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Bi-space Global Attractors for a Class of Second-Order Evolution Equations with Dispersive and Dissipative Terms in Locally Uniform Spaces

Fang-hong Zhang

Abstract. This paper deals with the asymptotic behavior of a class of second-order evolution equations with dispersive and dissipative terms' critical nonlinearity in locally uniform spaces. First of all, we prove the global well-posedness of solutions to the evolution equations in the locally uniform spaces $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$ and define a strong continuous analytic semigroup. Secondly, the existence of the $(H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N), H^1_{\rho}(\mathbb{R}^N) \times H^1_{\rho}(\mathbb{R}^N))$-global attractor is established. Finally, we obtain the asymptotic regularity of solutions which appear to be optimal and the existence of a bounded subset(in $H^2_{\mathrm{lu}}(\mathbb{R}^N) \times H^2_{\mathrm{lu}}(\mathbb{R}^N)$), which attracts exponentially every initial $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$-bounded set with respect to the $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$-norm.

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1. Introduction

In this paper, we investigate the long-time behavior of the solutions for the following second-order evolution equations with dispersive and dissipative terms in locally uniform spaces:

$$u_{tt} - \Delta u - \Delta u_t - \beta \Delta u_{tt} + \alpha u_t + \lambda u + f(u) = g(x), \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{1.1}$$

with the initial data

$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

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where $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$ with $N \geq 3$. The nonlinearity $f(s) \in \mathcal{C}^1(\mathbb{R})$ satisfies the following conditions: *Dissipative condition*

$$\liminf_{|s|\to\infty} \frac{F(s)}{s^2} \geq 0, \quad \text{where } F(s) = \int_0^s f(r)\mathrm{d}r, \tag{1.3}$$

$$\liminf_{|s|\to\infty} \frac{sf(s) - \alpha F(s)}{s^2} \geq 0, \quad \text{where } \alpha > 0. \tag{1.4}$$

Growth condition

$$\begin{aligned} |f(s) - f(h)| &\leq \beta|s-h|(1 + |s|^q + |h|^q), \quad \forall\, s, h \in \mathbb{R}, \\ &\text{where } \beta > 0,\ 0 \leq q \leq \frac{4}{N-2}. \end{aligned} \tag{1.5}$$

Equation (1.1) is a special form of the so-called improved Boussinesq equation (see [5,19–21,26]) with damped term $-\Delta u_t$, which was used to describe ion-sound waves in plasma, e.g., see [20,21], and also known to represent other sorts of 'propagation problems' of, for example, lengthways waves in nonlinear elastic rods and ion-sonic waves of space transformations by a weak nonlinear effect (see [5,14]).

In bounded domains, there is a vast literature concerning the attractors for the second-order evolution equations with dispersive and dissipative terms equations. For instance, in [27,28], Xie and Zhong investigated the existence of global attractors with critical exponential growth nonlinearity using the new method named "Condition C". Carvalho and Cholewa in [11] presented systematic results including the existence uniqueness and long-time behavior by using the semigroup approach. Sun et al. in [24] studied the asymptotic regularity of the solutions and obtained the existence of exponential attractors. For the (nonautonomous) semi-linear second-order evolution (1.1) with memory terms, Zhang et al. in [32] constructed the existence of robust family of exponential attractors, while the nonlinearity is critical. In our previous work [33], we showed the existence of pullback attractors in the Banach spaces for the multivalued process generated by a class of second-order nonautonomous evolution equations with hereditary characteristics and ill-posedness.

On unbounded domain, up to now, there are few results. Only Jones and Wang in [18] applied the cutoff method and a decomposition trick to obtain the existence of random attractor for the stochastic second-order evolution equations (1.1) with subcritical nonlinearity.

To our best knowledge, for critical nonlinearity, the long-time dynamics for Eq. (1.1) on unbounded domain have not been considered by any predecessors. There are some barriers encountered. On the one hand, the Sobolev embeddings are not compact on unbounded domains, and hence the asymptotic compactness of solutions cannot be obtained by simply using Sobolev embeddings and regularity of solutions. On the other hand, the number $q+1 = \frac{N+2}{N-2}$ in (1.5) is called a critical exponent, since the nonlinearity f is not compact even in the bounded case, and hence the methods for subcritical nonlinearity cannot be used to derive the asymptotic compactness for our problem. Thirdly, Eq. (1.1) contains the term $-\Delta u_{tt}$; if the initial data $z(0) = (u(0), u_t(0))$ belongs to $H^1_{\text{lu}}(\mathbb{R}^N) \times H^1_{\text{lu}}(\mathbb{R}^N)$, then the solution $z(t) = (u(t), u_t(t))$ is always in $H^1_{\text{lu}}(\mathbb{R}^N) \times H^1_{\text{lu}}(\mathbb{R}^N)$ and has no higher regularity, which will cause some difficulties.

The main contributions of this paper are that:

(i) We overcome the above difficulties (less regularity; lack of compactness; the equation itself), establish the well-posedness (Theorem 3.1), and prove the existence of bi-space global attractors for the second-order evolution equations with dispersive and dissipative terms Eq. (1.1) on $\mathbb{R}^N$ (Theorem 4.9).

(ii) We obtain the asymptotic regularity of solutions on $\mathbb{R}^N$, which appears to be optimal (Theorem 5.8). To our best knowledge, this is the first time to obtain the regularity for Eq. (1.1) on unbounded domain with both subcritical and critical nonlinearity, and maybe it is a basis for further considering the asymptotic behavior of the solutions.

The presentation of this paper is follows: In Sect. 2, we recall some basic definitions about the locally uniform spaces and iterate some definitions and abstract results concerning the global attractor. In Sect. 3, we prove the existence of global attractors for the second-order evolution equations with dispersive and dissipative terms in locally uniform spaces, and the asymptotic regularity of the solution will be established in Sect. 4.

2. Preliminaries

In this section, we first recall some basic definitions about the locally uniform spaces.

Following [1–3,7,8,22,29], we consider a strictly positive integrable we ighted function $\rho : \mathbb{R}^N \to (0, \infty)$: for $1 \le p < \infty$, setting

$$L^p_\rho(\mathbb{R}^N) = \left\{ \varphi \in L^p_{\text{loc}}(\mathbb{R}^N) : \|\varphi\|_{L^p_\rho(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \rho(x) |\varphi(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\},$$

let $\tau_y \rho(x) = \rho_y(x) = \rho(x - y)$, $y \in \mathbb{R}^N$, and consider the locally uniform spaces

$$L^p_{\text{lu}}(\mathbb{R}^N) = \left\{ \varphi \in L^p_{\text{loc}}(\mathbb{R}^N) : \|\varphi\|_{L^p_{\text{lu}}(\mathbb{R}^N)} = \sup_{y \in \mathbb{R}^N} \|\varphi\|_{L^p_{\rho_y}(\mathbb{R}^N)} < \infty \right\},$$
$$\dot{L}^p_{\text{lu}}(\mathbb{R}^N) = \{ \varphi \in L^p_{\text{lu}}(\mathbb{R}^N) : \|\tau_y \varphi - \varphi\|_{L^p_{\text{lu}}(\mathbb{R}^N)} \to 0 \text{ as } |y| \to 0 \},$$

where $\dot{L}^p_{\text{lu}}(\mathbb{R}^N)$ is the closed subspace of $L^p_{\text{lu}}(\mathbb{R}^N)$ consisting of all its elements that are translation continuous. The locally uniform Sobolev spaces $W^{m,p}_{\text{lu}}(\mathbb{R}^N)$ and $\dot{W}^{m,p}_{\text{lu}}(\mathbb{R}^N)$ are defined, respectively, by $L^p_{\text{lu}}(\mathbb{R}^N)$ and $\dot{L}^p_{\text{lu}}(\mathbb{R}^N)$ in a way similar to the standard $W^{m,p}_{\text{lu}}(\mathbb{R}^N)$.

We consider strictly positive integrable weighted functions $\rho \in \mathcal{C}^2(\mathbb{R}^N)$ satisfying

$$\left|\frac{\partial \rho}{\partial x_j}(x)\right| \leq \rho_0 \rho(x), \ \left|\frac{\partial^2 \rho}{\partial x_j \partial x_k}(x)\right| \leq \rho_1 \rho(x), \quad \forall x \in \mathbb{R}^N, j,k = 1,2,\ldots,N, \tag{2.1}$$

with certain positive constants ρ_0, ρ_1. In this paper, we consider the exemplary weighted functions

$$\rho(x) = (1 + \epsilon|x|^2)^{-s}, \quad \text{with } s > \frac{N}{2}, \ \epsilon > 0. \tag{2.2}$$

Obviously, $\rho \in \mathcal{C}^2(\mathbb{R}^N)$, then one can obtain the estimates that $|\nabla\rho| \leq c_1\sqrt{\epsilon}\rho$ and $|\Delta\rho| \leq \epsilon c_2 \rho$.

Now, we recall the uniform space $W_U^{s,p}(\mathbb{R}^N)$, $s \in \mathbb{R}^+ \cup \{0\}$, and the Banach space consisting of all $\phi \in W_{\text{loc}}^{s,p}(\mathbb{R}^N)$ such that

$$\|\phi\|_{W_U^{s,p}(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|\phi\|_{W_U^{s,p}(B(y,1))} < \infty, \tag{2.3}$$

where $B(y,1) = \{x \in \mathbb{R}^N : |x-y| \leq 1\}$. In addition, the following two norms are equivalent: there exist C_1, C_2 such that for all $u \in L_{\text{lu}}^p$,

$$\begin{aligned}\|u\|_{L_{\text{lu}}^p}^p &= \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} \rho(x-y)|u(x)|^p \mathrm{d}x \\ &\leq C_1 \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u(x)|^p \mathrm{d}x \leq C_2 \|u\|_{L_{\text{lu}}^p}^p.\end{aligned}$$

Note that for $k \in \mathbb{N} \cup \{0\}$, uniform space $W_U^{k,p}(\mathbb{R}^N)$ and locally uniform space $W_{\text{lu}}^{k,p}(\mathbb{R}^N)$ coincide algebraically and topologically when the weighted function ρ satisfies (2.1). Furthermore, by intermediate spaces, we know that the same holds for $W_U^{s,p}(\mathbb{R}^N)$ and $W_{\text{lu}}^{s,p}(\mathbb{R}^N)$ with $s \in \mathbb{R}^+ \cup \{0\}$, and we will use this equivalence frequently in this paper.

In addition, we need the following embedding lemma, interpolation inequalities in the weighted spaces and locally uniform space.

Lemma 2.1. [1]

(i) *If* $s_1 \geq s_2 \geq 0$, $1 < p_1 \leq p_2 < \infty$ *and* $s_1 - \frac{N}{p_1} \geq s_2 - \frac{N}{p_2}$, *then*

$$W_U^{s_1,p_1}(\mathbb{R}^N) \hookrightarrow W_U^{s_2,p_2}(\mathbb{R}^N)$$

is continuous.

(ii) *If* ρ *satisfies* (2.1), *then the inclusion*

$$W_U^{s_1,p_1}(\mathbb{R}^N) \hookrightarrow W_\rho^{s_2,p_2}(\mathbb{R}^N),$$

provided that $s_2 \in \mathbb{N}$, $s_1 > s_2$, $1 < p_1 \leq p_2 < \infty$ *and* $s_1 - \frac{N}{p_1} > s_2 - \frac{N}{p_2}$.

Lemma 2.2. [1] *For any* $p \in [2, \frac{2N}{N-2}]$ *and* $\theta \in [0,1]$, *we have*

$$\|\varphi\|_{L_\rho^p} \leq C\|\varphi\|_{H_{\text{lu}}^1}^\theta \|\varphi\|_{L_\rho^r}^{1-\theta},$$

and

$$\|\varphi\|_{L_\rho^p} \leq C\|\varphi\|_{H_\rho^1}^\theta \|\varphi\|_{L_{\text{lu}}^r}^{1-\theta},$$

where $\frac{1}{p} \leq \frac{\theta}{2} + \frac{1-\theta}{r}$ *and* $-\frac{N}{p} \leq \theta(1 - \frac{N}{2}) - (1-\theta)\frac{N}{r}$.

Lemma 2.3. [31] *there exist* C_1, C_2 *such that for all* $u \in L^p_\rho$ $(1 \le p < \infty)$,

$$C_1 \int_{\mathbb{R}^N} \rho(x)|u(x)|^p \mathrm{d}x \le \int_{\mathbb{R}^N} \rho(y) \int_{B(y,1)} |u(x)|^p \mathrm{d}x\mathrm{d}y \le C_2 \int_{\mathbb{R}^N} \rho(x)|u(x)|^p \mathrm{d}x.$$

Next, we iterate some definitions and abstract results concerning the global attractor, which are necessary to obtain our main results; we refer to [4,6,9,16,22,23,25] for more details.

