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Existence and Uniqueness of Periodic Solutions for a Class of Higher Order Differential Equations

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Abstract. In this paper, we study the existence, non-existence and uniqueness of periodic solutions for a class of higher order differential equations. The proof is based on the Mawhin's continuation theorem and averaging method. Finally, two examples are given to illustrate the applicability of the conclusions of this paper.

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Keywords. Higher order differential equation, periodic solution, averaging method, Mawhin's continuation theorem.

1. Introduction

In recent years, the existence of periodic solutions of second-order differential equations have received extensive attention, see for instance the papers [1, 2, 4, 8, 11, 12, 19, 26–28]. In Ref. [21], Morris studied the existence of periodic solutions of differential equation

$$x'' + 2x^3 = p(t), \tag{1}$$

where $p(t)$ is a continuous periodic function. Furthermore, the authors studied the existence and stability of periodic solution of Eq. (1) in Ref. [7, 23].

In Ref. [18], Llibre and Makhlouf extended the second-order differential Eq. (1) to the second-order differential equations of the form

$$x'' \pm x^n = \mu f(t), \tag{2}$$

where $n = 4, 5, \ldots$, $f(t)$ is a continuous T-periodic function with $\int_0^T f(t)\mathrm{d}t \neq 0$, μ is a positive small parameter, the authors using averaging theory established the existence and stability of T-periodic solutions of Eq. (2). In Ref. [20], Makhlouf and Djamel extended the second-order Eq. (2) to the third order, and obtained the existence and stability of periodic solutions.

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Furthermore, in Ref. [3], Cen, Llibre and Zhang extended the equations in Refs. [18,20] to more general higher order differential equations:

$$x^{(m)} + f_n(x) = \mu h(t), \tag{3}$$

where the integers $m, n \geqslant 2$, $f_n(x) = \delta x^n$ or $f_n(x) = \delta |x|^n$ with $\delta = \pm 1$, $h(t)$ is a continuous T-periodic function with $\int_0^T h(t)\mathrm{d}t \neq 0$, and μ is a positive small parameter. The authors obtained the existence and stability of T-periodic solutions of Eq. (3). At the same time, the existence, uniqueness and stability of periodic solutions of higher order differential equations have been extensively studied; see Refs. [13–16,22,24,25].

In Refs. [3,18], if $\mu = 1$, the studied problem degenerates to the case without parameters, for example, if $\mu = 1$, then Eq. (2) degenerates to Eq. (1). However, in Refs. [3,18], the authors studied the differential equation with small parameters, that is, $\mu < 1$, at this time, the studied equation cannot degenerate to the case without parameters. Therefore, it is meaningful to consider the periodic solutions of differential equations without parameters. At the same time, an interesting problem is to consider the periodic solution of differential equations when the parameters are large enough or small enough. To the best of our knowledge, there is no references discuss this issue.

The purpose of the current article is to investigate the existence of periodic solutions and non-existence of positive periodic solutions of higher order differential equations:

$$x^{(n)} + a(t)x^{\beta} = h(t), \tag{4}$$

where $n \geqslant 1$ is integer, $\beta > 0$, $a(t)$ and $h(t)$ are T-periodic L^1-functions. We also consider the uniqueness and non-existence of positive periodic solutions to higher order differential equations with parameters:

$$x^{(n)} + a(t)x^{\beta} = \nu h(t), \tag{5}$$

where $n \geqslant 1$ is integer, $\beta > 0$, $\nu > 0$, $a(t)$ and $h(t)$ are continuous T-periodic functions.

The remainder part of this paper is organized as follows. In Sect. 2, we collected some general results, given some notations and assumptions. In Sect. 3, we first apply the Mawhin's continuation theorem to obtain the existence of T-periodic solutions of Eq. (4). Then, inspired by Ref. [5], we establish the result that there is no positive T-periodic solution for Eq. (4). Further, we use averaging method to obtain the uniqueness of positive T-periodic solutions of Eq. (5). In Sect. 4, we apply the previous result to some examples to demonstrate the applicability of our main results.

2. Preliminaries

Throughout this paper, let Banach spaces $C_T = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) \equiv x(t),\ \forall\, t \in \mathbb{R}\}$ with the norm $|x|_\infty = \max_{t\in[0,T]}|x(t)|$. For a given L^1-function $w : [0,T] \to \mathbb{R}$, we denote $\bar{w} = \frac{1}{T}\int_0^T w(s)\mathrm{d}s$, when $w(t)$ does not change the sign, we stands for

$$w^+ = \max_{t\in[0,T]} |w(t)|,\ w^- = \min_{t\in[0,T]} |w(t)|.$$

When $w(t)$ changes sign, we represent

$$w^* = \max\{w(t) : t \in [0,T]\},\ w_* = \min\{w(t) : t \in [0,T]\}.$$

Now, we give expressions of constants C_1 and C_2. Let

$$C_1 = \left(\frac{\pi^{n-2}N_3}{a^+N_3^\beta + h^+}\right)^{\frac{1}{n}},$$
$$C_2 = \min\left\{\left(\frac{\pi^{n-2}N_4}{a^+N_4^\beta + h^*}\right)^{\frac{1}{n}}, \left(\frac{\pi^{n-2}N_4}{a^+N_4^\beta - h_*}\right)^{\frac{1}{n}}\right\},$$

where $n > 1$ is even integer, $N_3 > 2\big(\frac{h^+}{a^-}\big)^{\frac{1}{\beta}}$, $N_4 > \max\{2\big(\frac{h^*}{a^-}\big)^{\frac{1}{\beta}}, 2\big(-\frac{h_*}{a^-}\big)^{\frac{1}{\beta}}\}$ are constants.

