operations between Banach spaces.

We provide a complete description concerning c_0 and ℓ_∞ sums (see Theorems 2.1 and 2.2), which informally state that these sums of Banach spaces enjoy the SSD2P$_\kappa$ if, and only if, we can always find one component which satisfies a property which arbitrarily well approximates the SSD2P$_\kappa$. Thanks to these characterizations, we show that, for example, the Banach spaces $c_0(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$ and $\ell_\infty(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$ enjoy the SSD2P$_\kappa$ (see Example 2.4).

We also investigate the behavior of the SSD2P$_\kappa$ under projective tensor products. Namely, we prove that the Banach space $X \hat{\otimes}_\pi Y$ has the SSD2P$_\kappa$, whenever X and Y enjoy the property.

We conclude Sect. 2 by studying the difference in the behavior of the transfinite SSD2P compared to the finite SSD2P. In particular, we prove that, for the transfinite case, it is not possible to replace the functionals with relatively weakly open sets in Definition 1.2, even though it is possible for the traditional SSD2P (see Fact 2.5 (ii)). Moreover, we prove that an equivalent internal description of the SSD2P (see Fact 2.5 (iii)) also fails in the transfinite case.

Section 3 is dedicated to extending the class of known examples which possess the transfinite SSD2P. To this aim, we search for a description in the class of $C_0(X)$ spaces, whenever X is a Hausdorff locally compact space. The main result of this section states that the Banach space $C_0(X)$ fails the SSD2P$_\kappa$, where κ is the successor cardinal of the density character of X, but it enjoys the 1-ASSD2P$_\mu$, where μ is the cellularity of X (see Theorem 3.1).

Thanks to this result, new examples are provided, e.g. $C[0,1]$ and ℓ_∞ fail the SSD2P$_{\aleph_1}$, $C(\beta\mathbb{N}\backslash\mathbb{N})$ enjoys the 1-ASSD2P$_{2^{\aleph_0}}$ and $\ell_\infty(\kappa)$ has the 1-ASSD2P$_\kappa$, whenever $\kappa > \aleph_0$.

1.2. Notation

Given a sequence of Banach spaces (X_n) we define

$$\ell_\infty(\mathbb{N}, X_n) := \left\{ x \in \prod_{n=1}^{\infty} X_n : (\forall n \in \mathbb{N})\ x(n) \in X_n \text{ and } \sup_n \|x(n)\| < \infty \right\}$$

endowed with the usual supremum norm. Moreover, we set

$$c_0(\mathbb{N}, X_n) := \left\{ x \in \ell_\infty(\mathbb{N}, X_n) : \lim_n \|x(n)\| = 0 \right\}.$$

Finally, given a cardinal κ, we define $\mathrm{cf}(\kappa)$ its cofinality and κ^+ its successor cardinal.

2. Stability Results

In this section, we investigate the behavior of the transfinite SSD2P with respect to operations between Banach spaces.

2.1. Direct Sums

Given a sequence of Banach spaces (X_n), it is known that the c_0 sum $c_0(\mathbb{N}, X_n)$ is always ASQ [1, Example 3.1] and, therefore, has the SSD2P. Moreover, it was proved in [3, Proposition 4.3] that the c_0 sum $c_0(\mathscr{A}, X_\alpha)$ of a family of Banach spaces $\{X_\alpha : \alpha \in \mathscr{A}\}$ is $\mathrm{SQ}_{|\mathscr{A}|}$ and thus has the 1-ASSD2P$_{|\mathscr{A}|}$, whenever $|\mathscr{A}| > \aleph_0$. For these reasons, in the following, we will focus only on countable c_0 sums with respect to the SSD2P$_\kappa$, for $\kappa > \aleph_0$.

Theorem 2.1. *Let (X_n) be a sequence of Banach spaces and $\kappa > \aleph_0$. If, for every $r \in [0, 1)$, there is $n \in \mathbb{N}$ such that, for every set $A \subset S_{X_n^*}$ of cardinality $< \kappa$, there exist $B \subset B_{X_n}$ and $y \in B_{X_n}$ such that $\|y\| \geq r$, B r-norms A and $B \pm y \subset B_{X_n}$, then $c_0(\mathbb{N}, X_n)$ enjoys the* SSD2P$_\kappa$. *If in addition* $\mathrm{cf}(\kappa) > \aleph_0$, *then the converse also holds.*

Proof. Fix a set $A \subset S_{\ell_1(\mathbb{N}, X_n^*)}$ of cardinality $< \kappa$ and $\varepsilon > 0$. Let $\{x_\alpha^* : \alpha \in \mathscr{A}\}$ be an enumeration of A and find $m \in \mathbb{N}$ as in the statement for $r = (1-\varepsilon)^{\frac{1}{2}}$.