Definition 2.1. A set $\mathcal{A} \subset X$, which is invariant, closed in X, compact in Z and attracts the bounded subsets of X in the topology of Z, is called an (X, Z)-global attractor.

Definition 2.2. Let $\{S(t)\}_{t\ge 0}$ be a semigroup on Banach space X. A set $B_0 \subset Z$, satisfying that, for any bounded subset $B \subset X$, there is a $T = T(B)$, such that $S(t)B \subset B_0$, for any $t \ge T$, is called an (X, Z)-bounded absorbing set.

Definition 2.3. Let $\{S(t)\}_{t\ge 0}$ be a semigroup on Banach space X. $\{S(t)\}_{t\ge 0}$ is called (X, Z)-asymptotically compact, if for any bounded (in X) sequence $\{x_n\}_{n=1}^\infty \subset X$ and $t_n \ge 0$, $t_n \to \infty$ as $n \to \infty$, $\{S(t_n)x_n\}_{n=1}^\infty$ has a convergence subsequence with respect to the topology of Z.

With the usual notation, hereafter let $|u|$, $|\cdot|_p$, $\|\cdot\|_{\dot{W}^{m,p}_{\mathrm{lu}}}$, $\|\cdot\|_{W^{m,p}_{\mathrm{lu}}}$, $\|\cdot\|_{W^{m,p}_\rho}$ and $\|\cdot\|_{W^{m,p}}$ be the norm of $L^2(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$, $\dot{W}^{m,p}_{\mathrm{lu}}(\mathbb{R}^N)$, $W^{m,p}_{\mathrm{lu}}(\mathbb{R}^N)$, $W^{m,p}_\rho(\mathbb{R}^N)$ and $W^{m,p}(\mathbb{R}^N)$, respectively. Also, let $\langle\cdot,\cdot\rangle$ be the usual inner product in $L^2(\mathbb{R}^N)$. Let C be an arbitrary positive constant, which may be different from line to line and even in the same line. For convenience, without loss of generality, we always assume $\alpha = \beta = \lambda = 1$ hereafter.

3. Global Well-Posedness

In this section, we will investigate the well-posedness of system (1.1)–(1.2).

Theorem 3.1. (Global well-posedness) *Assume that* f *satisfies* (1.3)–(1.5), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$. *Then for any* $T > 0$ *and* $(u_0, u_1) \in H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$, *there is a unique solution* $(u(t), u_t(t))$ *of Eqs.* (1.1) *and* (1.2) *such that*

$$u(t) \in \mathcal{C}([0,T]; H^1_{\mathrm{lu}}(\mathbb{R}^N)), u_t(t) \in \mathcal{C}([0,T]; H^1_{\mathrm{lu}}(\mathbb{R}^N)).$$

Moreover, the solution continuously depends on the initial data.

Proof. We divide the proof into three steps:
Step 1 `Local well-posedness`

Setting $v = (I - \Delta)u$ and $v_t = w$, we can rewrite Eq. (1.1) into the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 0 & -I \\ I & I \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \mathcal{F}\begin{pmatrix} v \\ w \end{pmatrix}, \quad t > 0,$$

where

$$\mathcal{F}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ f \circ ((I + A)^{-1}v) + g(x) \end{pmatrix}.$$

By the growth condition (1.5), $f(\cdot)$ is local Lipschitz in $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$, which is equivalent to $(0, f \circ ((I+A)^{-1}v) + g(x))^T$. The abstract semigroup theory about local well-posedness (e.g., see [3,10,11,23,25,26,29]) of an abstract parabolic equation leads to a local solution to system (1.1)–(1.2).

Step 2 `Global existence`

By the a priori estimates given in Lemma 4.1 below, we infer

$$\|u(t)\|^2_{H^1_{\mathrm{lu}}} + \|u_t(t)\|^2_{H^1_{\mathrm{lu}}} \le Ce^{-\nu t}(\|u(0)\|^2_{H^1_{\mathrm{lu}}} + \|u_t(0)\|^2_{H^1_{\mathrm{lu}}}) + C\int_{\mathbb{R}^N} \rho_y(|g|^2+1).$$

This implies that for each local solution $(u(t), u_t(t))$ of system (1.1)–(1.2) corresponding to initial data $(u_0, u_1) \in H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$, its $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$-norm cannot blow up at finite time, which implies the global existence of solutions.

Step 3 `Lipschitz continuity`

Let $u^1(t)$, $u^2(t)$ be two solutions of system (1.1)–(1.2) corresponding to the initial data $(u^1_0, u^1_1), (u^2_0, u^2_1) \in H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$ and denote $z(t) = u^1(t) - u^2(t)$, then $z(t)$ satisfies

$$z_{tt} - \Delta z - \Delta z_t - \Delta z_{tt} + z_t + z + f(u^1) - f(u^2) = 0. \tag{3.1}$$

We set $m = z_t + \eta z (0 < \eta \ll 1)$ and rewrite the Eq. (3.1) as follows:

$$\begin{aligned} &m_t + (1-\eta)m + (1-\eta+\eta^2)z - (1-\eta+\eta^2)\Delta z \\ &\quad -(1-\eta)\Delta m - \Delta m_t + f(u^1) - f(u^2) = 0. \end{aligned} \tag{3.2}$$

Multiplying (3.2) by $\rho_y m$, we infer

$$\begin{aligned} &\langle m_t, \rho_y m\rangle + (1-\eta)\langle m, \rho_y m\rangle + (1-\eta+\eta^2)\langle z, \rho_y m\rangle - (1-\eta+\eta^2)\langle \Delta z, \rho_y m\rangle \\ &\quad -(1-\eta)\langle \Delta m, \rho_y m\rangle - \langle \Delta m_t, \rho_y m\rangle + \langle f(u^1) - f(u^2), \rho_y m\rangle = 0. \end{aligned} \tag{3.3}$$

Next, we deal with each term of (3.3) one by one as follows:

$$\langle m_t, \rho_y m\rangle + (1-\eta)\langle m, \rho_y m\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^N} \rho_y |m|^2 + (1-\eta)\int_{\mathbb{R}^N} \rho_y |m|^2, \tag{3.4}$$

$$\langle z, \rho_y m\rangle = \langle z, \rho_y(z_t + \eta z)\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^N} \rho_y|z|^2 + \eta\int_{\mathbb{R}^N} \rho_y |z|^2, \tag{3.5}$$

$$\begin{aligned} \langle -\Delta z, \rho_y m\rangle = \langle -\Delta z, \rho_y(z_t+\eta z)\rangle &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^N} \rho_y|\nabla z|^2 + \eta\int_{\mathbb{R}^N} \rho_y|\nabla z|^2 \\ &\quad + \int_{\mathbb{R}^N} \nabla z \nabla \rho_y z_t + \eta\int_{\mathbb{R}^N} \nabla z\nabla\rho_y z, \end{aligned} \tag{3.6}$$

$$\langle -\Delta m, \rho_y m\rangle = \int_{\mathbb{R}^N} \rho_y |\nabla m|^2 + \int_{\mathbb{R}^N} \nabla m \nabla \rho_y m, \tag{3.7}$$

$$\langle -\Delta m_t, \rho_y m\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^N} \rho_y |\nabla m|^2 + \int_{\mathbb{R}^N} \nabla m_t \nabla \rho_y m. \tag{3.8}$$

$$\int_{\mathbb{R}^N} \nabla z \nabla \rho_y z_t + \eta \int_{\mathbb{R}^N} \nabla z \nabla \rho_y z \le C\sqrt{\epsilon}\int_{\mathbb{R}^N} \rho_y(|\nabla z|^2 + |z_t|^2 + |z|^2), \tag{3.9}$$

$$\int_{\mathbb{R}^N} \nabla m_t \nabla \rho_y m \le C\sqrt{\epsilon} \int_{\mathbb{R}^N} \rho_y (|\nabla m_t|^2 + |m|^2), \tag{3.10}$$

$$(1-\eta) \int_{\mathbb{R}^N} \nabla m \nabla \rho_y m \le C\sqrt{\epsilon} \int_{\mathbb{R}^N} \rho_y (|\nabla m|^2 + |m|^2). \tag{3.11}$$

By the Sobolev embedding $H^1(B_1(r)) \hookrightarrow L^{\frac{2N}{N-2}}(B_1(r))$ and $\dot{H}^1_{\text{lu}}(\mathbb{R}^N) \hookrightarrow \dot{L}^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \rho_y (f(u^1) - f(u^2)) z \right| \\
&\quad \le \int_{\mathbb{R}^N} \rho_y(x) \left(1 + |u^1|^{\frac{4}{N-2}} + |u^2|^{\frac{4}{N-2}}\right) |z|^2 \mathrm{d}x \\
&\quad \le \int_{\mathbb{R}^N} \rho_y(r) \left(\int_{B_1(r)} \left(1 + |u^1|^{\frac{N+2}{N-2}} + |u^2|^{\frac{4}{N-2}}\right) |z|^2 \mathrm{d}x \right) \mathrm{d}r \\
&\quad \le C \int_{\mathbb{R}^N} \rho_y(r) \left(\int_{B_1(r)} |z|^{\frac{2N}{N-2}} \mathrm{d}x \right)^{\frac{N-2}{N}} \mathrm{d}r \\
&\quad \le C \int_{\mathbb{R}^N} \rho_y(r) |z|^2_{H^1(B_1(r))} \mathrm{d}r \\
&\quad \le C \int_{\mathbb{R}^N} \rho_y (|\nabla z|^2 + |z|^2) \mathrm{d}x,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \rho_y (f(u^1) - f(u^2)) z_t \right| \\
&\quad \le \int_{\mathbb{R}^N} \rho_y(x) \left(1 + |u^1|^{\frac{4}{N-2}} + |u^2|^{\frac{4}{N-2}}\right) |z||z_t| \mathrm{d}x \\
&\quad \le \int_{\mathbb{R}^N} \rho_y(r) \left(\int_{B_1(r)} \left(1 + |u^1|^{\frac{N+2}{N-2}} + |u^2|^{\frac{4}{N-2}}\right) |z||z_t| \mathrm{d}x \right) \mathrm{d}r \\
&\quad \le C \int_{\mathbb{R}^N} \rho_y(r) \left(\int_{B_1(r)} |z|^{\frac{2N}{N-2}} \mathrm{d}x \right)^{\frac{N-2}{2N}} \left(\int_{B_1(r)} |z_t|^{\frac{2N}{N-2}} \mathrm{d}x \right)^{\frac{N-2}{2N}} \mathrm{d}r \\
&\quad \le C \int_{\mathbb{R}^N} \rho_y(r) |z|_{H^1(B_1(r))} |z_t|_{H^1(B_1(r))} \mathrm{d}r \\
&\quad \le C_\eta \int_{\mathbb{R}^N} \rho_y (|\nabla z|^2 + |z|^2) \mathrm{d}x + \eta \int_{\mathbb{R}^N} \rho_y (|\nabla z_t|^2 + |z_t|^2) \mathrm{d}x.
\end{aligned} \tag{3.13}$$

From (3.3)–(3.13), we get

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\mathbb{R}^N} \rho_y |m|^2 + (1 - \eta + \eta^2) \int_{\mathbb{R}^N} \rho_y |z|^2 \right. \\
&\quad \left. + (1 - \eta + \eta^2) \int_{\mathbb{R}^N} \rho_y |\nabla z|^2 + \int_{\mathbb{R}^N} \rho_y |\nabla m|^2 \right) \\
&\quad + (1 - \eta) \int_{\mathbb{R}^N} \rho_y |m|^2 + \eta(1 - \eta + \eta^2) \int_{\mathbb{R}^N} \rho_y (|z|^2 + |\nabla z|^2) + \int_{\mathbb{R}^N} \rho_y |\nabla m|^2
\end{aligned}$$

$$\leq C\sqrt{\epsilon}\int_{\mathbb{R}^N}\rho_y(|\nabla z|^2+|z|^2+|z_t|^2+|\nabla m_t|^2+|m|^2+|\nabla m|^2)$$
$$+C\int_{\mathbb{R}^N}\rho_y(|\nabla z|^2+|z|^2)+\eta\int_{\mathbb{R}^N}\rho_y(|\nabla z_t|^2+|z_t|^2). \tag{3.14}$$

In particular, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\rho_y(|m|^2+|\nabla m|^2+|z|^2+|\nabla z|^2)$$
$$\leq C\int_{\mathbb{R}^N}\rho_y(|m|^2+|\nabla m|^2+|z|^2+|\nabla z|^2). \tag{3.15}$$