In order to study the existence of T-periodic solutions to Eq. (4), we list the following assumptions:

(H1) *Suppose $a(t)$ is a T-periodic L^1-function and does not change the sign on $[0,T]$, $h(t)$ is a T-periodic L^1-function with $h(t) \not\equiv 0$;*

(H2) *Suppose $\beta \in \{\frac{l}{k}\}$, where k, l are positive odd integers.*

For convenience, we introduce some notations and an abstract existence theorem about coincidence degree theory. For more details, see Ref. [9].

Let X, Y be two Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ be linear mapping and $N : X \to Y$ be continuous mapping. The mapping L is said to be a Fredholm mapping of index zero if $\operatorname{Im} L$ is closed in Y and $\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L < +\infty$. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. It follows that the restriction L_P of L to $\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of L_P by K_P. If Ω is a bounded open subset of X, N is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \Omega \to X$ is compact.

Lemma 2.1. (Mawhin's Continuation Theorem) *Let X and Y be two Banach spaces, L be a Fredholm mapping of index zero, $\Omega \subset X$ is an open bounded set and N is L-compact on $\overline{\Omega}$. If all the following conditions hold:*

1. *$Lx \neq \lambda Nx$ for all $x \in \partial\Omega \cap \operatorname{Dom} L$, and all $\lambda \in (0,1)$;*
2. *$QNx \neq 0$, for all $x \in \partial\Omega \cap \operatorname{Ker} L$;*
3. *$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism.*

Then, the equation $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$.

Next, we introduce an abstract averaging method. For more details, see Ref. [6].

Put $m \geqslant 1$ is an integer, $T > 0$, $\varepsilon_1 > 0$ and $I \subset \mathbb{R}$ is an open interval, let

$$f : [0,T] \times I \times \mathbb{R} \times \cdots \times \mathbb{R} \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R},$$
$$(t, u_0, u_1, \ldots, u_{m-1}, \varepsilon) \mapsto f(t, u_0, u_1, \ldots, u_{m-1}, \varepsilon)$$

is continuous and, for any $k = 0, \ldots, m-1$, $\frac{\partial f}{\partial u_k}$ exists and continuous.

Let $a_1, a_2, \ldots, a_{m-1} \in \mathbb{R}$, we consider the following problem:

$$\begin{cases} x^{(m)} + \sum\limits_{j=1}^{m-1} a_{m-j} x^{(m-j)} = \varepsilon f(t, x, x', \ldots, x^{(m-1)}, \varepsilon), \\ x^{(j)}(0) = x^{(j)}(T), \quad (j = 0, \ldots, m-1), \end{cases} \tag{6}$$

and define the periodic averaged function

$$F : I \to \mathbb{R}, \ c \mapsto F(c) := \frac{1}{T} \int_0^T f(s, c, 0, \ldots, 0, 0) \mathrm{d}s. \tag{7}$$

Then, $F \in C^1(I, \mathbb{R})$ and, for $\forall c \in I$, we have

$$F'(c) = \frac{1}{T} \int_0^T \frac{\partial f}{\partial u_0}(s, c, 0, \ldots, 0, 0) \mathrm{d}s. \tag{8}$$

Lemma 2.2. (Averaging method) *Assume that the linear problem*

$$x^{(m)} + \sum_{j=1}^{m-1} a_{m-j} x^{(m-j)} = 0, \ x^{(j)}(0) = x^{(j)}(T), j = 0, \ldots, m-1,$$

has only constant solutions. Then, for every $c_0 \in I$ such that

$$F(c_0) = 0, \ F'(c_0) \neq 0,$$

where F is given by (7), there exists $\varepsilon_0 \in [-\varepsilon_1, \varepsilon_1] \backslash \{0\}$ such that, for $0 < |\varepsilon| < \varepsilon_0$, the problem (6) has a unique solution $x(t, \varepsilon)$ such that $\lim\limits_{\varepsilon \to 0} x(t, \varepsilon) = c_0$ uniformly in $t \in [0, T]$.

Below we introduce two commonly used inequalities.

Lemma 2.3. [10, Lemma 5.2] *Let $x : [0, T] \to \mathbb{R}$ be an arbitrary absolutely continuous function with $x(0) = x(T)$. Then, the inequality*

$$\left(\max_{t \in [0,T]} x(t) - \min_{t \in [0,T]} x(t) \right)^2 \leqslant \frac{T}{4} \int_0^T |x'(t)|^2 \mathrm{d}t$$

holds.

Lemma 2.4. [17, Lemma 2.3] *Let $T > 0$ be a constant, $x \in C^m(\mathbb{R}, \mathbb{R}), m \geqslant 2$, and $x(t + T) \equiv x(t)$. Then,*

$$\int_0^T |x^{(i)}(t)|^p \mathrm{d}t \leqslant \left(\frac{T}{\pi_p} \right)^p \int_0^T |x^{(i+1)}(t)|^p \mathrm{d}t, \quad i = 1, 2, \ldots, m-1,$$

where $\pi_p = 2 \int_0^{\frac{p-1}{p}} \frac{\mathrm{d}s}{(1 - \frac{sp}{p-1})^{\frac{1}{p}}} = \frac{2\pi (p-1)^{\frac{1}{p}}}{p \sin(\frac{\pi}{p})}$.

3. Main Results

In this section, we will state and prove the main results of this article.