By assumption, there are $y^m \in B_{X_m}$ and $x_\alpha^m \in B_{X_m}$ such that $\|y^m\| \geq 1 - \varepsilon$, $x_\alpha^m \pm y^m \in B_{X_m}$ and $x_\alpha^*(m)(x_\alpha^m) \geq (1-\varepsilon)^{\frac{1}{2}} \|x_\alpha^*(m)\|$ hold for every $\alpha \in \mathscr{A}$.

For each $m \neq n \in \mathbb{N}$ and $\alpha \in \mathscr{A}$, find $x_\alpha^n \in B_{X_n}$ satisfying $x_\alpha^*(n)(x_\alpha^n) \geq (1-\varepsilon)^{\frac{1}{2}} \|x_\alpha^*(n)\|$. Moreover, since $x_\alpha^* \in \ell_1(\mathbb{N}, X_n^*)$, there exists $n_\alpha \geq m$ such that

$$\sum_{1 \leq n \leq n_\alpha} \|x_\alpha^*(n)\| \geq (1-\varepsilon)^{\frac{1}{2}}.$$

Now define

$$x_\alpha := \sum_{1 \leq n \leq n_\alpha} x_\alpha^n e_n \in B_{c_0(\mathbb{N}, X_n)}$$

and $y := y^m e_m \in B_{c_0(\mathbb{N}, X_n)}$.

Notice that

$$x_\alpha^*(x_\alpha) = \sum_{1 \leq n \leq n_\alpha} x_\alpha^*(n)(x_\alpha^n) \geq (1-\varepsilon)^{\frac{1}{2}} \sum_{1 \leq n \leq n_\alpha} \|x_\alpha^*(n)\| \geq 1 - \varepsilon,$$

which means that the set $\{x_\alpha : \alpha \in \mathscr{A}\}$ $(1-\varepsilon)$-norms A.

On the other hand, $\|y\| = \|y^m\| \geq 1 - \varepsilon$ and

$$\|x_\alpha \pm y\| = \max\left\{ \|x_\alpha^m \pm y^m\|, \sup_{1 \leq n \neq m \leq n_\alpha} \|x_\alpha^n\| \right\} \leq 1$$

holds for all $\alpha \in \mathscr{A}$. Therefore, $c_0(\mathbb{N}, X_n)$ enjoys the SSD2P$_\kappa$.

For the converse, fix $\varepsilon > 0$ and, for every $n \in \mathbb{N}$, $A_n \subset S_{X_n^*}$ of cardinality $< \kappa$. Define

$$A := \{x^* e_n : n \in \mathbb{N} \text{ and } x^* \in A_n\} \subset S_{\ell_1(\mathbb{N}, X_n^*)}$$

and notice that $|A| \leq \aleph_0 \cdot \sup |A_n| < \kappa$, because $\mathrm{cf}(\kappa) > \aleph_0$. Therefore, there exist a set $B \subset B_{c_0(\mathbb{N}, X_n)}$, which $(1-\varepsilon)$-norms A, and $y \in B_{c_0(\mathbb{N}, X_n)}$ such that $\|y\| \geq 1 - \varepsilon$ and $B \pm y \subset B_{c_0(\mathbb{N}, X_n)}$.

Since $\|y\| \geq 1-\varepsilon$, there exists $n \in \mathbb{N}$ satisfying $\|y(n)\| \geq 1-\varepsilon$. Moreover, from the fact that, in particular, B $(1-\varepsilon)$-norms the set $\{x^*e_n : x^* \in A_n\}$, we deduce that the set $B_n := \{x(n) : x \in B\} \subset B_{X_n}$ $(1-\varepsilon)$-norms A_n.

Finally, notice that, given $x(n) \in B_n$,

$$\|x(n) \pm y(n)\| \leq \|x \pm y\| \leq 1,$$

which concludes the proof. □

Notice that the same proof can be adjusted to ℓ_∞ sums too. As a matter of fact, it is not needed to find n_α as in the proof of Theorem 2.1 and one can define

$$x_\alpha := \sum_{n=1}^{\infty} x_\alpha^n e_n \in B_{\ell_\infty(\mathbb{N},X_n)}.$$

By doing so, the following theorem is easily proved, up to a few minor changes.

Theorem 2.2. *Let $\{X_\alpha : \alpha \in \mathscr{A}\}$ be a family of Banach spaces and $\kappa > \aleph_0$. If, for every $r \in [0,1)$, there is $\alpha \in \mathscr{A}$ such that, for every set $A \subset S_{X_\alpha^*}$ of cardinality $< \kappa$, there exist $B \subset B_{X_\alpha}$ and $y \in B_{X_\alpha}$ such that $\|y\| \geq r$, B r-norms A and $B \pm y \subset B_{X_\alpha}$, then $\ell_\infty(\mathscr{A}, X_\alpha)$ enjoys the* SSD2P$_\kappa$. *If in addition* $\mathrm{cf}(\kappa) > |\mathscr{A}|$, *then the converse also holds.*

Corollary 2.3. *Let X and Y be Banach spaces and $\kappa > \aleph_0$. Either X or Y enjoy the* SSD2P$_\kappa$ *if, and only if, $X \oplus_\infty Y$ enjoys the* SSD2P$_\kappa$.