By the Gronwall Lemma, for any $T\geq 0$, we get

$$\sup_{t\in[0,T]}(\|z(t)\|^2_{H^1_{\mathrm{lu}}}+\|z_t(t)\|^2_{H^1_{\mathrm{lu}}})\leq \mathrm{e}^{CT}(\|z(0)\|^2_{H^1_{\mathrm{lu}}}+\|z_t(0)\|^2_{H^1_{\mathrm{lu}}}). \tag{3.16}$$

This completes the proof. □

Remark 3.2. Theorem 3.1 implies that the solution of Eqs. (1.1)–(1.2) generates a $\mathcal{C}^0$ semigroup $\{S(t)\}_{t\geq 0}$ defined by

$$S(t): H^1_{\mathrm{lu}}(\mathbb{R}^N)\times H^1_{\mathrm{lu}}(\mathbb{R}^N)\to H^1_{\mathrm{lu}}(\mathbb{R}^N)\times H^1_{\mathrm{lu}}(\mathbb{R}^N)$$
$$\text{and}\quad S(t):(u_0,u_1)\mapsto (u(t),u_t(t)).$$

Moreover, the semigroup $\{S(t)\}_{t\geq 0}$ satisfying the Lipschitz continuity: given any $R>0$ and any two initial data $(u_0^1,u_1^1),(u_0^2,u_1^2)\in H^1_{\mathrm{lu}}(\mathbb{R}^N)\times H^1_{\mathrm{lu}}(\mathbb{R}^N)$ with $\|(u_0^i,u_1^i)\|_{H^1_{\mathrm{lu}}(\mathbb{R}^N)\times H^1_{\mathrm{lu}}(\mathbb{R}^N)}\leq R$, $i=1,2$, it holds that:

$$\|S(t)(u_0^1,u_1^1)-S(t)(u_0^2,u_1^2)\|_{H^1_{\mathrm{lu}}(\mathbb{R}^N)\times H^1_{\mathrm{lu}}(\mathbb{R}^N)}$$
$$\leq \mathrm{e}^{C_Rt}\|(u_0^1,u_1^1)-(u_0^2,u_1^2)\|_{H^1_{\mathrm{lu}}(\mathbb{R}^N)\times H^1_{\mathrm{lu}}(\mathbb{R}^N)},\quad \forall\, t\geq 0.$$

4. Global Attractor

In the section, we will prove the existence of global attractor for a class of second-order evolution equations with dispersive and dissipative terms in locally uniform spaces.

4.1. Dissipation Estimates

Lemma 4.1. *Assume that f satisfies (1.3)–(1.5), $g(x)\in L^2_{\mathrm{lu}}(\mathbb{R}^N)$. There is a positive constant ϱ_1 such that for any bounded subset $B\subset \dot{W}^{1,2}_{\mathrm{lu}}(\mathbb{R}^N)\times \dot{W}^{1,2}_{\mathrm{lu}}(\mathbb{R}^N)$, there exists a positive constant $T_1=T_1(B)$ such that*

$$\|u(t)\|_{H^1_{\mathrm{lu}}}+\|u_t(t)\|_{H^1_{\mathrm{lu}}}\leq \varrho_1\quad \textit{for all } t\geq T_1 \textit{ and } (u_0,u_1)\in B. \tag{4.1}$$

Proof. We set $v=u_t+\delta u (0<\delta\ll 1)$ and rewrite Eq. (1.1) as follows

$$v_t+(1-\delta)v+(1-\delta+\delta^2)u-(1-\delta+\delta^2)\Delta u$$
$$-(1-\delta)\Delta v-\Delta v_t+f(u)=g(x). \tag{4.2}$$

Multiplying (4.2) by $\rho_y v$, we infer

$$\langle v_t,\rho_y v\rangle+(1-\delta)\langle v,\rho_y v\rangle+(1-\delta+\delta^2)\langle u,\rho_y v\rangle-(1-\delta+\delta^2)\langle \Delta u,\rho_y v\rangle$$
$$-(1-\delta)\langle \Delta v,\rho_y v\rangle-\langle \Delta v_t,\rho_y v\rangle+\langle f(u),\rho_y v\rangle=\langle g(x),\rho_y v\rangle. \tag{4.3}$$

Next, we deal with each term of (4.3) one by one as follows:

$$\langle v_t, \rho_y v\rangle + (1-\delta)\langle v, \rho_y v\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\rho_y|v|^2 + (1-\delta)\int_{\mathbb{R}^N}\rho_y|v|^2, \tag{4.4}$$

$$\langle u, \rho_y v\rangle = \langle u, \rho_y(u_t+\delta u)\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\rho_y|u|^2 + \delta\int_{\mathbb{R}^N}\rho_y|u|^2, \tag{4.5}$$

$$\begin{aligned}\langle -\Delta u, \rho_y v\rangle &= \langle -\Delta u, \rho_y(u_t+\delta u)\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\rho_y|\nabla u|^2 + \delta\int_{\mathbb{R}^N}\rho_y|\nabla u|^2\\ &\quad + \int_{\mathbb{R}^N}\nabla u\nabla\rho_y u_t + \delta\int_{\mathbb{R}^N}\nabla u\nabla\rho_y u,\end{aligned} \tag{4.6}$$

$$\langle -\Delta v, \rho_y v\rangle = \int_{\mathbb{R}^N}\rho_y|\nabla v|^2 + \int_{\mathbb{R}^N}\nabla v\nabla\rho_y v, \tag{4.7}$$

$$\langle -\Delta v_t, \rho_y v\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\rho_y|\nabla v|^2 + \int_{\mathbb{R}^N}\nabla v_t\nabla\rho_y v, \tag{4.8}$$

$$\begin{aligned}\langle f(u), \rho_y v\rangle &= \langle f(u), \rho_y(u_t+\delta u)\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\rho_y F(u) + \delta\int_{\mathbb{R}^N}\rho_y f(u)u,\end{aligned} \tag{4.9}$$

$$\langle g(x), \rho_y v\rangle \le \varsigma\int_{\mathbb{R}^N}\rho_y|v|^2 + C_\varsigma\int_{\mathbb{R}^N}\rho_y|g|^2. \tag{4.10}$$

From (4.3)–(4.10), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\int_{\mathbb{R}^N}\rho_y|v|^2 + (1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|u|^2 + (1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|\nabla u|^2\\ &\qquad + \int_{\mathbb{R}^N}\rho_y|\nabla v|^2 + 2\int_{\mathbb{R}^N}\rho_y F(u)\Bigg)\\ &\qquad +2(1-\delta-\varsigma)\int_{\mathbb{R}^N}\rho_y|v|^2 + 2\delta(1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|u|^2\\ &\qquad +2\delta(1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|\nabla u|^2\\ &\qquad +2(1-\delta+\delta^2)\int_{\mathbb{R}^N}\nabla u\nabla\rho_y u_t\\ &\qquad +2\delta(1-\delta+\delta^2)\int_{\mathbb{R}^N}\nabla u\nabla\rho_y u\\ &\qquad +2(1-\delta)\int_{\mathbb{R}^N}\rho_y|\nabla v|^2 + 2(1-\delta)\int_{\mathbb{R}^N}\nabla v\nabla\rho_y v + 2\int_{\mathbb{R}^N}\nabla v_t\nabla\rho_y v\\ &\qquad +2\delta\int_{\mathbb{R}^N}\rho_y f(u)u\\ &\quad \le C_\varsigma\int_{\mathbb{R}^N}\rho_y|g|^2.\end{aligned} \tag{4.11}$$

Noting that

$$2(1-\delta+\delta^2)\int_{\mathbb{R}^N}\nabla u\nabla\rho_y u_t+2\delta(1-\delta+\delta^2)\int_{\mathbb{R}^N}\nabla u\nabla\rho_y u$$
$$\leq C\sqrt{\epsilon}\int_{\mathbb{R}^N}\rho_y(|\nabla u|^2+|u_t|^2+|u|^2), \tag{4.12}$$
$$2(1-\delta)\int_{\mathbb{R}^N}\nabla v\nabla\rho_y v+2\int_{\mathbb{R}^N}\nabla v_t\nabla\rho_y v$$
$$\leq C\sqrt{\epsilon}\int_{\mathbb{R}^N}\rho_y(|\nabla v|^2+|v_t|^2+|v|^2). \tag{4.13}$$

By (1.3)–(1.4), we infer

$$\int_{\mathbb{R}^N}\rho_y f(u)u\geq c_1\int_{\mathbb{R}^N}\rho_y F(u)+\mu\int_{\mathbb{R}^N}\rho_y|u|^2-C_\mu\int_{\mathbb{R}^N}\rho_y, \tag{4.14}$$

and

$$\int_{\mathbb{R}^N}\rho_y F(u)\geq -c_2\int_{\mathbb{R}^N}\rho_y. \tag{4.15}$$

Substituting the estimates (4.12)–(4.15) into (4.11), and choosing ϵ and ς small enough, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\int_{\mathbb{R}^N}\rho_y|v|^2+(1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|u|^2+(1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|\nabla u|^2$$
$$+\int_{\mathbb{R}^N}\rho_y|\nabla v|^2+2\int_{\mathbb{R}^N}\rho_y F(u)\Bigg)$$
$$+\nu\Bigg(\int_{\mathbb{R}^N}\rho_y|v|^2+(1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|u|^2+(1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|\nabla u|^2$$
$$+\beta\int_{\mathbb{R}^N}\rho_y|\nabla v|^2+2\int_{\mathbb{R}^N}\rho_y F(u)\Bigg)$$
$$\leq C_\delta\int_{\mathbb{R}^N}\rho_y|g|^2+C, \tag{4.16}$$

where ν is a positive constant which depends on δ and μ.

Denoting

$$\mathcal{E}_1(t)=\int_{\mathbb{R}^N}\rho_y|v|^2+(1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|u|^2$$
$$+(1-\delta+\delta^2)\int_{\mathbb{R}^N}\rho_y|\nabla u|^2+\int_{\mathbb{R}^N}\rho_y|\nabla v|^2+2\int_{\mathbb{R}^N}\rho_y F(u), \tag{4.17}$$

we can obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t)+\nu\mathcal{E}_1(t)\leq C\int_{\mathbb{R}^N}\rho_y(|g|^2+1). \tag{4.18}$$

Using the Gronwall lemma, we infer

$$\mathcal{E}_1(t)\leq \mathrm{e}^{-\nu t}\mathcal{E}_1(0)+C\int_{\mathbb{R}^N}\rho_y(|g|^2+1). \tag{4.19}$$

Noting that $\mathcal{E}_1(t) \sim \|u(t)\|^2_{H^1_{\mathrm{lu}}} + \|u_t(t)\|^2_{H^1_{\mathrm{lu}}}$, this completes the proof.

Remark 4.2. Lemma 4.1 implies that the $\mathcal{C}^0$ semigroup $\{S(t)\}_{t\geq 0}$ has a $(H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N), H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N))$-bounded absorbing set in the locally uniform space $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$.

Lemma 4.3. *Assume that f satisfies (1.3)–(1.5), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$. There is a positive constant ϱ_2 such that for any bounded subset $B \subset \dot{W}^{1,2}_{\mathrm{lu}}(\mathbb{R}^N) \times \dot{W}^{1,2}_{\mathrm{lu}}(\mathbb{R}^N)$, there exists a positive constant $T_2 = T_2(B)$ such that*

$$\|u_t(t)\|_{H^1_{\mathrm{lu}}} + \|u_{tt}(t)\|_{H^1_{\mathrm{lu}}} \leq \varrho_2 \quad \text{for all } t \geq T_2 \text{ and } (u_0, u_1) \in B. \tag{4.20}$$

Proof. Multiplying (4.2) by $\rho_y v_t$, we infer

$$\begin{aligned} &\langle v_t, \rho_y v_t\rangle + (1-\delta)\langle v, \rho_y v_t\rangle + (1-\delta+\delta^2)\langle u, \rho_y v_t\rangle \\ &\quad -(1-\delta+\delta^2)\langle \Delta u, \rho_y v_t\rangle - (1-\delta)\langle \Delta v, \rho_y v_t\rangle - \langle \Delta v_t, \rho_y v_t\rangle \\ &\quad +\langle f(u), \rho_y v_t\rangle = \langle g(x), \rho_y v_t\rangle. \end{aligned} \tag{4.21}$$