Theorem 3.1. *Suppose that assumptions* (H1) *and* (H2) *hold.*

If $n = 1$*. Then,* Eq. (4) *has at least one* T*-periodic solution.*

If $n > 1$ *is even integer and the period* T *satisfies* $0 < T \leqslant C$*, where* $C = \min\{C_1, C_2\}$*. Then,* Eq. (4) *has at least one* T*-periodic solution.*

Remark 3.1. In Theorem 3.1, when $n > 1$ is even integer, we take $C = \min\{C_1, C_2\}$. In fact, the value of C is related to sign $a(t), h(t)$, for example, if $a(t) > 0$ and $h(t) > 0$, take $C = C_1$; if $a(t) > 0$ and $h(t)$ changes sign, take $C = C_2$, we will give the reasons in the proof of Theorem 3.1.

Proof. Let Banach spaces $X = Y = C_T$. Define linear operator $L : \operatorname{Dom} L \subset X \to Y$:

$$Lx = x^{(n)}, \quad x \in \operatorname{Dom} L,$$

where $\operatorname{Dom} L = \{x | x \in X, x^{(n)} \in C(\mathbb{R}, \mathbb{R})\}$. It is easily seen that $\operatorname{Ker} L = \mathbb{R}$ and $\operatorname{Im} L = \{y \mid y \in Y, \int_0^T y(s)\mathrm{d}s = 0\}$, hence $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$. It is easy to verify that $\operatorname{Im} L$ is a closed set in Y. Thus the operator L is a Fredholm operator with index zero.

From (H2), we can define nonlinear operator $N : X \to Y$:

$$Nx = h(t) - a(t)x^{\beta}.$$

Now, we define the projectors $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y$:

$$Px(t) = x(0),$$
$$Qx(t) = \frac{1}{T}\int_0^T x(s)\mathrm{d}s.$$

Clearly, $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Then, $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ can be given by

$$K_P y(t) = \int_0^T G(s,t)y(s)\mathrm{d}s,$$

where $G(t, s)$ be the Green's function of

$$\begin{cases} x^{(n)} = 0, \ t \in [0, T], \\ x(0) = 0, \ x^{(i)}(0) = x^{(i)}(T), \ i = 0, 1, \ldots, n-1. \end{cases}$$

It is immediate to prove that $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ is a linear completely continuous operator and $N : X \to Y$ is continuous bounded operator, therefore, N is L-compact on $\overline{\Omega}$ with any open bounded subset $\Omega \subset X$.

Next, we will discuss the cases where $n = 1$ and $n > 1$ is even integer, respectively.

Case 1: When $n = 1$. From the assumption (H1), we know that the function $a(t)$ has two cases: $a(t) > 0$ or $a(t) < 0$, and function $h(t)$ has three cases: $h(t) > 0$, $h(t) < 0$ or $h(t)$ changes sign. Without loss of generality, for the function $a(t)$, we only discuss the case of $a(t) > 0$, $a(t) < 0$ is similar. Next, we classify and discuss the three cases of function $h(t)$.

Case 1.1: If $a(t) > 0$ and $h(t) > 0$, then Eq. (4) is equivalent to equation

$$x' + a(t)x^{\beta} - h(t) = 0, \tag{9}$$

where $0 < a^- \leqslant a(t) \leqslant a^+$, $0 < h^- \leqslant h(t) \leqslant h^+$.

Let

$$\Omega_0 := \{x \in X \mid M_0 < x(t) < N_0\}, \tag{10}$$

where $N_0 > \left(\frac{h^+}{a^-}\right)^{\frac{1}{\beta}}$, $0 < M_0 < \left(\frac{h^-}{a^+}\right)^{\frac{1}{\beta}}$ are constants. For $\forall t \in [0, T]$, we have

$$0 < M_0 < \left(\frac{h^-}{a^+}\right)^{\frac{1}{\beta}} \leqslant \left(\frac{h(t)}{a(t)}\right)^{\frac{1}{\beta}} \leqslant \left(\frac{h^+}{a^-}\right)^{\frac{1}{\beta}} < N_0. \tag{11}$$

Obviously, M_0 and N_0 are well defined and $\Omega_0 \subset X$ is a bounded open set.

Now, we prove that condition (1) of Lemma 2.1 holds. Suppose the converse: there exist $0 < \lambda < 1$ and $x \in \partial\Omega_0 \cap \operatorname{Dom} L$ such that

$$x' + \lambda a(t)x^{\beta} - \lambda h(t) = 0. \tag{12}$$

Let $\underline{t}$ and $\bar{t}$, respectively, denote the global minimum and maximum points $x(t)$ on $t \in [0, T]$, that is

$$x(\bar{t}) = \max_{t \in [0,T]} x(t), \quad x(\underline{t}) = \min_{t \in [0,T]} x(t).$$

Obviously, we have

$$x'(\bar{t}) = 0, \quad x'(\underline{t}) = 0.$$

Suppose $x(t)$ is arbitrary T-periodic solution of Eq. (9), we claim that

$$M_0 < x(\underline{t}) \leqslant x(t) \leqslant x(\bar{t}) < N_0. \tag{13}$$

In fact, if (13) does not hold, then $x(\bar{t}) \geqslant N_0$ or $x(\underline{t}) \leqslant M_0$ at least one holds. If $x(\bar{t}) \geqslant N_0$, we get

$$\begin{aligned} 0 =& x'(\bar{t}) + \lambda a(\bar{t})x(\bar{t})^{\beta} - \lambda h(\bar{t}) \\ =& \lambda(a(\bar{t})x(\bar{t})^{\beta} - h(\bar{t})) \\ \geqslant& \lambda(a^- x(\bar{t})^{\beta} - h^+) \\ >& 0. \end{aligned}$$