Proof. Apply Theorem 2.2 with $|\mathscr{A}| = 2$. □

Example 2.4. Let $\kappa > \aleph_0$. We claim that $c_0(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$ enjoys the SSD2P$_\kappa$, despite the fact that it is a sum of reflexive spaces. To this aim, observe that the claim is immediately achieved thanks to [3, Example 3.1], but, for the sake of providing an application of Theorem 2.1, let us prove it here again. Notice that, it suffices to show that the Banach spaces $\ell_n(\kappa)$'s satisfy the hypothesis of Theorem 2.1. To this purpose, fix $\varepsilon > 0$ and choose any $m \in \mathbb{N}$ satisfying $2^{\frac{1}{m}} \leq 1+\varepsilon$. Now fix a set $A \subset S_{\ell_m(\kappa)^*}$ of cardinality $< \kappa$ and let $\{x_\alpha^* : \alpha \in \mathscr{A}\}$ be an enumeration for A. Moreover, find, for each $\alpha \in \mathscr{A}$, $x_\alpha \in S_{\ell_m(\kappa)}$ satisfying $x_\alpha^*(x_\alpha) \geq 1-\varepsilon$.

Since the support of the x_α's is at most countable and $\kappa > \aleph_0$, there exists an ordinal $\mu < \kappa$ such that $x_\alpha(\mu) = 0$ holds for all $\alpha \in \mathscr{A}$. Let $y \in B_{\ell_m(\kappa)}$ be defined by $y(\mu) := \delta_\lambda^\mu$ and notice that

$$\|x_\alpha \pm y\| = \left(\sum_{\lambda<\kappa} |x_\alpha(\lambda)|^m + 1\right)^{\frac{1}{m}} = 2^{\frac{1}{m}} \leq 1+\varepsilon$$

holds for every $\alpha \in \mathscr{A}$. Notice that, up to a small perturbation argument, we showed that the Banach spaces $\ell_n(\kappa)$'s satisfy the hypothesis of Theorem 2.1, thus the claim is proved.

It is then clear that also the Banach space $\ell_\infty(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$ enjoys the SSD2P$_\kappa$, thanks to Theorem 2.2.

2.2. Tensor Product

It is known that the SSD2P is preserved by taking projective tensor products [10, Theorem 2.2]. In the cited paper, the authors' proof relies on the following characterization of the SSD2P.

Fact 2.5. [8, Theorem 2.1] *Let X be a Banach space. The following assertions are equivalent:*

(i) *X has the* SSD2P.
(ii) *Given non-empty relatively weakly open sets $U_1, \dots, U_n$ in B_X and $\varepsilon > 0$, there exist $x_1, \dots, x_n, y \in B_X$ such that $\|y\| \geq 1 - \varepsilon$, $x_i \pm y \in B_X$ and $x_i \in U_i$ for all $1 \leq i \leq n$.*
(iii) *Given $x_1, \dots x_n \in S_X$, there exist nets (y^i_α) and (z_α) in S_X such that $\lim \|y^i_\alpha \pm z_\alpha\| = 1$ and, with respect to the weak topology on X, $\lim z_\alpha = 0$ and $\lim y^i_\alpha = x_i$ hold for all $1 \leq i \leq n$.*

As we will later demonstrate, Fact 2.5 doesn't hold true for the SSD2P$_\kappa$ whenever $\kappa > \aleph_0$. Therefore, a different proof is required to extend [10, Theorem 2.2] to the transfinite setting.

Theorem 2.6. *Let X and Y be Banach spaces and $\kappa > \aleph_0$. If X and Y have the* SSD2P$_\kappa$*, then the projective tensor product $X \hat{\otimes}_\pi Y$ enjoys the* SSD2P$_\kappa$*.*

Proof. Fix a set $\mathscr{B} \subset S_{(X \hat{\otimes}_\pi Y)^*}$ of cardinality $< \kappa$ and $\varepsilon > 0$. Recall that the Banach space $(X \hat{\otimes}_\pi Y)^*$ is isometrically isomorphic to the space of bounded bilinear forms acting on $X \times Y$ [13, Theorem 2.9], hence, for every $B \in \mathscr{B}$, there exists $x_B \otimes y_B \in S_X \otimes S_Y$ satisfying $B(x_B \otimes y_B) \geq (1 - \varepsilon)^{\frac{1}{3}}$.