Next, we deal with each term of (4.21) one by one as follows:

$$\langle v_t, \rho_y v_t\rangle + (1-\delta)\langle v, \rho_y v_t\rangle = \int_{\mathbb{R}^N} \rho_y |v_t|^2 + \frac{1-\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \rho_y |v|^2, \tag{4.22}$$

$$\begin{aligned} |(1-\delta+\delta^2)\langle u, \rho_y v_t\rangle| &\leq \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + C_\varsigma \int_{\mathbb{R}^N} \rho_y |u|^2 \\ &\leq \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + C_{\varsigma,\varrho_1}, \end{aligned} \tag{4.23}$$

$$\begin{aligned} &(1-\delta+\delta^2)\langle -\Delta u, \rho_y v_t\rangle \\ &\quad = (1-\delta+\delta^2)\int_{\mathbb{R}^N} \nabla u \nabla \rho_y v_t + (1-\delta+\delta^2)\int_{\mathbb{R}^N} \nabla u \rho_y \nabla v_t \\ &\quad \leq \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + \varsigma \int_{\mathbb{R}^N} \rho_y |\nabla v_t|^2 + C_\varsigma \int_{\mathbb{R}^N} \rho_y |\nabla u|^2 \\ &\quad \leq \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + \varsigma \int_{\mathbb{R}^N} \rho_y |\nabla v_t|^2 + C_{c_1,\varsigma,\varrho_1}, \end{aligned} \tag{4.24}$$

$$\begin{aligned} &(1-\delta)\langle -\Delta v, \rho_y v_t\rangle \\ &\quad = (1-\delta)\int_{\mathbb{R}^N} \nabla v \nabla \rho_y v_t + (1-\delta)\int_{\mathbb{R}^N} \nabla v \rho_y \nabla v_t \\ &\quad \leq \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + C_\varsigma \int_{\mathbb{R}^N} \rho_y |\nabla v|^2 + \frac{1-\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \rho_y |\nabla v|^2 \\ &\quad \leq \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + \frac{1-\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \rho_y |\nabla v|^2 + C_{c_1,\varsigma,\varrho_1}, \end{aligned} \tag{4.25}$$

$$-\langle \Delta v_t, \rho_y v_t\rangle = \int_{\mathbb{R}^N} \nabla v_t \nabla \rho_y v_t + \int_{\mathbb{R}^N} \rho_y |\nabla v_t|^2, \tag{4.26}$$

$$\int_{\mathbb{R}^N} \nabla v_t \nabla \rho_y v_t \leq C_{c_1}\sqrt{\epsilon}(|\nabla v_t|^2 + |v_t|^2). \tag{4.27}$$

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N} \rho_y f(u) v_t\right| &\le \int_{\mathbb{R}^N} \rho_y(r) \left(\int_{B_1(r)} (1+|u|^{\frac{N+2}{N-2}})|v_t| \mathrm{d}x\right) \mathrm{d}r \\
&\le \int_{\mathbb{R}^N} \rho_y(r) \left(\int_{B_1(r)} (1+|u|^{\frac{N+2}{N-2}})^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}} \left(\int_{B_1(r)} |v_t|^{\frac{2N}{N-2}} \mathrm{d}x\right)^{\frac{N-2}{2N}} \mathrm{d}r \\
&\le \int_{\mathbb{R}^N} \rho_y(r) \left(\int_{B_1(r)} \left(1+|u|^{\frac{N+2}{N-2}}\right)^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}} \left(\int_{B_1(r)} |v_t|^{\frac{2N}{N-2}} \mathrm{d}x\right)^{\frac{N-2}{2N}} \mathrm{d}r \\
&\le \int_{\mathbb{R}^N} \rho_y(r) \|u\|_{H^1(B_1(r))} \|v_t\|_{B_1(r)} \mathrm{d}r \\
&\le \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + C_\varsigma \|u\|^2_{H^1_{\mathrm{lu}}} \\
&\le \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + C_{\varsigma,\varrho_1}.
\end{aligned} \tag{4.28}
$$

$$
\langle g(x), \rho_y v_t\rangle \le \varsigma \int_{\mathbb{R}^N} \rho_y |v_t|^2 + C_\varsigma \int_{\mathbb{R}^N} \rho_y |g|^2. \tag{4.29}
$$

Substituting the estimates (4.22)–(4.29) into (4.21), and choosing ϵ and ς small enough, we get that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left((1-\delta)\int_{\mathbb{R}^N} \rho_y |v|^2 + (1-\delta)\int_{\mathbb{R}^N} \rho_y |\nabla v|^2\right) \\
&\qquad +2(1-5\varsigma - C_{c_1,\sqrt{\epsilon}})\int_{\mathbb{R}^N} \rho_y |v_t|^2 + 2(1-\varsigma - C_{c_1,\sqrt{\epsilon}})\int_{\mathbb{R}^N} \rho_y |\nabla v_t|^2 \\
&\quad \le C_\varsigma \int_{\mathbb{R}^N} \rho_y |g|^2 + C_{c_1,\varsigma,\varrho_1}.
\end{aligned} \tag{4.30}
$$

In particular, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left((1-\delta)\int_{\mathbb{R}^N} \rho_y |v|^2 + (1-\delta)\int_{\mathbb{R}^N} \rho_y |\nabla v|^2\right) \\
&\quad \le C_\varsigma \int_{\mathbb{R}^N} \rho_y |g|^2 + C_{c_1,\varsigma,\varrho_1}.
\end{aligned} \tag{4.31}
$$

We infer that

$$
\int_{\mathbb{R}^N} \rho_y |v(t)|^2 + \int_{\mathbb{R}^N} \rho_y |\nabla v(t)|^2 \le C + C\int_{\mathbb{R}^N} \rho_y |g|^2. \tag{4.32}
$$

This completes the proof. □

4.2. Decomposition of the Equations

For the nonlinear term f, following the idea in [12,13,15,29–31], for a $\mathcal{C}^1$-function $f(\cdot)$ satisfying (1.3)–(1.5), the following decomposing properties hold: there are constants $C > 0$ and γ satisfying $0 < \gamma < q+1$ such that f can be decomposed as

$$
f = f_0 + f_1
$$

with $f_0,\ f_1 \in \mathcal{C}^1(\mathbb{R})$ satisfying

$$|f_0(s)| \le C(|s| + |s|^{q+1}),\ \forall\ s \in \mathbb{R}, \tag{4.33}$$

$$f_0(s)s \ge 0, \quad \forall\ s \in \mathbb{R}, \tag{4.34}$$

$$|f_1(s)| \le C(1 + |s|^{\gamma}), \quad \forall\ s \in \mathbb{R} \text{ with some } \gamma < q+1, \tag{4.35}$$

$$\exists\ l \in \mathbb{R},\ F_1(s) \ge -l, \quad \forall\ s \in \mathbb{R}, \tag{4.36}$$

$\exists k_i \ge 1,\ \tilde{\mu}_i \ge 0$ such that $\forall \mu_i \in (0, \tilde{\mu}_i]$, $\exists C_{\mu_i} \in \mathbb{R}$,

$$k_i F_i(s) + \mu_i s^2 - C_{\mu_i} \le s f_i(s), \quad \text{for all } s \in \mathbb{R}, \tag{4.37}$$

where $F_i(s) = \int_0^s f_i(r)\mathrm{d}r$, $i = 1, 2$.

Now, we decompose the solution into the sum

$$S(t)(u_0, u_1) = D(t)(u_0, u_1) + K(t)(u_0, u_1),$$

where $(z(t), z_t(t)) = D(t)(u_0, u_1)$ and $(w(t), w_t(t)) = K(t)(u_0, u_1)$ solves the following equations, respectively:

$$\begin{cases} z_{tt} - \Delta z - \Delta z_t - \Delta z_{tt} + z_t + z + f_0(z) = 0, \\ z(0) = u_0, z_t(0) = u_1, \end{cases} \tag{4.38}$$

and

$$\begin{cases} w_{tt} - \Delta w - \Delta w_t - \Delta w_{tt} + w_t + w + f(u) - f_0(z) = g(x), \\ w(0) = 0, w_t(0) = 0. \end{cases} \tag{4.39}$$

Note that $\{D(t)\}_{t \ge 0}$ also forms a semigroup, but $\{K(t)\}_{t \ge 0}$ may not.

4.3. A Priori Estimates

Lemma 4.4. *Assume that f_0 satisfies (4.33)–(4.34), (4.37). Then there is a positive constant ϱ_3 such that for any bounded subset $B \subset \dot{W}^{1,2}_{\mathrm{lu}}(\mathbb{R}^N) \times \dot{W}^{1,2}_{\mathrm{lu}}(\mathbb{R}^N)$, there exists a positive constant $T_3 = T_3(B)$ such that*

$$\|z_t(t)\|^2_{H^1_{\mathrm{lu}}} + \|z_{tt}(t)\|^2_{H^1_{\mathrm{lu}}} \le \varrho_3, \quad \forall t \ge T_3,\ (u_0, u_1) \in B. \tag{4.40}$$

The proof of this Lemma is a repeat of Lemma 4.3, and we omit the details.

Remark 4.5. $D(t)$ maps the bounded set of $\dot{W}^{1,2}_{\mathrm{lu}}(\mathbb{R}^N) \times \dot{W}^{1,2}_{\mathrm{lu}}(\mathbb{R}^N)$ to be a uniformly (w.r.t. time t) bounded set; that is, for any $(u_0, u_1) \in H^1_{\mathrm{lu}} \times H^1_{\mathrm{lu}}$,

$$\begin{aligned} \|D(t)(u_0, u_1)\|^2_{H^1_{\mathrm{lu}}} &= \|z(t)\|^2_{H^1_{\mathrm{lu}}} + \|z_t(t)\|^2_{H^1_{\mathrm{lu}}} \\ &\le \mathcal{Q}(\|(u_0, u_1)\|_{H^1_{\mathrm{lu}} \times H^1_{\mathrm{lu}}}), \quad \text{for all } t \ge 0. \end{aligned} \tag{4.41}$$

For the solutions $(z(t), z_t(t)) = D(t)(u_0, u_1)$ of Eq. (4.38), we also need the following exponential decay result.

Lemma 4.6. *Assume that f_0 satisfies (4.33)–(4.34), (4.37). Then there exists a positive constant ν such that for every $t \ge T_3$,*

$$\|D(t)(u_0, u_1)\|^2_{H^1_{\mathrm{lu}}} = \|z(t)\|^2_{H^1_{\mathrm{lu}}} + \|z_t(t)\|^2_{H^1_{\mathrm{lu}}} \le \mathcal{Q}_1(\|(u_0, u_1)\|_{H^1_{\mathrm{lu}} \times H^1_{\mathrm{lu}}})\mathrm{e}^{-\nu t}, \tag{4.42}$$

where $\mathcal{Q}_1(\cdot)$ is an increasing function on $[0, \infty)$.

Proof. We set $q = z_t + \delta z (0 < \delta \ll 1)$ and rewrite the Eq. (4.38) as follows:

$$q_t + (1-\delta)q + (1-\delta+\delta^2)z - (1-\delta+\delta^2)\Delta z - (1-\delta)\Delta q - \Delta q_t \\ + f_0(z) = 0. \tag{4.43}$$

Multiplying (4.43) by $\rho_y q$, we infer that

$$\langle q_t, \rho_y q\rangle + (1-\delta)\langle q, \rho_y q\rangle + (1-\delta+\delta^2)\langle z, \rho_y q\rangle - (1-\delta+\delta^2)\langle \Delta z, \rho_y q\rangle \\ -(1-\delta)\langle \Delta q, \rho_y q\rangle - \langle \Delta q_t, \rho_y q\rangle + \langle f(z), \rho_y q\rangle = 0. \tag{4.44}$$

By some standard calculations, we infer that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N} \rho_y |q|^2 + (1-\delta+\delta^2)\int_{\mathbb{R}^N} \rho_y |z|^2 + (1-\delta+\delta^2)\int_{\mathbb{R}^N} \rho_y |\nabla z|^2 \right.\\ &\qquad \left. + \int_{\mathbb{R}^N} \rho_y |\nabla q|^2 + 2\int_{\mathbb{R}^N} \rho_y F_0(z)\right) \\ &\qquad +2(1-\delta)\int_{\mathbb{R}^N} \rho_y |q|^2 + 2\delta(1-\delta+\delta^2)\int_{\mathbb{R}^N} \rho_y |z|^2 \\ &\qquad +2\delta(1-\delta+\delta^2)\int_{\mathbb{R}^N} \rho_y |\nabla z|^2 \\ &\qquad +2(1-\delta+\delta^2)\int_{\mathbb{R}^N} \nabla z \nabla \rho_y z_t + 2\delta(1-\delta+\delta^2)\int_{\mathbb{R}^N} \nabla z \nabla \rho_y z \\ &\qquad +2(1-\delta)\int_{\mathbb{R}^N} \rho_y |\nabla q|^2 + 2(1-\delta)\int_{\mathbb{R}^N} \nabla v \nabla \rho_y q \\ &\qquad +2\int_{\mathbb{R}^N} \nabla q_t \nabla \rho_y v + 2\delta\int_{\mathbb{R}^N} \rho_y f_0(z) z \\ &\quad = 0. \end{aligned} \tag{4.45}$$