When $x(\underline{t}) \leqslant 0$, obviously, we have

$$0 = x'(\underline{t}) + \lambda a(\underline{t})x(\underline{t})^{\beta} - \lambda h(\underline{t}) = \lambda(a(\underline{t})x(\underline{t})^{\beta} - h(\underline{t})) < 0.$$

When $0 < x(\underline{t}) \leqslant M_0$, we obtain

$$\begin{aligned} 0 &= x'(\underline{t}) + \lambda a(\underline{t})x(\underline{t})^{\beta} - \lambda h(\underline{t}) \\ &= \lambda(a(\underline{t})x(\underline{t})^{\beta} - h(\underline{t})) \\ &\leqslant \lambda(a^+ x(\underline{t})^{\beta} - h^-) \\ &< 0. \end{aligned}$$

But these are contradiction with Eq. (12). Therefore, each possible periodic solution $x(t)$ of Eq. (12) satisfies $M_0 < x(t) < N_0$. Hence, the condition (1) of Lemma 2.1 holds.

Next, we prove that condition (2) of Lemma 2.1 holds. For $\forall t \in [0, T]$, we obtain

$$a(t)M_0^\beta - h(t) \leqslant a^+ M_0^\beta - h^- < 0, \tag{14}$$
$$a(t)N_0^\beta - h(t) \geqslant a^- N_0^\beta - h^+ > 0. \tag{15}$$

Take $x \in \partial\Omega_0 \cap \operatorname{Ker} L$, then we have $x = M_0$ or $x = N_0$, by (14) and (15), for $\forall x \in \partial\Omega_0 \cap \operatorname{Ker} L$, we get

$$\begin{aligned} QNx =& \frac{1}{T}\int_0^T \Big(h(t) - a(t)x^\beta\Big)\mathrm{d}t \\ \neq& 0. \end{aligned}$$

Hence, condition (2) of Lemma 2.1 holds.

Finally, we prove that condition (3) of Lemma 2.1 holds. Define a continuous function

$$G(x, \theta) = (\theta - 1)\left(x - \frac{M_0 + N_0}{2}\right) + \theta \frac{1}{T}\int_0^T \Big(h(t) - a(t)x^\beta\Big)\mathrm{d}t, \quad \theta \in [0, 1].$$

Obviously, we obtain

$$G(x, \theta) \neq 0, \quad \forall x \in \partial\Omega_0 \cap \operatorname{Ker} L.$$

Using the homotopy invariance theorem, we get

$$\begin{aligned} \deg(QN, \Omega_0 \cap \operatorname{Ker} L, 0) &= \deg\left(G(x, 1), \Omega_0 \cap \operatorname{Ker} L, 0\right) \\ &= \deg\left(G(x, 0), \Omega_0 \cap \operatorname{Ker} L, 0\right) \\ &\neq 0. \end{aligned}$$

Therefore, condition (3) of Lemma 2.1 holds.

Therefore, we conclude from Lemma 2.1 that Eq. (9) has a T-periodic solution in $\overline{\Omega_0}$.

Case 1.2: If $a(t) > 0$ and $h(t)$ changes sign, in this case, Eq. (4) is equivalent to equation

$$x' + a(t)x^\beta - h(t) = 0, \tag{16}$$

where $0 < a^- \leqslant a(t) \leqslant a^+$, $h_* \leqslant h(t) \leqslant h^*$. Obviously, $h_* \leqslant 0, h^* \geqslant 0$ and h_*, h^* are not zero at the same time.

Let

$$\Omega_1 := \{x \in X \mid M_1 < x(t) < N_1\}, \tag{17}$$

where $N_1 > \left(\frac{h^*}{a^-}\right)^{\frac{1}{\beta}}$ and $M_1 < \left(\frac{h_*}{a^-}\right)^{\frac{1}{\beta}}$ are constants. Obviously, $\Omega_1 \subset X$ is a bounded open set.

The following proof is similar to the proof of Case 1.1, and so we omit it.

Case 1.3: If the function $a(t) > 0$ and $h(t) < 0$. Let $\tilde{h}(t) = -h(t)$, then Eq. (4) is equivalent to equation

$$x' + a(t)x^\beta + \tilde{h}(t) = 0, \tag{18}$$

where $0 < a^- \leqslant a(t) \leqslant a^+, 0 < h^- \leqslant \tilde{h}(t) \leqslant h^+$. Let

$$\Omega_2 := \{x \in X \mid M_2 < x(t) < N_2\},$$

where $0 > N_2 > \left(-\frac{h^-}{a^+}\right)^{\frac{1}{\beta}}$, $M_2 < \left(-\frac{h^+}{a^-}\right)^{\frac{1}{\beta}}$ are constants. Obviously, $M_2 < N_2 < 0$ and $\Omega_2 \subset X$ is a bounded open set.

The following proof is similar to the proof of case 1.1, and so we omit it.

When $a(t) < 0$ and $h(t) > 0$, the prove is the same as when $a(t) > 0$ and $h(t) < 0$; similarly, when $a(t) < 0$ and $h(t) < 0$ the proof is the same as when $a(t) > 0$, $h(t) > 0$; when $h(t)$ changes sign and $a(t) < 0$ the proof is the same as case 1.2.

Next, we discuss the case of $n > 1$ is even integer.

Case 2: When $n > 1$ is even integer. Analogously, we classify and discuss three cases of function $h(t)$.