Given $B \in \mathscr{B}$, define

$$B' := \frac{B(\cdot \otimes y_B)}{\|B(\cdot \otimes y_B)\|} \in S_{X^*}.$$

Since X has the SSD2P$_\kappa$, there are x and x'_B's in B_X such that $\|x\| \geq (1 - \varepsilon)^{\frac{1}{2}}$ and, for all $B \in \mathscr{B}$, $x'_B \pm x \in B_X$ and $B'(x'_B) \geq (1 - \varepsilon)^{\frac{1}{3}}$.

Now, given $B \in \mathscr{B}$, define

$$B'' := \frac{B(x'_B \otimes \cdot)}{\|B(x'_B \otimes \cdot)\|} \in S_{Y^*}.$$

Since Y has the SSD2P$_\kappa$, there are y and y''_B's in B_Y such that $\|y\| \geq (1 - \varepsilon)^{\frac{1}{2}}$ and, for all $B \in \mathscr{B}$, $y''_B \pm y \in B_Y$ and $B''(y''_B) \geq (1 - \varepsilon)^{\frac{1}{3}}$.

Define $u_B := x'_B \otimes y''_B \in B_{X \hat{\otimes}_\pi Y}$ and $v := x \otimes y \in B_{X \hat{\otimes}_\pi Y}$. Notice that $\|v\| = \|x\|\|y\| \geq 1 - \varepsilon$. Moreover, the fact that $u_B \pm v \in B_{X \hat{\otimes}_\pi Y}$ is due to [12, Lemma 2.2]. Finally, let us prove that the set $\{u_B : B \in \mathscr{B}\}$ $(1 - \varepsilon)$-norms $\mathscr{B}$.

$$\begin{aligned} B(u_B) = B(x'_B \otimes y''_B) &\geq (1 - \varepsilon)^{\frac{1}{3}} \|B(x'_B \otimes \cdot)\| \geq (1 - \varepsilon)^{\frac{1}{3}} B(x'_B \otimes y_B) \\ &\geq (1 - \varepsilon)^{\frac{2}{3}} \|B(\cdot \otimes y_B)\| \geq (1 - \varepsilon)^{\frac{2}{3}} B(x_B \otimes y_B) \geq 1 - \varepsilon, \end{aligned}$$

which proves the claim and thus concludes the proof. □

Remark 2.7. It is known that requiring only one component to have the SSD2P is not enough in order to ensure the projective tensor product enjoys the SSD2P [11, Corollary 3.9]. Up to a few changes, the same ideas can be used to show that requiring in the statement of Theorem 2.6 only one component to enjoy the SSD2P$_\kappa$ is not enough. Let us sketch the argument required to prove this statement.

We will later show that $\ell_\infty(\kappa)$ has the 1-ASSD2P$_\kappa$ (see Example 3.3), nevertheless, we claim that the Banach space $X := \ell_\infty(\kappa)\hat{\otimes}_\pi \ell_3^3$ doesn't enjoy the SSD2P$_\kappa$.

Since ℓ_3^3 is not finitely representable in ℓ_1, it is not finitely representable in $\ell_1(\kappa)$ either (notice that each finite-dimensional subspace of $\ell_1(\kappa)$ is isometrically isomorphic to some finite-dimensional subspace of ℓ_1, and converse). Thanks to a simple transfinite analogue of [11, Lemma 3.7] (replacing finite dimensional spaces with spaces of density $< \kappa$) we conclude that $\ell_1(\kappa)\hat{\otimes}_\varepsilon(\ell_3^3)^*$ is not κ-octahedral (see [4, Definition 5.3]), moreover, we can infer that $X = (\ell_1(\kappa)\hat{\otimes}_\varepsilon(\ell_3^3)^*)^*$ [13, Theorem 5.3]. Therefore, by applying [5, Theorem 3.2], we conclude that X fails the SSD2P$_\kappa$.

2.3. Some More Remarks

Previously we claimed that the transfinite analog of Fact 2.5 doesn't hold true. Let us now prove this statement for the implication (i) $\iff$ (ii) by continuing the investigation that we began in Example 2.4. As a matter of fact, the Banach space $c_0(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$ enjoys the SSD2P$_\kappa$. Nevertheless, we claim that it fails condition (ii) from Fact 2.5 with respect to $\aleph_1$. This claim follows from the following theorem:

Theorem 2.8. *Let (X_n) be a sequence of Banach spaces. If, given any sequence of relatively weakly open sets (U_n) in $B_{c_0(\mathbb{N},X_n)}$ and $\varepsilon > 0$, there exist (x_n) and y in $B_{c_0(\mathbb{N},X_n)}$ such that $\|y\| \geq 1-\varepsilon$, $x_n \pm y \in B_{c_0(\mathbb{N},X_n)}$ and $x_n \in U_n$ for all $n \in \mathbb{N}$, then there exists $m \in \mathbb{N}$ such that X_m is not uniformly convex.*