Noting that

$$\begin{aligned} &|2(1-\delta+\delta^2)\int_{\mathbb{R}^N} \nabla z \nabla \rho_y z_t + 2\delta(1-\delta+\delta^2)\int_{\mathbb{R}^N} \nabla z \nabla \rho_y z| \\ &\quad \le C\sqrt{\epsilon}\int_{\mathbb{R}^N} \rho_y (|\nabla z|^2 + |z_t|^2 + |z|^2), \end{aligned} \tag{4.46}$$

$$\begin{aligned} &2(1-\delta)\int_{\mathbb{R}^N} \nabla z \nabla \rho_y z + 2\int_{\mathbb{R}^N} \nabla z_t \nabla \rho_y z \\ &\quad \le C\sqrt{\epsilon}\int_{\mathbb{R}^N} \rho_y (|\nabla z|^2 + |z_t|^2 + |z|^2). \end{aligned} \tag{4.47}$$

By (4.33), Lemma 4.4 and Remark 4.5, we infer that

$$\begin{aligned} \int_{\mathbb{R}^N} \rho_y F_0(z) &\le C\int_{\mathbb{R}^N} \rho_y (|z|^2 + |z|^{\frac{2N}{N-2}}) \\ &\le C\|z\|_{H^1_{\mathrm{lu}}}^{\frac{4}{N-2}} \int_{\mathbb{R}^N} \rho_y (|z|^2 + |\nabla z|^2) \\ &\le C_{\|(u_0,u_1)\|_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}} \int_{\mathbb{R}^N} \rho_y (|z|^2 + |\nabla z|^2). \end{aligned} \tag{4.48}$$

Note that

$$f_0(z)z \geq 0, \quad \forall\, z \in \mathbb{R}; \qquad F_0(z) \geq 0, \quad \forall\, z \in \mathbb{R}. \tag{4.49}$$

Hence, choosing ϵ small enough, we infer that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\int_{\mathbb{R}^N} \rho_y|q|^2 + (1-\delta+\delta^2)\int_{\mathbb{R}^N} \rho_y|z|^2 + (1-\delta+\delta^2)\int_{\mathbb{R}^N} \rho_y|\nabla z|^2 \\
&\quad + \int_{\mathbb{R}^N} \rho_y|\nabla q|^2 + 2\int_{\mathbb{R}^N} \rho_y F_0(z)\Bigg) \\
&\quad +\nu\Bigg(\int_{\mathbb{R}^N} \rho_y|q|^2 + (1-\delta+\delta^2)\int_{\mathbb{R}^N} \rho_y|z|^2 + (1-\delta+\delta^2)\int_{\mathbb{R}^N} \rho_y|\nabla z|^2 \\
&\quad + \int_{\mathbb{R}^N} \rho_y|\nabla q|^2 + 2\int_{\mathbb{R}^N} \rho_y F_0(z)\Bigg) \leq 0,
\end{aligned} \tag{4.50}$$

where ν is a positive constant which depends on δ.

Applying the Gronwall Lemma, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \rho_y|q|^2 + \int_{\mathbb{R}^N} \rho_y|z|^2 + \int_{\mathbb{R}^N} \rho_y|\nabla z|^2 + \int_{\mathbb{R}^N} \rho_y|\nabla q|^2 \\
&\qquad \leq \mathcal{Q}_1(\|(u_0,u_1)\|_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}})\mathrm{e}^{-\nu t}.
\end{aligned} \tag{4.51}$$

This completes the proof. □

Remark 4.7. Based on Lemmas 4.1, 4.3–4.6, the solutions $(w(t), w_t(t)) = K(t)(u_0, u_1)$ of Eq. (4.39), we infer that there exists a positive constant ϱ_4 such that

$$\|w(t)\|^2_{H^1_{\mathrm{lu}}} + \|w_t(t)\|^2_{H^1_{\mathrm{lu}}} + \|w_{tt}(t)\|^2_{H^1_{\mathrm{lu}}} \leq \varrho_4, \quad \forall t \geq 0 \text{ and } (u_0,u_1) \in B. \tag{4.52}$$

Lemma 4.8. *Assume that f satisfies (1.3)–(1.5), f_1 satisfies (4.35)–(4.37), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$. Then there exists a positive constant k such that for every $t \geq 0$,*

$$\begin{aligned}
\|K(t)(u_0,u_1)\|^2_{H^{1+\sigma}_{\mathrm{lu}}} &= \|w(t)\|^2_{H^{1+\sigma}_{\mathrm{lu}}} + \|w_t(t)\|^2_{H^{1+\sigma}_{\mathrm{lu}}} \\
&\leq \mathcal{Q}_2(\|(u_0,u_1)\|_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}, \|g\|_{L^2_{\mathrm{lu}}})\mathrm{e}^{kt},
\end{aligned} \tag{4.53}$$

where $\mathcal{Q}_2(\cdot)$ is an increasing function on $[0,\infty)$, $\sigma = \min\{\frac{1}{4}, \frac{N+2-(N-2)\gamma}{2}\}$, where γ is given in (4.35).

Proof. Let θ be a smooth function satisfying $0 \leq \theta(s) \leq 1$ for $s \in [0,\infty)$ and

$$\theta(s) = 1 \quad \text{for } 0 \leq s \leq \frac{1}{2}; \qquad \theta(s) = 0 \quad \text{for } s \geq 1.$$

Set $\theta_y(x) = \theta(|x-y|)$ and $A = -\Delta$.

We set $m = w_t + \delta w (0 < \delta \ll 1)$ and rewrite the Eq. (4.39) as follows:

$$\begin{aligned}
&m_t + (1-\delta)m + (1-\delta+\delta^2)w - (1-\delta+\delta^2)\Delta w - (1-\delta)\Delta m - \Delta m_t \\
&\quad + f(u) - f(z) + f_1(z) = g(x).
\end{aligned} \tag{4.54}$$

Multiplying (4.54) by $\theta_y A^\sigma(\theta_y m)$, we infer that

$$\begin{aligned}&\langle m_t, \theta_y A^\sigma(\theta_y m)\rangle + (1-\delta)\langle m, \theta_y A^\sigma(\theta_y m)\rangle + (1-\delta+\delta^2)\langle w, \theta_y A^\sigma(\theta_y m)\rangle \\ &\quad -(1-\delta+\delta^2)\langle \Delta w, \theta_y A^\sigma(\theta_y m)\rangle - (1-\delta)\langle \Delta m, \theta_y A^\sigma(\theta_y m)\rangle \\ &\quad -\langle \Delta m_t, \theta_y A^\sigma(\theta_y m)\rangle + \langle f(u) - f(z) + f_1(z), \theta_y A^\sigma(\theta_y m)\rangle \\ &= \langle g(x), \theta_y A^\sigma(\theta_y m)\rangle. \end{aligned} \tag{4.55}$$

We deal with each term above, one by one, as follows:

$$\begin{aligned}&\langle m_t, \theta_y A^\sigma(\theta_y m)\rangle + (1-\delta)\langle m, \theta_y A^\sigma(\theta_y m)\rangle \\ &\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} |\theta_y A^{\frac{\sigma}{2}}(\theta_y m)|^2 + (1-\delta)\int_{\mathbb{R}^N} |A^{\frac{\sigma}{2}}(\theta_y m)|^2, \end{aligned} \tag{4.56}$$

$$\begin{aligned}&\langle w, \theta_y A^\sigma(\theta_y m)\rangle \\ &\quad = \langle w, \theta_y A^\sigma(\theta_y(w_t + \delta w))\rangle \\ &\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} |\theta_y A^{\frac{\sigma}{2}}(\theta_y w)|^2 + \delta\int_{\mathbb{R}^N} |\theta_y A^{\frac{\sigma}{2}}(\theta_y w)|^2, \end{aligned} \tag{4.57}$$

$$\begin{aligned}&-\langle \Delta w, \theta_y A^\sigma(\theta_y m)\rangle \\ &\quad = \langle Aw, \theta_y A^\sigma(\theta_y(w_t + \delta w))\rangle \\ &\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y w)|^2 + 2\int_{\mathbb{R}^N} \nabla w \nabla\theta_y A^\sigma(\theta_y w_t) + \langle \Delta\theta_y w, A^\sigma(\theta_y w_t)\rangle \\ &\qquad +\delta\int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y w)|^2 + 2\delta\int_{\mathbb{R}^N} \nabla w\nabla\theta_y A^\sigma(\theta_y w) + \delta\langle \Delta\theta_y w, A^\sigma(\theta_y w)\rangle, \end{aligned} \tag{4.58}$$

$$\begin{aligned}\langle -\Delta m_t, \theta_y A^\sigma(\theta_y m)\rangle &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y m)|^2 + 2\int_{\mathbb{R}^N} \nabla m_t\nabla\theta_y A^\sigma(\theta_y m) \\ &\quad +\langle \Delta\theta_y m_t, A^\sigma(\theta_y m_t)\rangle, \end{aligned} \tag{4.59}$$

$$\begin{aligned}-\langle \Delta m, \theta_y A^\sigma(\theta_y m)\rangle &= \int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y m)|^2 + 2\int_{\mathbb{R}^N} \nabla m\nabla\theta_y A^\sigma(\theta_y m) \\ &\quad +\langle \Delta\theta_y m, A^\sigma(\theta_y m)\rangle. \end{aligned} \tag{4.60}$$

Noting that $\sigma < \frac{1}{2}$, by Remark 4.7, we have

$$\begin{aligned}&2\int_{\mathbb{R}^N} \nabla w\nabla\theta_y A^\sigma(\theta_y w_t) + \langle \Delta\theta_y w, A^\sigma(\theta_y w_t)\rangle \\ &\quad \le C_{c_1}\|w\|_{H^1_{\mathrm{lu}}}\left(\int_{\mathbb{R}^N} |A^\sigma(\theta_y w_t)|^2\right)^{\frac{1}{2}} \\ &\quad \le C_{c_1}\|w\|_{H^1_{\mathrm{lu}}}\|w_t\|_{H^1_{\mathrm{lu}}} \\ &\quad \le C_{c_1,\varrho_4}, \end{aligned} \tag{4.61}$$

$$
\begin{aligned}
&2\delta \int_{\mathbb{R}^N} \nabla w \nabla \theta_y A^\sigma(\theta_y w) + \delta \langle \Delta \theta_y w, A^\sigma(\theta_y w)\rangle \\
&\quad \le C_{c_1} \|w\|_{H^1_{\rm lu}} \left(\int_{\mathbb{R}^N} |A^\sigma(\theta_y w)|^2 \right)^{\frac{1}{2}} \\
&\quad \le C_{c_1} \|w\|^2_{H^1_{\rm lu}} \\
&\quad \le C_{c_1,\varrho_4},
\end{aligned} \tag{4.62}
$$

$$
\begin{aligned}
&2 \int_{\mathbb{R}^N} \nabla m_t \nabla \theta_y A^\sigma(\theta_y m) + \langle \Delta \theta_y m_t, A^\sigma(\theta_y m_t)\rangle \\
&\quad \le C_{c_1} \|m_t\|_{H^1_{\rm lu}} \left(\int_{\mathbb{R}^N} |A^\sigma(\theta_y m)|^2 \right)^{\frac{1}{2}} \\
&\quad \le C_{c_1} \|m_t\|_{H^1_{\rm lu}} \|m\|_{H^1_{\rm lu}} \\
&\quad \le \varsigma \|m_t\|^2_{H^1_{\rm lu}} + C_{c_1,\tau} \|m\|^2_{H^1_{\rm lu}} \\
&\quad \le C_{c_1,\varrho_4,\epsilon},
\end{aligned} \tag{4.63}
$$

and

$$
\begin{aligned}
&2 \int_{\mathbb{R}^N} \nabla m \nabla \theta_y A^\sigma(\theta_y m) + \langle \Delta \theta_y m, A^\sigma(\theta_y m)\rangle \\
&\quad \le C_{c_1} \|m\|_{H^1_{\rm lu}} \left(\int_{\mathbb{R}^N} |A^\sigma(\theta_y m)|^2 \right)^{\frac{1}{2}} \\
&\quad \le C_{c_1} \|m\|^2_{H^1_{\rm lu}} \\
&\quad \le C_{c_1,\varrho_4,\epsilon}.
\end{aligned} \tag{4.64}
$$

Note that $\sigma \le \frac{N+2-(N-2)\gamma}{2}$, by (4.35), we get

$$
\begin{aligned}
&|\langle f_1(z), \theta_y A^\sigma(\theta_y m)\rangle| \\
&\quad \le C \int_{\mathbb{R}^N} \theta_y (1+|z|^\gamma) |A^\sigma(\theta_y m)| \\
&\quad \le C \left(\int_{\mathbb{R}^N} |A^\sigma(\theta_y m)|^{\frac{2N}{N-2+2\sigma}} \right)^{\frac{N-2+2\sigma}{2N}} \left(\int_{B(y,1)} |1+|z|^\gamma|^{\frac{2N}{N+2-2\sigma}} \right)^{\frac{N+2-2\sigma}{2N}} \\
&\quad \le C(1+\|z\|^\gamma_{H^1_U})(\|\theta_y m\|_{L^2} + \|A^{\frac{\sigma+1}{2}}(\theta_y m)\|_{L^2}),
\end{aligned} \tag{4.65}
$$

where $B(y,1) = \{x \in \mathbb{R}^N : |x-y| \le 1\}$.