Case 2.1: If functions $a(t) > 0$ and $h(t) > 0$, in this case, Eq. (4) is equivalent to equation

$$x^{(n)} + a(t)x^\beta - h(t) = 0, \tag{19}$$

where $0 < a^- \leqslant a(t) \leqslant a^+$, $0 < h^- \leqslant h(t) \leqslant h^+$.

Let

$$\Omega_3 := \{x \in X \mid |x|_\infty < N_3\}, \tag{20}$$

where $N_3 > 2\left(\frac{h^+}{a^-}\right)^{\frac{1}{\beta}}$ is constant.

Now, we prove that condition (1) of Lemma 2.1 holds. Suppose the converse: there exist $0 < \lambda < 1$ and $x \in \partial\Omega_0 \cap \mathrm{Dom}\, L$ such that

$$x^{(n)} + \lambda a(t)x^\beta - \lambda h(t) = 0. \tag{21}$$

If $n > 1$ is even integer, then $\int_0^T x^{(n)}x\mathrm{d}t = (-1)^{\frac{n}{2}} \int_0^T (x^{(\frac{n}{2})})^2\mathrm{d}t$. Multiplying (21) by x and the integrating from 0 to T, we have

$$\int_0^T (x^{(\frac{n}{2})})^2\mathrm{d}t + (-1)^{\frac{n}{2}} \int_0^T [\lambda a(t)x^\beta x - \lambda h(t)x]\mathrm{d}t = 0. \tag{22}$$

By (20) we know that for $\forall x \in \partial\Omega_3$, there is $|x|_\infty = N_3$. For $|x|_\infty = N_3$, we have $|x_{\max} - x_{\min}| \geqslant \frac{N_3}{2}$ or $|x_{\max} - x_{\min}| < \frac{N_3}{2}$. Further, when $|x_{\max} - x_{\min}| < \frac{N_3}{2}$, we also have $\frac{N_3}{2} < x \leqslant N_3$ or $-N_3 \leqslant x < -\frac{N_3}{2}$. Next, we will discuss these situations in categories. When $\frac{N_3}{2} < x \leqslant N_3$, integrating (21) from 0 to T,

$$\begin{aligned} 0 &= \int_0^T [\lambda a(t)x^\beta - \lambda h(t)]\mathrm{d}t \\ &> \int_0^T (a_-(\frac{N_3}{2})^\beta - h^+)\mathrm{d}t \\ &> 0. \end{aligned}$$

When $-N_3 \leqslant x \leqslant -\frac{N_3}{2}$, integrating (21) from 0 to T,

$$0 = \int_0^T [\lambda a(t)x^\beta - \lambda h(t)]\mathrm{d}t < 0.$$

If $|x_{\max} - x_{\min}| \geqslant \frac{N_3}{2}$, by Lemmas 2.3 and 2.4, we get

$$\begin{aligned}
0 &= \int_0^T \left(x^{\left(\frac{n}{2}\right)}\right)^2 \mathrm{d}t + (-1)^{\frac{n}{2}} \int_0^T [\lambda a(t)x^{\beta+1} - \lambda h(t)x]\mathrm{d}t \\
&\geqslant \int_0^T \left(x^{\left(\frac{n}{2}\right)}\right)^2 \mathrm{d}t - \int_0^T [a(t)x^{\beta+1} + h(t)x]\mathrm{d}t \\
&\geqslant \left(\frac{\pi}{T}\right)^{n-2} \int_0^T (x')^2 \mathrm{d}t - \int_0^T [a^+ N_3^{\beta+1} + h^+ N_3]\mathrm{d}t \\
&\geqslant \frac{\pi^{n-2}}{T^{n-2}} \frac{N_3^2}{T} - T(a^+ N_3^{\beta+1} + h^+ N_3) \\
&= N_3 T \left(\frac{\pi^{n-2}}{T^n} N_3 - (a^+ N_3^\beta + h^+)\right) \\
&> 0.
\end{aligned}$$

All of the above situations are contradictory to the facts, so the condition (1) of Lemma 2.1 holds.

For $\forall t \in [0, T]$, we obtain

$$a(t)(-N_3)^\beta - h(t) < 0, \tag{23}$$

$$a(t)N_3^\beta - h(t) \geqslant a^- N_3^\beta - h^+ > 0. \tag{24}$$

The following proof is similar to the proof of Case 1.1, and so we omit it. Therefore, we conclude from Lemma 2.1 that Eq. (19) has a T-periodic solution in $\overline{\Omega_3}$.

Case 2.2: If $a(t) > 0$ and $h(t)$ changes sign, in this case, Eq. (4) is equivalent to equation

$$x^{(n)} + a(t)x^\beta - h(t) = 0, \tag{25}$$

where $0 < a^- \leqslant a(t) \leqslant a^+$, $h_* \leqslant h(t) \leqslant h^*$. Obviously, $h_* \leqslant 0, h^* \geqslant 0$ and h_*, h^* are not zero at the same time.

Let

$$\Omega_4 := \{x \in X \mid |x|_\infty < N_4\}, \tag{26}$$

where $N_4 > \max\{2\left(\frac{h^*}{a^-}\right)^{\frac{1}{\beta}}, 2\left(-\frac{h_*}{a^-}\right)^{\frac{1}{\beta}}\}$ is constant.