Proof. Let $A := \{x_n^* : n \in \mathbb{N}\} \subset S_{\ell_1(\mathbb{N},X_n^*)}$, where the x_n^*'s are any chosen elements satisfying the following conditions:

$$x_n^*(m) \neq 0 \text{ and } \|x_n^*(n)\| \geq \|x_n^*(m)\| \text{ for all } n, m \in \mathbb{N}.$$

Now consider the relatively weakly open sets

$$U_{n,m} := \{x \in B_{c_0(\mathbb{N},X_n)} : x_n^*(x) > 1 - m^{-1}\|x_n^*(m)\|\}$$

Fix $\varepsilon > 0$ and find $x_{n,m} \in U_{n,m}$ and $y \in B_{c_0(\mathbb{N},X_n)}$ such that $\|y\| \geq 1-\varepsilon$ and $y \pm x_{n,m} \in B_{c_0(\mathbb{N},X_n)}$ hold for all $n, m \in \mathbb{N}$.

Since $\|y\| \geq 1-\varepsilon$, we can find $p \in \mathbb{N}$ such that $\|y(p)\| \geq 1-\varepsilon$. On the other hand, since $x_{p,m} \in U_{p,m}$, we have that

$$\begin{aligned} 1 - m^{-1}\|x_p^*(m)\| \leq x_p^*(x_{p,m}) &\leq \sum_{n\neq p} \|x_p^*(n)\| + x_p^*(p)(x_{p,m}(p)) \\ &= 1 - \|x_p^*(p)\| + x_p^*(p)(x_{p,m}(p)), \end{aligned}$$

hence

$$x_p^*(p)(x_{p,m}(p)) \geq \|x_p^*(p)\| - m^{-1}\|x_p^*(m)\|,$$

therefore

$$\|x_{p,m}(p)\| \geq 1 - m^{-1}\frac{\|x_p^*(m)\|}{\|x_p^*(p)\|} \geq 1 - m^{-1}.$$

Now, the fact that $x_{p,m} \pm y \in B_{c_0(\mathbb{N},X_n)}$ implies

$$1 \geq \|x_{p,m} \pm y\| \geq \|x_{p,m}(p) \pm y(p)\|.$$

Finally, let us compute the modulus of convexity of X_p.

$$\begin{aligned}
\delta_{X_p}(2-2\varepsilon) &:= \inf\left\{1 - \left\|\frac{u+v}{2}\right\| : u,v \in B_{X_p} \text{ and } \|u-v\| \geq 2-2\varepsilon\right\}\\
&\leq \inf_m \left(1 - \frac{\|(x_{p,m}(p)+y(p)) + (x_{p,m}(p)-y(p))\|}{2}\right)\\
&= \inf_m(1 - \|x_{p,m}(p)\|) \leq \inf_m m^{-1}\\
&= 0,
\end{aligned}$$

which implies that X_p is not uniformly convex. □

Let us now turn our attention to the implication (i) $\Longleftrightarrow$ (iii) from Fact 2.5. We claim that also this fails in the transfinite context.

Example 2.9. We will prove that $\ell_\infty(\kappa)$ fails the SSD2P$_{\kappa^+}$ (see Example 3.3). Nevertheless, condition (iii) from Fact 2.5 is satisfied in a very strong way. In fact, fix $x \in S_{\ell_\infty(\kappa)}$, an ordinal $\mu < \kappa$ and define $y_\mu^x := x - x(\mu)e_\mu \in B_{\ell_\infty(\kappa)}$ and $z_\mu := e_\mu \in S_{\ell_\infty(\kappa)}$. It is then clear that $y_\mu^x \pm z_\mu \in S_X$ and that, with respect to the weak topology, $\lim z_\mu = 0$ and $\lim y_\mu^x = x$ holds for every $x \in S_{\ell_\infty(\kappa)}$. In other words, since $|\ell_\infty(\kappa)| = 2^\kappa$, we showed that $\ell_\infty(\kappa)$ satisfies condition (iii) from Fact 2.5, where, instead of fixing a finite set in the unit sphere, we can fix any subset of the unit sphere of cardinality at most 2^κ.

Despite Theorem 2.8 and Example 2.9, it is possible to recover some transfinite analog of Fact 2.5, but only for the 1-ASSD2P$_\kappa$.