By virtue of (1.5), we have

$$
\begin{aligned}
&|\langle f(u) - f(z), \theta_y A^\sigma(\theta_y m)\rangle| \\
&\quad \le C \int_{\mathbb{R}^N} \theta_y |w| \left(1 + |u|^{\frac{4}{N-2}} + |z|^{\frac{4}{N-2}}\right) |A^\sigma(\theta_y m)| \\
&\quad \le C \left(\int_{\mathbb{R}^N} |A^\sigma(\theta_y m)|^{\frac{2N}{N-2+2\sigma}} \right)^{\frac{N-2+2\sigma}{2N}} \left(\int_{\mathbb{R}^N} ||\theta_y w|^{\frac{2N}{N-2-2\sigma}} \right)^{\frac{N-2-2\sigma}{2N}}
\end{aligned}
$$

$$\times \left(\int_{B(y,1)} |1 + |u|^{\frac{4}{N-2}} + |z|^{\frac{4}{N-2}}|^{\frac{N}{2}} \right)^{\frac{2}{N}}$$
$$\leq C \left(\|\theta_y m\|_{L^2} + \|A^{\frac{\sigma+1}{2}}(\theta_y m)\|_{L^2} \right) \left(\|\theta_y w\|_{L^2} + \|A^{\frac{\sigma+1}{2}}(\theta_y w)\|_{L^2} \right), \tag{4.66}$$

where $\sigma \leq \frac{1}{4} < \frac{N-2}{2}$ and $B(y,1) = \{x \in \mathbb{R}^N : |x-y| \leq 1\}$.

Since $\sigma < \frac{1}{2}$,

$$\langle g, \theta_y A^\sigma(\theta_y m) \rangle \leq \|g\|_{L^2_U} \|m\|_{H^1_U}. \tag{4.67}$$

Therefore, by virtue of Lemmas 4.1, 4.3 and Remark 4.7, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\mathbb{R}^N} |A^{\frac{\sigma}{2}}(\theta_y m)|^2 + (1-\delta+\delta^2) \int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y w)|^2 + \int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y m)|^2 \right)$$
$$\leq C \left(\int_{\mathbb{R}^N} |A^{\frac{\sigma}{2}}(\theta_y m)|^2 + (1-\delta+\delta^2) \int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y w)|^2 + \int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y m)|^2 \right)$$
$$+ C_{\|g\|_{L^2_U}, \varrho_1, \varrho_2, \varrho_3, \varrho_4}. \tag{4.68}$$

Applying the Gronwall lemma, we infer that

$$\int_{\mathbb{R}^N} |A^{\frac{\sigma}{2}}(\theta_y m)|^2 + \int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y w)|^2 + \int_{\mathbb{R}^N} |A^{\frac{\sigma+1}{2}}(\theta_y m)|^2$$
$$\leq \mathcal{Q}_2(\|(u_0,u_1)\|_{H^1_{\mathrm{lu}}, \|g\|_{L^2_{\mathrm{lu}}}}) \mathrm{e}^{kt}. \tag{4.69}$$

This completes the proof. □

Now, we state our main results:

Theorem 4.9. (Existence of global attractor) *Assume that f satisfies* (1.3)–(1.5), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$. *Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by the weak solutions of Eqs.* (1.1) *and* (1.2) *with the initial data $(u_0,u_1) \in H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$ has an unique $(H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N), H^1_\rho(\mathbb{R}^N) \times H^1_\rho(\mathbb{R}^N))$ global attractor $\mathcal{A}$, which satisfies*:

- (i) *$\mathcal{A}$ is closed and compact in $H^1_\rho(\mathbb{R}^N) \times H^1_\rho(\mathbb{R}^N))$*;
- (ii) *$\mathcal{A}$ attracts every bounded subset of $(H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$ with respect to $H^1_\rho(\mathbb{R}^N) \times H^1_\rho(\mathbb{R}^N))$-norms*;
- (iii) *$\mathcal{A}$ is invariant; that is, $S(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$.*

5. Asymptotic Regularity

In this section, we will prove the regularity of the $(H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N), H^1_\rho(\mathbb{R}^N) \times H^1_\rho(\mathbb{R}^N))$ global attractor by some bootstrap arguments. Similar to that in Zelik [30,31], based on Lemmas 4.6 and 4.8 above, for the solution $(u(t), u_t(t))$, we can decompose it as follows.

Lemma 5.1. *Assume that f satisfies* (1.3)–(1.5) *and $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$, and let $u(t)$ be the solution of Eqs.* (1.1)–(1.2) *with the initial data $(u_0,u_1) \in B$. Then for any $\varsigma > 0$, there are positive constants C_ς and K_ς, such that*

$$u(t) = z_1(t) + w_1(t), \quad \textit{for all } t \geq 0, \tag{5.1}$$

where $z_1(t)$, $w_1(t)$ *satisfy the estimates as follows*:

$$\|w_1(t)\|^2_{H^{1+\sigma}_U} \leq K_\varsigma, \quad t \geq 0, \tag{5.2}$$

and for every $t \geq s \geq 0$,

$$\int_s^t \|z_1(r)\|^2_{H^1_U} \mathrm{d}r \leq \varsigma(t-s) + C_\varsigma, \tag{5.3}$$

where the constants C_ς *and* K_ς *depend on* ς, σ.

Proof. Note that

$$\mathcal{B} = \bigcup_{t \geq T_{B_0}} S(t)B_0;$$

by Lemma 4.1, we infer that

$$\sup_{t \geq 0} \|S(t)(u_0, u_1)\|^2_{H^1_{\mathrm{lu}}} \leq \varrho_1, \quad \text{for all } (u_0, u_1) \in \mathcal{B}.$$

Now, taking $T \geq \frac{1}{k_0} \ln \frac{\mathcal{Q}_1(\varrho_1)}{\varepsilon}$(where $\mathcal{Q}_1(\cdot)$ the function in Lemma 4.6), and in every interval $[mT, (m+1)T)$, $m = 1, 2, \ldots,$ we set

$$z_1(t) = z(t) \quad \text{and} \quad w_1(t) = w(t),$$

where $z(t)$ is the solution of Eq. (4.38) in the interval $[(m-1)T, (m+1)T)$ with the initial data $(z((m-1)T), z_t((m-1)T)) = (u((m-1)T), u_t((m-1)T))$, and $w(t)$ is the solution of Eq. (4.39) in the interval $[(m-1)T, (m+1)T)$ with the initial data $(w((m-1)T), w_t((m-1)T)) = (0, 0)$.

In the interval $[0, T)$, we set

$$z_1(t) = z(t) \quad \text{and} \quad w_1(t) = w(t),$$

where $z(t)$ is the solution of Eq. (4.38) with the initial data $(z(0), z_t(0)) = (u_0, u_1)$, and $(w(t), w_t(t))$ is the solution of Eq. (4.39) with the initial data $(w(0), w_t(0)) = (0, 0)$.

Then from Lemma 4.6, we infer that

$$\int_s^t \|z_1(r)\|^2_{H^1_U} \mathrm{d}r \leq \varsigma(t-s) + \chi_{[0,T)}(s)\mathcal{Q}_1(\varrho_1), \quad \text{for all } t \geq s \geq 0,$$

where $\chi_{[0,T)}(s)$ is the characteristic function of set $[0, T)$. According to Lemma 4.8, we infer that

$$\|w_1(t)\|^2_{H^{1+\sigma}_U} \leq \mathcal{Q}_2(\|(u_0, u_1)\|_{H^1_{\mathrm{lu}} \times H^1_{\mathrm{lu}}}, \|g\|_{L^2_{\mathrm{lu}}})\mathrm{e}^{2k_1 T}, \quad \text{for all } t \geq 0.$$

This completes the proof. □

Remark 5.2. According to the proof of Lemma 5.1, we infer that the decomposition $z_1(t)$ can also further satisfy that

$$\|z_1(t)\|^2_{H^1_U} \leq \mathcal{Q}_1(\varrho_1), \quad \text{for all } t \geq 0.$$

Lemma 5.3. *Assume that f satisfies (1.3)–(1.5), and f_1 satisfies (4.35)–(4.37), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$. For any bounded set $B \subset H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$, there exists a positive constant $J_{\|B\|_{H^1_{\mathrm{lu}} \times H^1_{\mathrm{lu}}}}$ which depends only on the $H^1_{\mathrm{lu}} \times H^1_{\mathrm{lu}}$-bounds of B, such that*

$$\|K(t)(u_0,u_1)\|^2_{H^{1+\sigma}_{\mathrm{lu}}} \leq J_{\|B\|_{H^1_{\mathrm{lu}} \times H^1_{\mathrm{lu}}}}, \quad \text{for all } t \geq 0 \text{ and } (u_0,u_1) \in B, \tag{5.4}$$

where σ is given in Lemma 4.8.

Proof. Multiplying (4.54) by $\theta_y A^\sigma(\theta_y m)$, similar to the proof of Lemma 4.8, here we only need to deal with the nonlinear term in a different way:

$$\begin{aligned}&\langle f(u) - f(z) + f_1(z), \theta_y A^\sigma(\theta_y m)\rangle \\ &\quad = \langle f(u) - f(z) + f_1(z), \theta_y A^\sigma(\delta\theta_y w + \theta_y w_t)\rangle.\end{aligned} \tag{5.5}$$

By (1.5), we have

$$\begin{aligned}&\langle f(u) - f(z) + f_1(z), \delta\theta_y A^\sigma(\theta_y w)\rangle \\ &\quad = \langle f(u) - f(z), \delta\theta_y A^\sigma(\theta_y w)\rangle + \langle f_1(z), \delta\theta_y A^\sigma(\theta_y w)\rangle \\ &\quad \leq C\int_{\mathbb{R}^N} (1 + |u|^{\frac{4}{N-2}} + |z|^{\frac{4}{N-2}})\theta_y|w||A^\sigma(\theta_y w)| + \langle f_1(z), \theta_y A^\sigma(\theta_y w)\rangle \\ &\quad \triangleq \sum_{i=1}^{4} I_i.\end{aligned} \tag{5.6}$$

For I_1, we infer that

$$I_1 = C\int_{\mathbb{R}^N} \theta_y|w||A^\sigma(\theta_y w)| \leq \|w\|_{L^2_U}\left(\int_{\mathbb{R}^N} |A^\sigma(\theta_y w)|^2\right)^{\frac{1}{2}}. \tag{5.7}$$

For I_2, using Lemma 4.1, we get

$$\begin{aligned}I_2 &= C\int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}}\theta_y|w||A^\sigma(\theta_y w)| \\ &\leq C\int_{\mathbb{R}^N} (|z_1|^{\frac{4}{N-2}} + |w_1|^{\frac{4}{N-2}})\theta_y|w||A^\sigma(\theta_y w)|.\end{aligned} \tag{5.8}$$