Now, we prove that condition (1) of Lemma 2.1 holds. Let $0 < \lambda < 1$ and $\forall x \in \partial\Omega_4 \cap \operatorname{Dom} L$ such that

$$x^{(n)} + \lambda a(t)x^\beta - \lambda h(t) = 0. \tag{27}$$

Multiplying (27) by x and the integrating from 0 to T, we have

$$\int_0^T \left(x^{\left(\frac{n}{2}\right)}\right)^2 \mathrm{d}t + (-1)^{\frac{n}{2}} \int_0^T [\lambda a(t)x^{\beta+1} - \lambda h(t)x]\mathrm{d}t = 0. \tag{28}$$

Similarly, we need to discuss $\frac{N_4}{2} < x \leqslant N_4$, $-N_4 \leqslant x < -\frac{N_4}{2}$ and $|x_{\max} - x_{\min}| \geqslant \frac{N_4}{2}$, respectively. When $\frac{N_4}{2} < x \leqslant N_4$, integrating (28) from 0 to T, we obtain

$$\begin{aligned} 0 &= \int_0^T [\lambda a(t) x^\beta - \lambda h(t)] \mathrm{d}t \\ &> \int_0^T \left(a_- \left(\frac{N_4}{2} \right)^\beta - h^* \right) \mathrm{d}t \\ &> 0. \end{aligned}$$

When $-N_4 \leqslant x < -\frac{N_4}{2}$, integrating (28) from 0 to T, we get

$$\begin{aligned} 0 &= \int_0^T [\lambda a(t) x^\beta - \lambda h(t)] \mathrm{d}t \\ &\leqslant \int_0^T \left(a_- \left(-\frac{N_4}{2} \right)^\beta - h_* \right) \mathrm{d}t \\ &< 0. \end{aligned}$$

When $|x_{\max} - x_{\min}| \geqslant \frac{N_4}{2}$ and $h^* \geqslant |h_*|$, by Lemmas 2.3 and 2.4, we obtain

$$\begin{aligned} 0 &= \int_0^T \left(x^{\left(\frac{n}{2}\right)} \right)^2 \mathrm{d}t + (-1)^{\frac{n}{2}} \int_0^T [\lambda a(t) x^{\beta+1} - \lambda h(t) x] \mathrm{d}t \\ &\geqslant \int_0^T \left(x^{\left(\frac{n}{2}\right)} \right)^2 \mathrm{d}t - \int_0^T [a(t) x^{\beta+1} + h(t) x] \mathrm{d}t \\ &\geqslant \left(\frac{\pi}{T} \right)^{n-2} \int_0^T (x')^2 \mathrm{d}t - \int_0^T [a^+ N_4^{\beta+1} + h^* N_4] \mathrm{d}t \\ &\geqslant \frac{\pi^{n-2}}{T^{n-1}} N_4^2 - T(a^+ N_4^{\beta+1} + h^* N_4) \\ &= N_4 T \left(\frac{\pi^{n-2}}{T^n} N_4 - (a^+ N_4^\beta + h^*) \right) \\ &> 0. \end{aligned}$$

When $|x_{\max} - x_{\min}| \geqslant \frac{N_4}{2}$ and $h^* < |h_*|$, from Lemmas 2.3 and 2.4, we get

$$\begin{aligned} 0 &= \int_0^T \left(x^{\left(\frac{n}{2}\right)} \right)^2 \mathrm{d}t + (-1)^{\frac{n}{2}} \int_0^T [\lambda a(t) x^{\beta+1} - \lambda h(t) x] \mathrm{d}t \\ &\geqslant \int_0^T \left(x^{\left(\frac{n}{2}\right)} \right)^2 \mathrm{d}t - \int_0^T [a(t) x^{\beta+1} + h(t) x] \mathrm{d}t \\ &\geqslant \left(\frac{\pi}{T} \right)^{n-2} \int_0^T (x')^2 \mathrm{d}t - \int_0^T [a^+ N_4^{\beta+1} - h_* N_4] \mathrm{d}t \\ &\geqslant \frac{\pi^{n-2}}{T^{n-1}} N_4^2 - T(a^+ N_4^{\beta+1} - h_* N_4) \\ &= N_4 T \left(\frac{\pi^{n-2}}{T^n} N_4 - (a^+ N_4^\beta - h_*) \right) \\ &> 0. \end{aligned}$$

The following proofs are similar, so we omit it.

Case 2.3: If $a(t) < 0$ and $h(t) < 0$, then we have that $-a^+ \leqslant a(t) \leqslant -a^- \leqslant 0$, $-h^+ \leqslant h(t) \leqslant h^- \leqslant 0$.

Let $\tilde{a}(t) = -a(t), \tilde{h}(t) = -h(t)$, then Eq. (4) is equivalent to equation

$$x^{(n)} - \tilde{a}(t)x^\beta + \tilde{h}(t) = 0, \tag{29}$$

where $0 < a^- \leqslant \tilde{a}(t) \leqslant a^+, 0 < h^- \leqslant \tilde{h}(t) \leqslant h^+$.

Let

$$\Omega_5 := \{x \in X \mid |x|_\infty < N_5\},$$

where $N_5 = N_3 > 2\left(\frac{h^+}{a^-}\right)^{\frac{1}{\beta}}$ is constant.

The following proof is similar to the proof of case 1.1, and so we omit it.

In view of all the above discussion, we know that Eq. (4) has at least one T-periodic solutions. This completes the proof. □

Next, we obtain the following result that there is no positive T-periodic solution to Eq. (4).

Theorem 3.2. *Suppose that $\beta > 0$ and $a(t), h(t)$ are T-periodic L^1-function. If one of the following conditions hold:*

1. $a(t) \leqslant 0$ *and* $h(t) > 0$*;*
2. $a(t) < 0$ *and* $h(t) \geqslant 0$*;*
3. $a(t) \geqslant 0$ *and* $h(t) < 0$*;*
4. $a(t) > 0$ *and* $h(t) \leqslant 0$*.*

Then, Eq. (4) has no positive T-periodic solution under periodic boundary conditions $x^{(j)}(0) = x^{(j)}(T), j = 0, \dots, n-1$.