Proposition 2.10. *Let X be a Banach space and $\kappa > \aleph_0$. Consider the following statements:*

- (i) *X has the* 1-ASSD2P$_\kappa$.
- (ii) *Given a family $\mathscr{U}$ consisting of $< \kappa$ many relatively weakly open sets in B_X, a relatively weakly open neighborhood V of 0 in B_X and $\varepsilon > 0$, there are $\{x_U : U \in \mathscr{U}\}$ and $y \in V \cap S_X$ satisfying $x_U \in U$ and $x_U \pm y \in B_X$ for all $U \in \mathscr{U}$.*
- (iii) *Given $A \subset S_X$ of cardinality $< \kappa$, there are nets $\{(y_\alpha^x) : x \in A\}$ and (z_α) in S_X satisfying $\lim \|z_\alpha \pm y_\alpha^x\| = 1$ and, with respect to the weak topology, $\lim z_\alpha = 0$ and $\lim y_\alpha^x = x$ for all $x \in A$.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. (i) $\Longrightarrow$ (ii). Fix a family $\mathscr{U}$ consisting of $< \kappa$ many relatively weakly open sets in B_X, a relatively weakly open neighborhood V of 0 in B_X and $\varepsilon > 0$. For every $U \in \mathscr{U}$, thanks to Bourgain's lemma [7, Lemma II.1], we can find functionals $x^*_{1,U}, \dots, x^*_{n_U,U} \in S_{X^*}$, $\varepsilon_U > 0$ and convex coefficients $r_{1,U}, \dots, r_{n_U,U}$ such that

$$\left\{ \sum_{i=1}^{n_U} r_i x_i : x^*_{i,U}(x_i) > 1 - \varepsilon_U \text{ for all } 1 \le i \le n_U \right\} \subset U.$$

Moreover, we can find $x^*_{1,V}, \dots, x^*_{n_V,V} \in S_{X^*}$ and $\varepsilon_V > 0$ satisfying

$$\{x \in B_X : |x^*_{i,V}(x)| \le \varepsilon_V \text{ for all } 1 \le i \le n_V\} \subset V.$$

Since X has the 1-ASSD2P$_\kappa$ and $|\{x^*_{i,U} : 1 \le i \le n_U \text{ and } U \in \mathscr{U} \cup \{V\}\}| \le \aleph_0 \cdot |\mathscr{U}| < \kappa$, there exist $\{x_{i,U} : 1 \le i \le n_U \text{ and } U \in \mathscr{U} \cup \{V\}\}$ and y in S_X satisfying $x^*_{i,U}(x_{i,U}) \ge 1 - \varepsilon_U$ and $x_{i,U} \pm y \in S_X$ for all $1 \le i \le n_U$ and $U \in \mathscr{U} \cup \{V\}$. Now, given $U \in \mathscr{U}$, define

$$x_U := \sum_{i=1}^{n_U} r_i x_{i,U} \in B_X$$

and notice that $x_U \in U$. Moreover,

$$\|x_U \pm y\| = \left\| \sum_{i=1}^{n_U} r_i (x_{i,U} \pm y) \right\| \le \sum_{i=1}^{n_U} r_i \|x_{i,U} \pm y\| = 1.$$

In order to conclude, it only remains to prove that $y \in V$. But this is clear because for every $1 \le i \le n_V$, we have that

$$1 = \|x_{i,V} \pm y\| \ge x^*_{i,V}(x_{i,V} \pm y) \ge 1 - \varepsilon_V \pm x^*_{i,V}(y),$$

which means that $|x^*_{i,V}(y)| \le \varepsilon_V$, hence $y \in V$.

(ii) $\implies$ (iii). Fix a set $A \subset S_X$ of cardinality $< \kappa$ and temporarily fix a weak neighborhood U of 0. Define $\mathscr{U} := \{(x+U) \cap B_X : x \in A\}$ and find $\{y^x_U : x \in A\} \subset B_X$ and $z_U \in U \cap S_X$ satisfying $y^x_U \in x + U$ and $y^x_U \pm z_U \in B_X$ for all $x \in A$.

Now semi-order the family of weakly open neighborhoods of 0 with respect to the inclusion and consider the nets (y^x_U) and (z_U). It is clear that $\lim y^x_U = x$, $\lim z_U = 0$ and $\lim \|z_U \pm y^x_U\| = 1$ holds for all $x \in A$. Moreover, up to a perturbation argument, we can assume that all y^x_U's belong to S_X. Thus the claim is proved. □

Let us show that the implication (iii) $\implies$ (ii) from Proposition 2.10 fails. As already witnessed by Example 2.9, $\ell_\infty(\kappa)$ satisfies condition (iii) in a very strong way, nevertheless it fails the SSD2P$_{\kappa^+}$. Therefore, we only need to notice that condition (ii) with respect to κ^+ clearly implies possessing the SSD2P$_{\kappa^+}$, thus the claim is proved.

It remains unclear whether the implication (ii) $\implies$ (i) holds.

Remark 2.11. One might have wondered whether Theorem 2.1 can be pushed further and used to obtain c_0 sums which possess the 1-ASSD2P$_\kappa$. Unfortunately, this doesn't happen, as a matter of fact, the space $c_0(\mathbb{N}_{\ge 2}, \ell_n(\kappa))$ fails the 1-ASSD2P$_\kappa$ because, if it had the property, then Theorem 2.10 would apply and this would lead to a contradiction when combined with Theorem 2.8.