By Remark 4.7 and the interpolation inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |z_1|^{\frac{4}{N-2}}\theta_y|w||A^\sigma(\theta_y w)| \\ &\quad = \int_{B(y,1)} |z_1|^{\frac{4}{N-2}}|\theta_y w||A^\sigma(\theta_y w)| \\ &\quad \leq C\left(\int_{B(y,1)} |z_1|^{\frac{2N}{N-2}}\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^N} |\theta_y w|^{\frac{2N}{N-2-2\sigma}}\right)^{\frac{N-2-2\sigma}{2N}} \\ &\qquad \times \left(\int_{\mathbb{R}^N} |A^\sigma(\theta_y w)|^{\frac{2N}{N-2+2\sigma}}\right)^{\frac{N-2+2\sigma}{2N}} \\ &\quad \leq C\|z_1\|^{\frac{4}{N-2}}_{H^1_U}(\|A^{\frac{1+\sigma}{2}}(\theta_y w)\|_{L^2} + \|\theta_y w\|_{L^2})(\|A^{\frac{1+\sigma}{2}}(\theta_y w)\|_{L^2} + \|A^\sigma(\theta_y w)\|_{L^2}) \\ &\quad \leq C_{\varrho_1,\epsilon}\|z_1\|^2_{H^1_U}\int_{\mathbb{R}^N} |A^{\frac{1+\sigma}{2}}(\theta_y w)|^2 + \epsilon\int_{\mathbb{R}^N} |A^{\frac{1+\sigma}{2}}(\theta_y w)|^2 + C_{\varrho_1,\epsilon}(1 + \|\theta_y w\|^2_{L^2}),\end{aligned} \tag{5.9}$$

and by Lemma 5.1 and the interpolation inequality, we infer that

$$\begin{aligned}
\int_{\mathbb{R}^N} |w_1|^{\frac{4}{N-2}}\theta_y|w||A^\sigma(\theta_y w)| &\le C\left(\int_{B(y,1)}|w_1|^{\frac{2N}{N-2-2\sigma}}\right)^{\frac{2(N-2-2\sigma)}{N(N-2)}}\\
&\quad\times\left(\int_{\mathbb{R}^N}|\theta_y w|^{\frac{2N(N-2)}{(N-2)^2-2(N-6)\sigma}}\right)^{\frac{(N-2)^2-2(N-6)\sigma}{2N(N-2)}}\left(\int_{\mathbb{R}^N}|A^\sigma(\theta_y w)|^{\frac{2N}{N-2+2\sigma}}\right)^{\frac{N-2+2\sigma}{2N}}\\
&\le C\|w_1\|^{\frac{4}{N-2}}_{H_U^{1+\sigma}}\|\theta_y w\|_{L^{\frac{2N(N-2)}{(N-2)^2-2(N-6)\sigma}}}(\|A^{\frac{1+\sigma}{2}}(\theta_y w)\|_{L^2}+\|A^\sigma(\theta_y w)\|_{L^2})\\
&\le \epsilon\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2+CK_{\varsigma,\epsilon}(1+\|\theta_y w\|^2_{L^2}). \qquad (5.10)
\end{aligned}$$

Hence,

$$\begin{aligned}
I_2 &\le \epsilon\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2+C_{\varrho_1,\epsilon}\|z_1\|^2_{H_U^1}\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2\\
&\quad +C_{K_\varsigma,\varrho_1,\epsilon}(1+\|\theta_y w\|^2_{L^2}). \qquad (5.11)
\end{aligned}$$

For I_3, we get

$$\begin{aligned}
I_3 &= C\int_{\mathbb{R}^N}|z|^{\frac{4}{N-2}}\theta_y|w||A^\sigma(\theta_y w)|C\left(\int_{B(y,1)}|z|^{\frac{2N}{N-2}}\right)^{\frac{2}{N}}\\
&\qquad\times\left(\int_{\mathbb{R}^N}|\theta_y w|^{\frac{2N}{N-2-2\sigma}}\right)^{\frac{N-2-2\sigma}{2N}}\left(\int_{\mathbb{R}^N}|A^\sigma(\theta_y w)|^{\frac{2N}{N-2+2\sigma}}\right)^{\frac{N-2+2\sigma}{2N}}\\
&\le C\|z\|^{\frac{4}{N-2}}_{H_U^1}\left(\left\|A^{\frac{1+\sigma}{2}}(\theta_y w)\right\|_{L^2}+\|\theta_y w\|_{L^2}\right)\\
&\qquad\times\left(\|A^{\frac{1+\sigma}{2}}(\theta_y w)\|_{L^2}+\|A^\sigma(\theta_y w)\|_{L^2}\right)\\
&\le C_{\varrho_1}\|z\|^{\frac{4}{N-2}}_{H_U^1}\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2+C_{\varrho_1}(1+\|\theta_y w\|^2_{L^2}). \qquad (5.12)
\end{aligned}$$

Note that from Lemma 5.1 and Remark 5.2, we can take T large enough such that

$$\|z\|^{\frac{4}{N-2}}_{H_U^1}\le\frac{\epsilon}{C_{\varrho_1}},\quad \text{for all } t\ge T. \qquad (5.13)$$

For I_4, by (4.35), we infer that

$$\begin{aligned}
I_4 &= \langle f_1(z),\theta_y A^\sigma(\theta_y w)\rangle\\
&\le C\int_{\mathbb{R}^N}(1+|z|^\gamma)\theta_y|A^\sigma(\theta_y w)|\\
&\le C\left(1+\int_{B(y,1)}\theta_y|z|^{\frac{2N\gamma}{N+2-2\sigma}}\right)^{\frac{N+2-2\sigma}{2N}}\|A^\sigma(\theta_y w)\|_{L^{\frac{2N}{N-2+2\sigma}}}\\
&\le C(1+\|z\|^\gamma_{H_U^1})\left(\|A^{\frac{1+\sigma}{2}}(\theta_y w)\|_{L^2}+\|A^\sigma(\theta_y w)\|_{L^2}\right)\\
&\le C_{\varrho_3,\epsilon}(1+\|\theta_y w\|^2_{L^2})+\epsilon\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2. \qquad (5.14)
\end{aligned}$$

Thus,

$$\begin{aligned}&\langle f(u)-f(z)+f_1(z),\delta\theta_y A^\sigma(\theta_y w)\rangle\\&\quad\le 5\epsilon\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2+C_{\varrho_1,\epsilon}\|z_1\|^2_{H^1_U}\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2\\&\qquad+C_{K_\varsigma,\varrho_3,\epsilon}(1+\|\theta_y w\|^2_{L^2}).\end{aligned}\tag{5.15}$$

Similarly, we get that

$$\begin{aligned}&\langle f(u)-f(z)+f_1(z),\theta_y A^\sigma(\theta_y w_t)\rangle\\&\le 5\epsilon\left(\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2+\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w_t)|^2\right)\\&\quad+C_{\varrho_3,\epsilon}\|z_1\|^2_{H^1_U}\int_{\mathbb{R}^N}|A^{\frac{1+\sigma}{2}}(\theta_y w)|^2+C_{K_\varsigma,\varrho_3,\epsilon}(1+\|\theta_y w\|^2_{L^2}+\|\theta_y w_t\|^2_{L^2}).\end{aligned}\tag{5.16}$$

We denote that

$$\mathcal{E}_2(t)=\int_{\mathbb{R}^N}|A^{\frac{\sigma}{2}}(\theta_y m)|^2+(1-\delta+\delta^2)\int_{\mathbb{R}^N}|A^{\frac{\sigma+1}{2}}(\theta_y w)|^2+\int_{\mathbb{R}^N}|A^{\frac{\sigma+1}{2}}(\theta_y m)|^2).$$

Therefore, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2(t)+C_\epsilon(1-C_{\varrho_1}\|z_1\|^2_{H^1_U})\mathcal{E}_2(t)\le C_{K_\varsigma,\|g\|_{L^2_U},\varrho_1,\varrho_2,\varrho_3,\varrho_4}.\tag{5.17}$$

Applying the Gronwall Lemma and integrating over $[1+T,t]$, we infer

$$\begin{aligned}\mathcal{E}_2(t)&\le \mathrm{e}^{-\int_{T+1}^t C_\epsilon(1-C_{\varrho_1}\|z_1(s)\|^2_{H^1_U})\mathrm{d}s}\mathcal{E}_2(T+1)\\&\quad+C_{K_\varsigma,\|g\|_{L^2_U},\varrho_1,\varrho_2,\varrho_3,\varrho_4}\int_{T+1}^t \mathrm{e}^{\int_t^s C_\epsilon(1-C_{\varrho_1}\|z_1(\tau)\|^2_{H^1_U})\mathrm{d}\tau}\mathrm{d}s.\end{aligned}\tag{5.18}$$

According to Lemma 5.1, for every $t\ge s\ge 0$,

$$\int_s^t\|z_1(r)\|^2_{H^1_U}\mathrm{d}r\le\varsigma(t-s)+C_\varsigma.$$

Now, we choose $\varsigma<\frac{1}{2C_{\varrho_1}}$ and have

$$\begin{aligned}&C_{K_\varsigma,\|g\|_{L^2_U},\varrho_1,\varrho_2,\varrho_3,\varrho_4}\int_{T+1}^t \mathrm{e}^{\int_t^s C_\epsilon(1-C_{\varrho_1}\|z_1(\tau)\|^2_{H^1_U})d\tau}\mathrm{d}s\\&\quad\le C_{K_\varsigma,\|g\|_{L^2_U},\varrho_1,\varrho_2,\varrho_3,\varrho_4}\mathrm{e}^{C_{\varrho_1}C_\epsilon}\int_{T+1}^t \mathrm{e}^{C_\epsilon(1-C_{\varrho_1}\varsigma)(s-t)}\mathrm{d}s\\&\quad\le C_{K_\varsigma,\|g\|_{L^2_U},\epsilon,\varrho_1,\varrho_2,\varrho_3,\varrho_4}\int_{T+1}^t \mathrm{e}^{\frac{s-t}{2}}\mathrm{d}s\\&\quad\le C_{K_\varsigma,\|g\|_{L^2_U},\epsilon,\varrho_1,\varrho_2,\varrho_3,\varrho_4},\end{aligned}\tag{5.19}$$

and

$$\begin{aligned}&\mathrm{e}^{-\int_{T+1}^t C_\epsilon(1-C_{\varrho_1}\|z_1(s)\|^2_{H^1_U})\mathrm{d}s}\mathcal{E}_2(T+1)\\&\quad\le \mathrm{e}^{-\frac{1}{2}(t-T-1)}\mathrm{e}^{C_{\varrho_1,\varsigma,\epsilon}}\mathcal{E}_2(T+1).\end{aligned}\tag{5.20}$$

Note that T is fixed, and using Lemma 4.8, we complete the proof. □

Lemma 5.4. *Assume that f satisfies (1.3)–(1.5), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$. Assume B_σ is an arbitrary bounded set in $H^{1+\sigma}_{\mathrm{lu}} \times H^{1+\sigma}_{\mathrm{lu}}$. Then there exists a constant $M_\sigma(>0)$ which depends only on the $H^{1+\sigma}_{\mathrm{lu}} \times H^{1+\sigma}_{\mathrm{lu}}$-bound of B_σ such that*

$$\|S(t)(u_0,u_1)\|^2_{H^{1+\sigma}_{\mathrm{lu}} \times H^{1+\sigma}_{\mathrm{lu}}} \le M_\sigma, \quad \textit{for all } t \ge 0 \textit{ and } (u_0,u_1) \in B_\sigma. \tag{5.21}$$

Proof. The proof of this lemma is completely similar to that of Lemma 5.3, and we can deal with the nonlinear term by similar calculations used in Lemma 5.3, so we omit it here. □

In the following, we can perform the bootstrap arguments to obtain the asymptotic regularity of the solutions. Similar to the proof Lemma 5.3, we infer the following two lemmas.

Lemma 5.5. *Assume that f satisfies (1.3)–(1.5), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$ and $\sigma \le \theta \le 1$. Then for any bounded $B_\theta \subset H^{1+\theta}_{\mathrm{lu}} \times H^{1+\theta}_{\mathrm{lu}}$. Then there exists a constant $M_\theta(>0)$ which depends only on the $H^{1+\theta}_{\mathrm{lu}} \times H^{1+\theta}_{\mathrm{lu}}$-bound of B_θ such that*

$$\|S(t)(u_0,u_1)\|^2_{H^{1+\theta}_{\mathrm{lu}} \times H^{1+\theta}_{\mathrm{lu}}} \le M_\theta, \\ \textit{for all } t \ge 0 \textit{ and } (u_0,u_1) \in B_\theta. \tag{5.22}$$

Lemma 5.6. *Assume that f satisfies (1.3)–(1.5), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$ and $\theta \in [\sigma, 1-\min\{\sigma, \frac{4\sigma}{n-2}\}]$, and assume that the initial data set B_θ is bounded in $H^{1+\theta}_{\mathrm{lu}} \times H^{1+\theta}_{\mathrm{lu}}$, then the decomposed ingredient $(w(t), w_t(t))$ satisfies that*

$$\|K(t)(u_0,u_1)\|^2_{H^{1+\theta+s_0}_{\mathrm{lu}}} \le J_\theta, \quad \textit{for all } t \ge 0 \textit{ and } (u_0,u_1) \in B_\theta, \tag{5.23}$$

where $s_0 = \min\{\sigma, \frac{4\sigma}{n-2}\}$ and the constant $J_\theta(>0)$ which depends only on the $H^{1+\theta}_{\mathrm{lu}} \times H^{1+\theta}_{\mathrm{lu}}$-bound of B_θ.

We also need the following attraction transitivity lemma.

Lemma 5.7. [17] *Let K_1, K_2, K_3 be subsets of H such that*

$$\mathrm{dist}_H(S(t)K_1, K_2) \le L_1 \mathrm{e}^{-\nu_1 t}, \ \mathrm{dist}_H(S(t)K_2, K_3) \le L_2 \mathrm{e}^{-\nu_2 t},$$

for some $\nu_1, \ \nu_2 > 0$ and $L_1, \ L_2 > 0$. Assume that for all $z_1, z_2 \in \bigcup_{t\ge 0} S(t)K_j$ $(j = 1,2,3)$, there holds

$$\|S(t)z_1 - S(t)z_2\| \le L_0 \mathrm{e}^{\nu_0 t}\|z_1 - z_2\|$$

for some $\nu_0 > 0$ and some $L_0 > 0$. Then it follows that

$$\mathrm{dist}_H(S(t)K_1, K_3) \le L\mathrm{e}^{-\nu t},$$

where $\nu = \frac{\nu_1\nu_2}{\nu_0+\nu_1+\nu_2}$ and $L = L_0L_1 + L_2$.