Proof. We only prove the result that there is no positive periodic solution to Eq. (4) under condition (1), the other cases are similar.

Suppose Eq. (4) has positive periodic solution $x_1(t)$, then we get

$$x_1^{(n)} + a(t)x_1^\beta = h(t). \tag{30}$$

Integrating on $[0, T]$ the Eq. (30), and taking into account the periodic boundary conditions $x^{(j)}(0) = x^{(j)}(T), j = 0, \dots, n-1$ and conditions $a(t) \leqslant 0$, $h(t) > 0$, we have

$$\begin{aligned}
0 &= \int_0^T [x_1^{(n)} + a(t)x_1^\beta - h(t)]\mathrm{d}t \\
&= \int_0^T [a(t)x_1^\beta - h(t)]\mathrm{d}t \\
&< 0,
\end{aligned}$$

this contradicts the fact. Therefore, when $a(t) \leqslant 0$, $h(t) > 0$, Eq. (4) has no positive T-periodic solution under periodic boundary condition $x^{(j)}(0) = x^{(j)}(T), j = 0, \dots, n-1$. This completes the proof. □

Similarly, we can get the result that Eq. (5) has no positive T-periodic solution.

Theorem 3.3. *Suppose that $\nu > 0$, $\beta > 0$ and $a(t), h(t)$ are T-periodic L^1-function. If one of the following conditions hold:*

1. $a(t) \leqslant 0$ *and* $h(t) > 0$*;*
2. $a(t) < 0$ *and* $h(t) \geqslant 0$*;*
3. $a(t) \geqslant 0$ *and* $h(t) < 0$*;*
4. $a(t) > 0$ *and* $h(t) \leqslant 0$*.*

Then Eq. (5) *has no positive T-periodic solution under periodic boundary conditions* $x^{(j)}(0) = x^{(j)}(T), j = 0, \ldots, n-1$.

Finally, we introduce the result of uniqueness of positive T-periodic solution of Eq. (5).

Theorem 3.4. *Suppose that $\beta > 0$, $\beta \neq 1$, $h(t), a(t)$ are continuous T-periodic functions with $\bar{a} \cdot \bar{h} > 0$ and $\frac{\bar{h}}{\bar{a}} \neq \beta^\beta$. Then* Eq. (5) *has only positive T-periodic solution $x(t,\nu)$ under periodic boundary condition $x^{(j)}(0) = x^{(j)}(T), j = 0, \ldots, n-1$, if one of the following conditions hold:*

(i) *Either $\frac{\beta-1}{\beta} > 0$ and $\nu > 0$ is small enough,*
(ii) *or $\frac{\beta-1}{\beta} < 0$ and $\nu > 0$ is large enough.*

Moreover, the following asymptotic behavior holds in both cases:

$$\lim_{\nu \to 0^+} \frac{1}{\nu^{\frac{1}{\beta}}} x(t,\nu) = \left(\frac{\bar{h}}{\bar{a}}\right)^{\frac{1}{\beta}}$$

uniformly in $t \in [0,T]$.

Proof. Put $x = \xi^{\frac{1}{\beta-1}} v$, then Eq. (5) becomes

$$\xi^{\frac{1}{\beta-1}} v^{(n)} = \nu h(t) - a(t) \xi^{\frac{\beta}{\beta-1}} v^\beta,$$

and choosing $\nu = \xi^{\frac{\beta}{\beta-1}}$, we get

$$v^{(n)} = \xi(h(t) - a(t) v^\beta). \tag{31}$$

It is readily seen that the linear problem

$$v^{(n)} = 0,\ v^{(j)}(0) = v^{(j)}(T), j = 0, 1, 2, 3, \ldots, n-1$$

has only constant solutions. For every $c > 0$, we get

$$F(c) = \bar{h} - \bar{a} c^\beta,\ F'(c) = \bar{h} - \beta \bar{a} c^{\beta-1},$$

Take $c_0 = \left(\frac{\bar{h}}{\bar{a}}\right)^{\frac{1}{\beta}} > 0$, according to $\frac{\bar{h}}{\bar{a}} \neq \beta^\beta$, we obtain

$$F(c_0) = 0,\ F'(c_0) \neq 0.$$

Therefore, Eq. (31) satisfies the conditions of Lemma 2.2 under periodic boundary condition $v^{(j)}(0) = v^{(j)}(T), j = 0, 1, 2, 3, \ldots, n-1$, then Eq. (5) has the only solution $x(t,\nu)$, and for $\forall t \in [0,T]$, we have