3. $C_0(X)$ Spaces

In [2], it was proved that $C_0(X)$, for X infinite Hausdorff locally compact, always has the SSD2P. In this section, we aim to extend the class of examples that enjoy the transfinite SSD2P by trying to characterize under which conditions $C_0(X)$ spaces have this property. Before doing so, let us introduce a bit of notation about some cardinal functions.

Let X be a topological space. Define the *density character* of X as

$$\mathrm{d}(X) := \min\{|\mathscr{D}| : \mathscr{D} \subset X \text{ is dense}\} + \aleph_0.$$

A cellular family in X is a family of mutually disjoint open sets in X. Define the *cellularity* of X as

$$\mathrm{c}(X) := \sup\{|\mathscr{C}| : \mathscr{C} \text{ is a cellular family in } X\} + \aleph_0.$$

It is well known that $\mathrm{c}(X) \leq \mathrm{d}(X)$. We refer the reader to [9] for a detailed treatment of these cardinal functions and more.

Before stating the main result of this section, let us recall that, thanks to the Riesz–Markov representation theorem, every continuous linear functional on $C_0(X)$ admits a unique representation as a regular countably additive Borel measure on X.

Theorem 3.1. *Let X be a Hausdorff locally compact space.*
- (i) *$C_0(X)$ fails the* $\mathrm{SSD2P}_{\mathrm{d}(X)^+}$.
- (ii) *If $\mathrm{c}(X) > \aleph_0$, then $C_0(X)$ has the* 1-$\mathrm{ASSD2P}_{\mathrm{c}(X)}$.

Proof. (i). Let $\mathscr{D}$ be dense in X. Consider the set $\{\delta_x : x \in \mathscr{D}\} \subset S_{C_0(X)^*}$ and suppose for contradiction that $C_0(X)$ has the $\mathrm{SSD2P}_{\mathrm{d}(X)^+}$. Then we can find functions $\{f_x : x \in \mathscr{D}\} \subset B_{C_0(X)}$ and $g \in B_{C_0(X)}$ satisfying

$$\|g\| \geq 2/3,\ f_x(x) \geq 2/3 \text{ and } \|f_x \pm g\| \leq 1.$$

Since $\mathscr{D}$ is dense, then we can find $x \in \mathscr{D}$ such that $|g(x)| > 1/3$, which contradicts the fact that $|f_x(x) \pm g(x)| \leq 1$.

(ii). Fix $\lambda < \mathrm{c}(X)$ and a set $\mathscr{M} \subset S_{C_0(X)^*}$ of cardinality λ. Find a cellular family $\mathscr{C}$ in X of size $\lambda < |\mathscr{C}| \leq \mathrm{c}(X)$ and, given any $m \in \mathbb{N}$ and $\mu \in \mathscr{M}$, define

$$\mathscr{C}_{m,\mu} := \{C \in \mathscr{C} : |\mu|(C) > m^{-1}\}.$$

Notice that

$$|\{\mathscr{C}_{m,\mu} : m \in \mathbb{N} \text{ and } \mu \in \mathscr{M}\}| \leq \aleph_0 \cdot \lambda < |\mathscr{C}|.$$

Therefore, there is $C \in \mathscr{C}$ satisfying $|\mu|(C) = 0$ for every $\mu \in \mathscr{M}$. Notice that, without loss of generality, we can assume that $|\mu|(\overline{C}) = 0$. In fact, if that's not the case, then we can replace C with some non-empty open set C' satisfying $\overline{C'} \subset C$.

Find functions $\{f_{m,\mu} : m \in \mathbb{N} \text{ and } \mu \in \mathscr{M}\} \subset S_{C_0(X)}$ such that $\mu(f_{m,\mu}) \geq 1 - (3m)^{-1}$ and, since μ's are regular, compact sets $\{K_{m,\mu} : m \in \mathbb{N} \text{ and } \mu \in \mathscr{M}\} \subset X \backslash \overline{C}$ satisfying $|\mu|(K_{m,\mu}) \geq 1 - (3m)^{-1}$. Now construct Urysohn's functions $\{g_{m,\mu} : m \in \mathbb{N} \text{ and } \mu \in \mathscr{M}\}$ and h in $S_{C_0(X)}$ satisfying

$$g_{m,\mu}|_{K_{m,\mu}} = 1,\ g_{m,\mu}|_{\overline{C}} = 0 \text{ and } h|_{X\backslash C} = 0.$$