Now, we state the following asymptotic regularity results:

Theorem 5.8. (Asymptotic Regularity) *Assume that f satisfies* (1.3)–(1.5), $g(x) \in L^2_{\mathrm{lu}}(\mathbb{R}^N)$, *and let* $\{S(t)\}_{t\geq 0}$ *be the semigroup generated by the weak solutions of Eqs.* (1.1)–(1.2) *with the initial data* $(u_0, u_1) \in H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$. *Then, there exists a set* $\mathcal{B} \subset H^2_{\mathrm{lu}}(\mathbb{R}^N) \times H^2_{\mathrm{lu}}(\mathbb{R}^N)$ *(closed and bounded in* $H^2_{\mathrm{lu}}(\mathbb{R}^N) \times H^2_{\mathrm{lu}}(\mathbb{R}^N)$*), a positive constant ν and a monotonically increasing function $\mathcal{Q}(\cdot)$ such that: for any bounded set* $B \subset H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$*, the following estimate holds:*

$$\mathrm{dist}_*(S(t)B, \mathcal{B}) \leq \mathcal{Q}(\|B\|_{H^1_{\mathrm{lu}}(\mathbb{R}^N)\times H^1_{\mathrm{lu}}(\mathbb{R}^N)})\mathrm{e}^{-\nu t},$$

where dist_* *denotes the usual Hausdorff semidistance in* $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$.

Proof. We denote

$$\mathcal{B} = \bigcup_{t\geq T_{B_0}} S(t)B_0,$$

where B_0 be the bounded absorbing set stated in Remark 4.2 and $T_{B_0} = \max\{T_1(B), T_2(B), T_3(B)\}$.

According to Lemmas 4.6 and 5.3, we know that there is a set A_σ which is bounded in $H^{1+\sigma}_{\mathrm{lu}}(\mathbb{R}^N) \times H^{1+\sigma}_{\mathrm{lu}}(\mathbb{R}^N)$ such that

$$\mathrm{dist}_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}(S(t)\mathcal{B}, A_\sigma) \leq \mathrm{dist}_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}(D(t)\mathcal{B}, A_\sigma) \leq \mathcal{Q}_1(\|\mathcal{B}\|_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}})\mathrm{e}^{-k_0 t}.$$

Applying Lemmas 4.6 and 5.6 to A_σ, we see that there is a set $A_{\sigma+s}$ which is bounded in $H^{1+\sigma+s}_{\mathrm{lu}}(\mathbb{R}^N) \times H^{1+\sigma+s}_{\mathrm{lu}}(\mathbb{R}^N)$, such that

$$\mathrm{dist}_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}(S(t)\mathcal{B}, A_{\sigma+s}) \leq \mathrm{dist}_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}(D(t)\mathcal{B}, A_{\sigma+s}) \leq \mathcal{Q}_1(\|\mathcal{B}\|_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}})\mathrm{e}^{-k_0 t},$$

where k_0 depends only on the $H^1_{\mathrm{lu}} \times H^1_{\mathrm{lu}}$-bound of A_σ. Combining this with Remark 3.2, we know that the conditions in Lemma 5.7 are all satisfied. Hence we have

$$\mathrm{dist}_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}(S(t)\mathcal{B}, A_{\sigma+s}) \leq C\mathcal{Q}_1(\|\mathcal{B}\|_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}})\mathrm{e}^{-k_0 t}, \tag{5.24}$$

for two appropriate constants C and k_0.

Note that $\sigma = \min\{\frac{1}{4}, \frac{N+2-(N-2)\gamma}{2}\}$ and $s_0 = \min\{\sigma, \frac{4\sigma}{n-2}\}$ are fixed, by finite steps (e.g., at most by $[\frac{1}{s}] + 2$ steps) we can infer that there is a bounded (in $H^2_{\mathrm{lu}}(\mathbb{R}^N) \times H^2_{\mathrm{lu}}(\mathbb{R}^N)$) set $B_1 \subset H^2_{\mathrm{lu}}(\mathbb{R}^N) \times H^2_{\mathrm{lu}}(\mathbb{R}^N)$ such that

$$\mathrm{dist}_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}(S(t)\mathcal{B}, B_1) \leq \mathcal{Q}_1(\|\mathcal{B}\|_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}})\mathrm{e}^{-\nu t}.$$

Now, for any bounded set $B \in H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$, by Lemma 4.1 and Remark 4.2, we see that there exist a T such that

$$S(t)B \subset \mathcal{B}, \quad \text{for all } t \geq T.$$

Hence,

$$\mathrm{dist}_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}}(S(t)\mathcal{B}, B_1) \leq Me^{\nu T}\mathrm{e}^{-\nu t}, \tag{5.25}$$

where $M = \sup\{\|S(t)B\|_{H^1_{\mathrm{lu}}\times H^1_{\mathrm{lu}}},\ 0 \leq t \leq T\} < \infty$.

Now, we apply the attraction transitivity lemma, i.e., Lemma 5.7, then again to (5.24) and (5.25), and this completes the proof. □

Remark 5.9. The $(H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N), H^1_\rho(\mathbb{R}^N) \times H^1_\rho(\mathbb{R}^N))$-global attractor given in Theorem 4.9 is bounded in the locally uniform space $H^2_{\mathrm{lu}}(\mathbb{R}^N) \times H^2_{\mathrm{lu}}(\mathbb{R}^N)$, which appears to be optimal.

Remark 5.10. There exists a bounded (in $(H^2_{\mathrm{lu}}(\mathbb{R}^N) \times H^2_{\mathrm{lu}}(\mathbb{R}^N))$ subset which attracts exponentially every initial $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$-bounded set with respect to the $H^1_{\mathrm{lu}}(\mathbb{R}^N) \times H^1_{\mathrm{lu}}(\mathbb{R}^N)$-norm.

Remark 5.11. To our best knowledge, this is the first time we obtain the regularity for Eqs. (1.1) and (1.2) with critical nonlinearity on the unbounded domain. Maybe it is a basis for further considering the asymptotic behavior, e.g., based on this result, whether the exponential attractors exist for Eqs. (1.1) and (1.2) with critical nonlinearity on unbounded domain is still open.

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Declarations

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References

[1] Arrieta, J.M., Cholewa, J.W., Dlotko, T., Rodrguez-Bernal, A.: Dissipative parabolic equations in locally uniform spaces. Math. Nachr. **280**, 1643–1663 (2007)

[2] Arrieta, J., Carvalho, A.N., Hale, J.K.: A damped hyperbolic equation with critical exponent. Commun. Partial Differ. Equ. **17**, 841–866 (1992)

[3] Babin, A.V., Vishik, M.I.: Attractors of partial differential evolution equations in an unbounded domain. Proc. R. Soc. Edinb. Sect. A **116**, 221–243 (1990)

[4] Babin, A.V., Vishik, M.I.: Attractors of Evolution Equations. Nakua, Moscow (1989). [English translation, North Holland (1992)]

[5] Bogolubsky, I.L.: Some examples of inelastic soliton interaction. Comput. Phys. Commun. **13**, 149–155 (1977)

[6] Cholewa, J.W., Dlotko, T.: Global Attractors in Abstract Parabolic Problems. Cambridge University Press, Cambridge (2000)

[7] Cholewa, J.W., Dlotko, T.: Strongly damped wave equation in uniform spaces. Nonlinear Anal. TMA **64**, 174–187 (2006)

[8] Cholewa, J.W., Dlotko, T.: Hyperbolic equations in uniform spaces. Bull. Pol. Acad. Sci. Math. **52**, 249–263 (2004)

[9] Cholewa, J.W., Dlotko, T.: Bi-spaces global attractors in abstract parabolic equations. Evol. Equ. Banach Center Publ. **60**, 13–26 (2003)

[10] Carvalho, A.N., Cholewa, J.W.: Local well posedness for strongly damped wave equations with critical nonlinearities. Bull. Aust. Math. Soc. **66**, 443–463 (2002)

[11] Carvalho, A.N., Cholewa, J.W.: Local well posedness, asymptotic behavior and asymptotic bootstrapping for a class of semilinear evolution equations of the second order in time. Trans. Am. Math. Soc. **361**(5), 2567–2586 (2009)

[12] Carvalho, A.N., Cholewa, J.W.: Attractors for strongly damped wave equations with critical nonlinearities. Pac. J. Math. **207**, 287–310 (2002)

[13] Carvalho, A.N., Cholewa, J.W., Dlotko, T.: Strongly damped wave problems: bootstrapping and regularity of solutions. J. Differ. Equ. **244**, 2310–2333 (2008)

[14] Clarkson, P.A., Leveque, R.J., Saxton, R.A.: Solitary wave interaction in elastic rods. Stud. Appl. Math. **75**, 95–122 (1986)

[15] Conti, M., Pata, V.: On the regularity of global attractors. Discret. Contin. Dyn. Syst. **25**, 1209–1217 (2009)

[16] Efendiev, M.A., Zelik, S.V.: The attractor for a nonlinear reaction–diffusion system in an unbounded domain. Commun. Pure Appl. Math. **54**, 625–688 (2001)

[17] Fabrie, P., Galushinski, C., Miranville, A., Zelik, S.: Uniform exponential attractors for a singular perturbed damped wave equation. Discret. Contin. Dyn. Syst. **10**, 211–238 (2004)

[18] Jones, R., Wang, B.: Asymptotic behavior of a class of stochastic nonlinear wave equations with dispersive and dissipative terms. Nonlinear Anal. Real World Appl. **14**(3), 1308–1322 (2013)

[19] Kano, T., Nishida, T.: A mathematical justification for Korteweg–de Vries equation and Boussinesq equation of water surface waves. Osaka J. Math. **23**, 389–413 (1986)

[20] Makhankov, V.G.: On stationary solutions of the Schrödinger equation with a self-consistent potential satisfying Boussinesq's equation. Phys. Lett. A **50**, 42–44 (1974)

[21] Makhankov, V.G.: Dynamics of classical solitons (in non-integrable systems). Phys. Rep. **35**, 1–128 (1978)

[22] Miranville, A., Zelik, S.: Attractors for dissipative partial differential equations in bounded and unbounded domains. Handb. Differ. Equ. Evolut. Equ. **4**(08), 103–200 (2008)

[23] Robinson, C.: Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors. Cambridge University Press, Cambridge (2001)

[24] Sun, C., Yang, L., Duan, J.: Asymptotic behavior for a semilinear second order evolution equation. Tran. Am. Math. Soc. **363**(11), 6085–6109 (2011)

[25] Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physic. Springer, New York (1997)

[26] Wang, S., Chen, G.: The Cauchy problem for the generalized IMBq equation in $W^{s,p}(\mathbb{R}^n)$. J. Math. Anal. Appl. **266**, 38–54 (2002)

[27] Xie, Y., Zhong, C.: The existence of global attractors for a class nonlinear evolution equation. J. Math. Anal. Appl. **336**, 54–69 (2007)

[28] Xie, Y., Zhong, C.: Asymptotic behavior of a class of nonlinear evolution equations. Nonlinear Anal. TMA **71**, 5095–5105 (2009)

[29] Yang, M., Sun, C.: Dynamics of strongly damped wave equations in locally uniform spaces: attractors and asymptotic regularity. Trans. Am. Math. Soc. **361**, 1069–1101 (2009)

[30] Zelik, S.V.: Asymptotic regularity of solutions of a nonautonomous damped wave equation with a critical growth exponent. Commun. Pure Appl. Anal. **3**, 921–934 (2004)

[31] Zelik, S.V.: The attractor for a nonlinear hyperbolic equation in the unbounded domain. Discret. Contin. Dyn. Syst. **7**(3), 593–641 (2012)

[32] Zhang, F.H., Wang, S., Wang, L.: Robust exponential attractors for a class of non-autonomous semi-linear second-order evolution equation with memory and critical nonlinearity. Appl. Anal. **98**(6), 1052–1084 (2019)

[33] Zhang, F.H., Chen, X.: Pullback attractors for a class of semilinear second-order nonautonomous evolution equations with hereditary characteristics. J. Math. **2022**, 1–11 (2022)

Fang-hong Zhang
Regional Circular Economy Key Laboratory of Gansu Higher Institutions
Lanzhou
China
e-mail: zhangfanghong2010@126.com

and

Department of Mathematics
Lanzhou Technology and Business College
Lanzhou
China

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