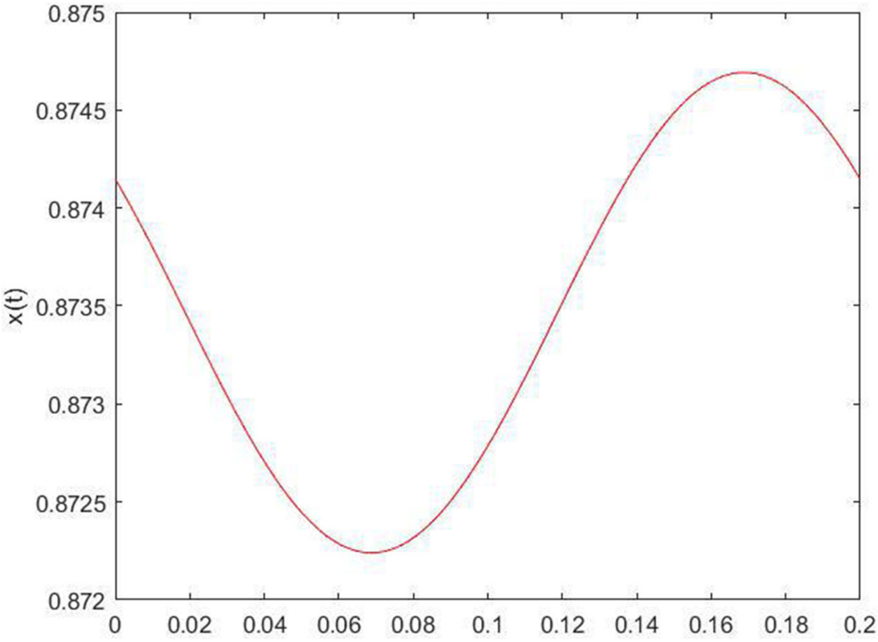

Figure 1. Periodic solution of Eq. (32)

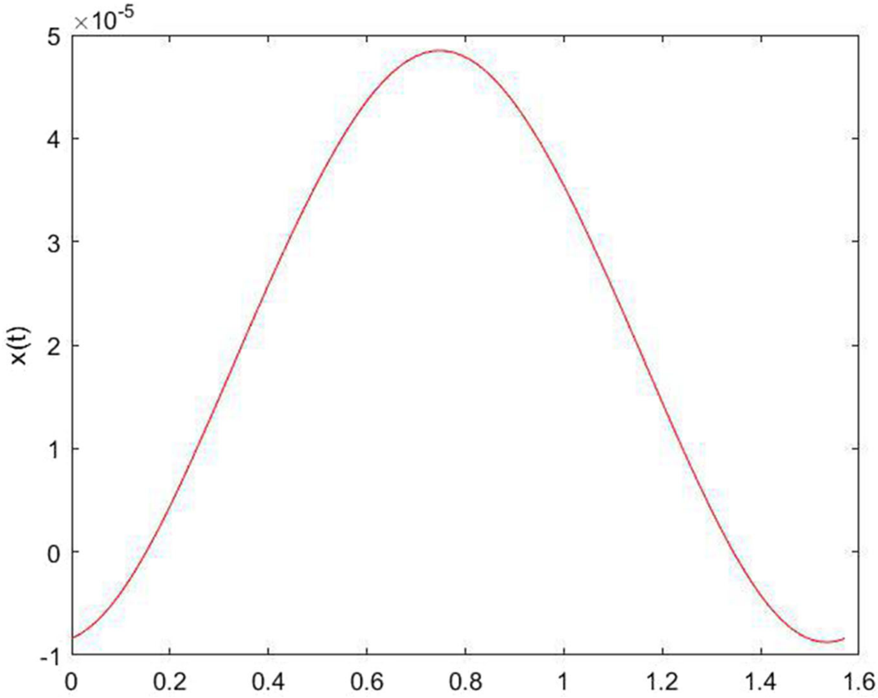

Figure 2. Periodic solution of Eq. (33)

$$\lim_{\nu\to 0^+}\frac{1}{\nu^{\frac{1}{\beta}}}x(t,\nu)=\left(\frac{\bar{h}}{\bar{a}}\right)^{\frac{1}{\beta}}.$$

This completes the proof. □

4. Examples

In this section, we apply the main results of this paper to two concrete examples, obtain the existence of periodic solutions of equations, and verify the correctness of the conclusions of this paper through numerical simulations.

Example 4.1. Consider the following second-order equations:

$$x''+(\cos 10\pi t+3)x^3=\sin 10\pi t+2. \tag{32}$$

It is clear that Eq. (32) is the case of Eq. (4) when $a(t)=\cos 10\pi t+3$, $h(t)=\sin 10\pi t+2$, $\beta=3$, $n=2$. Obviously, the functions $a(t)>0$, $h(t)>0$ and $a^-=2$, $a^+=4$, $h^-=1$, $h^+=3$. It is not difficult to see that functions $a(t),h(t)$ satisfies assumption (H1) and β satisfies assumption (H2). By simple calculation, we can take $N_3'=2.3$ and period $T=0.2$.

Then, Theorem 3.1 guarantees that the Eq. (32) has at least one periodic solutions in $\overline{\Omega_3'}$, where $\Omega_3':=\{x\in X\mid |x|_\infty<N_3'\}$. Next, we use numerical simulations to show the existence of periodic solutions of Eq. (32).

Example 4.2. Consider Eq. (4) with $n=6$, $\beta=\frac{1}{3}$, $a(t)=\cos 4t+2$, $h(t)=\cos 4t$, that is

$$x^{(6)}+(\cos 4t+2)x^{\frac{1}{3}}=\cos 4t. \tag{33}$$

Obviously, the functions $h(t)$ change sign and $a(t)>0$, we have $a^-=1$, $a^+=3$, $h_*=-1$, $h^*=1$. It is not difficult to see that coefficient functions $a(t),h(t)$ satisfies assumption (H1) and β satisfies assumption (H2). By simple calculation, we can take $N_4'=2.1$ and period $T=\frac{\pi}{2}$.

From Theorem 3.1 we conclude that Eq. (33) has at least one periodic solutions in $\overline{\Omega_4'}$, where $\Omega_4':=\{x\in X\mid |x|_\infty<N_4'\}$. We also obtain the existence of periodic solutions of Eq. (33) by numerical simulation.

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