Define

$$i_{m,\mu} := \frac{f_{m,\mu} \cdot g_{m,\mu}}{\|f_{m,\mu} \cdot g_{m,\mu}\|} \in S_{C_0(X)}$$

and notice that $i_{m,\mu} \pm h \in S_{C_0(X)}$. Moreover, given any $m \in \mathbb{N}$ and $\mu \in \mathscr{M}$,

$$\begin{aligned}\mu(i_{m,\mu}) &\geq \int_X f_{m,\mu} \cdot g_{m,\mu} d\mu \geq \int_{K_{m,\mu}} f_{m,\mu} d\mu - (3m)^{-1} \\ &\geq \int_X f_{m,\mu} d\mu - 2 \cdot (3m)^{-1} \geq 1 - m^{-1}.\end{aligned}$$

□

It remains unclear whether the statement of Theorem 3.1 can be written using only one cardinal function. Namely, we don't know the answer to the following two questions:

Question 3.2. *Let X be a Hausdorff locally compact space. Is it true that $C_0(X)$ fails the* $\mathrm{SSD2P}_{\mathrm{c}(X)^+}$*? Is it true that $C_0(X)$ enjoys the* 1-$\mathrm{ASSD2P}_{\mathrm{d}(X)}$*, whenever* $\mathrm{d}(X) > \aleph_0$*?*

Example 3.3. Let us now employ Theorem 3.1 to produce some new examples of spaces enjoying or failing the transfinite SSD2P.

(i) Let X be a separable locally compact Hausdorff space. It is clear that $\mathrm{c}(X) \leq \mathrm{d}(X) = \aleph_0$, hence $C_0(X)$ fails the $\mathrm{SSD2P}_{\aleph_1}$.
(ii) It is known that $\mathrm{c}(\beta\mathbb{N} \setminus \mathbb{N}) = 2^{\aleph_0}$ [9, 7.22], therefore $C(\beta\mathbb{N} \setminus \mathbb{N})$ enjoys the 1-$\mathrm{ASSD2P}_{2^{\aleph_0}}$.
(iii) Let B be a Boolean algebra and let $S(B)$ be the Stone space associated to B. It is clear that the set $\{\{b\} : b \in B\} \subset S(B)$ defines a cellular family in $S(B)$.
Now let us consider a regular positive Borel measure μ over some Hausdorff locally compact space X. define $\mathfrak{B}_\mu$ the set of measurable sets modulo the negligible sets in X. It is known that $L_\infty(\mu)$ is isometrically isomorphic to $C(S(\mathfrak{B}_\mu))$ (see e.g. pages 27–29 in [6]), therefore we conclude that $L_\infty(\mu)$ enjoys the 1-$\mathrm{ASSD2P}_{|\mathfrak{B}_\mu|}$, whenever $|\mathfrak{B}_\mu| > \aleph_0$.
In particular, whenever $\kappa > \aleph_0$ and μ is the counting measure over κ, $|\mathfrak{B}_\mu| = \kappa$, thus it follows that $\ell_\infty(\kappa)$ enjoys the 1-$\mathrm{ASSD2P}_\kappa$, but it fails the $\mathrm{SSD2P}_{\kappa^+}$, because $\mathrm{d}(\ell_\infty(\kappa)) = \kappa$.

To conclude this section, let us provide a criterion to identify cellular families in particular classes of topological spaces, including Alexandrov-discrete spaces.

Proposition 3.4. *Let X be a $T_{2\frac{1}{2}}$ space and κ an infinite cardinal. If there are κ many points in X such that every non-empty intersection of at most κ many neighborhoods is still a neighborhood, then* $\mathrm{c}(X) \geq \kappa$.

Proof. Let $A \subset X$ be a set of cardinality κ such that every non-empty intersection of at most κ many neighborhoods of x is still a neighborhood for every $x \in A$. Since X is $T_{2\frac{1}{2}}$, for every distinct $x, y \in A$ we can find a closed neighborhood $U_{x,y}$ of x which doesn't contain y. By assumption

$$U_x := \left(\bigcap_{y \in A \setminus \{x\}} U^\circ_{x,y} \right) \cap \left(\bigcap_{y \in A \setminus \{x\}} X \setminus U_{y,x} \right)$$

is an open neighborhood of x. Notice that, given distinct $x, y \in A$ we have that

$$U_x \cap U_y \subset U^\circ_{x,y} \cap (X \setminus U_{y,x}) \cap U^\circ_{y,x} \cap (X \setminus U_{x,y}) = \emptyset$$

In other words, $\{U_x : x \in A\}$ defines a cellular family of size κ. □

Notice that the assumption in Proposition 3.4 is far from being necessary. It is consistent with ZFC that $\beta\mathbb{N} \setminus \mathbb{N}$ contains no P-points, that is, points for which every G_δ containing them is a neighborhood, nevertheless, as already recalled in Example 3.3, $\beta\mathbb{N} \setminus \mathbb{N}$ has a cellular family of cardinality $2^{\aleph_0}$.

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Declarations

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