Mediterr. J. Math. (2023) 20:192
https://doi.org/10.1007/s00009-023-02398-y
1660-5446/23/040001-17
published online April 24, 2023

Mediterranean Journal of Mathematics

Local Isometries on Subspaces of Continuous Functions

Juan J. Font and Maliheh Hosseini

Abstract. In this paper, we provide a representation of local isometries when defined between certain general subspaces of scalar-valued and vector-valued continuous functions. Based on the description mentioned above, we are able to prove the algebraic reflexivity of the group of isometries of the subspace of absolutely continuous vector-valued functions and of the subspace of continuously differentiable complex-valued functions.

Mathematics Subject Classification. Primary 47B38; Secondary 46E25.

Keywords. Spaces of continuous functions, local isometry, peak point, surjective linear isometry.

1. Introduction

A linear map T defined between two normed spaces A and B is said to be *locally surjective* or simply *local* if, for every $a \in A$, there exists a surjective linear map $T_a : A \longrightarrow B$, such that $T(a) = T_a(a)$.

Local isometries have attracted considerable attention recently since the publication of [8, 18, 19]. The main goal when dealing with local isometries is usually to obtain their surjectivity, which is equivalent to the study of the algebraic reflexivity of the set of surjective isometries of such spaces. This is a very basic problem in analysis: getting global conclusions from local hypothesis.

Molnár and Zalar [19] proved that for the space $C(K, \mathbb{C})$ of complex-valued continuous functions on a first countable compact space K (see [17] for the locally compact case), any local isometry is a surjection. First countability is essential in this result [18, Remark 3.2.2]. The proofs of such results strongly depend on the Gleason–Kahane–Żelazko theorem or on the

Juan J. Font and Maliheh Hosseini contributed equally to this work. Juan J. Font is supported by Spanish Government under Grant No. PID2019-106529GB-I00 and by Universitat Jaume I (Projecte UJI-B2019-08). This work was partially supported by a grant from the IMU-CDC.

Russo–Dye theorem, which are only applicable to complex-valued functions. Therefore, much less is known about local isometries of spaces of real-valued continuous functions. Even so, Cabello-Sánchez [7] proved similar results for various classes of locally compact spaces K in this real-valued context and showed that any local isometry on $C_0(K, \mathbb{R})$ is surjective. Such classes include totally disconnected locally compact spaces whose one-point compactification is metrizable and manifolds with and without boundary. More recently, Oi [20] has studied local isometries between complex Banach algebras using a new version of the Kowalski–Słodkowski theorem.

In this paper, we provide, using straightforward concepts, a representation of local isometries when defined between certain general subspaces of continuous functions for both scalar-valued and vector-valued functions. However, it seems intricate to obtain a general result concerning the algebraic reflexivity for the sets of surjective isometries of subspaces of continuous functions. Despite this and based on the description mentioned above, we are able to prove the algebraic reflexivity of the group of isometries of the subspace of absolutely continuous vector-valued functions and of the subspace of continuously differentiable complex-valued functions.

2. Preliminaries

Let X be a compact Hausdorff space and E be a real or complex normed space. By $C(X,E)$, we denote the space of all continuous E-valued functions defined on X. When E is the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we simply write $C(X)$.

Let A be a linear subspace of $C(X,E)$. It is said that $x \in X$ is a *peak point* for A if there is a function $f \in A$, such that $\|f\|_\infty = \|f(x)\| > \|f(y)\|$ for all $y \in X\backslash\{x\}$, where $\|f\|_\infty$ stands for the supremum norm of f. We shall write $Pk(A)$ to denote the set of peak points for A.

It is said that $x \in X$ is a *strong boundary point* (or *weak peak point*) for A if, for every neighborhood U of x, there is a function $f \in A$, such that $\|f\|_\infty = \|f(x)\| > \|f(y)\|$ for all $y \in X\backslash U$.

Let A be a linear subspace of $C(X)$. It is said that A separates (resp. strongly) the points of X if, given two distinct points $x, y \in X$, then there exists $f \in A$ with $f(x) \neq f(y)$ (resp. $|f(x)| \neq |f(y)|$). We also recall that a non-empty subset X' of X is called a *boundary* for A if each function in A attains its maximum modulus within X'. In particular, $Ch(A)$ stands for the Choquet boundary of A, i.e., a boundary for A consisting of the set of all $x \in X$, such that the evaluation functional δ_x at x is an extreme point of the closed unit ball of the dual space of $(A, \|.\|_\infty)$.

Furthermore, a closed subspace A of $C(X)$ is said to be *completely regular* if any $x \in X$ is a strong boundary point for A.

It is worth mentioning that, in a first countable context, for a closed subspace A of $C(X)$, strong boundary points coincide with peak points. To see this, assume that x is a strong boundary point for A. Let $\{U_n\}$ be a countable neighborhood basis at x, such that $U_n \subseteq U_m$ when $m \leq n$. For each $n \in \mathbb{N}$, take $f_n \in A$, such that $f_n(x) = 1 = \|f_n\|_\infty$ and $|f_n| < 1$ on

$X \backslash U_n$. Now, define $f = \sum_{n=1}^{\infty}(f_n/2^n)$. Clearly, $f \in A$, since A is closed, $f(x) = 1$, and for each $x' \neq x$, $|f(x')| < 1$. Thus, x is a peak point for A, as claimed (see also [10, p. 447] or [6, p. 97]).

3. Local Isometries on Subspaces of Scalar-Valued Continuous Functions

Let X and Y be first countable compact Hausdorff spaces. Let A and B be closed strongly separating subspaces of $C(X)$ and $C(Y)$, respectively, such that $Pk(A)$ and $Pk(B)$ are non-empty. Let us remark that the existence of peak points is not a strong assumption. For example, by Bishop [4], for a closed separating subspace A of complex $C(X)$ which contains the constants, $Pk(A)$ is not only non-empty, but it is also a boundary for A. Furthermore, when X is metrizable, according to [22, Corollary 8.4], the set of peak points of each closed separating subspace A of $C(X)$ containing the constants is dense in the Choquet boundary of A. Moreover, from [3, Proposition 7], for any closed separating subspace A of real $C(X)$ which contains the constants, not only $Pk(A) \neq \emptyset$ but also the closure of $Pk(A)$ contains the Choquet boundary (see also [3, Theorem 3] for more results giving function spaces for which the set of peak points is not empty). Similarly, in the absence of the constants, Rao and Roy proved that if A is a separable, closed strongly separating subspace A of complex $C(X)$, such that functions in A have no common zero, then $Pk(A)$ is dense in $Ch(A)$ [23, Proposition 4].

Let $T : A \longrightarrow B$ be a linear isometry. For a peak point x_0 for A, we can define

$$C_{x_0} = \{f \in A : 1 = \|f\|_\infty = |f(x_0)|\}.$$

For any $f \in A$, let

$$L(f) = \{y \in Y : \|Tf\|_\infty = |(Tf)(y)|\}$$

and let

$$I_{x_0} = \bigcap_{f \in C_{x_0}} L(f).$$

By [2, Lemmas 2.3 and 2.4], we know that $I_{x_0} \neq \emptyset$, and furthermore

$$I_{x_0} = \{y \in Y : |(Tf)(y)| = |f(x_0)|, \forall f \in A\}.$$

To make the paper self-contained, we adapt the techniques used in [2, Theorem 3.1] to prove the following theorem.

Theorem 3.1. *Let X and Y be first countable compact Hausdorff spaces. Let A and B be closed strongly separating subspaces of $C(X)$ and $C(Y)$, respectively, such that $Pk(A)$ and $Pk(B)$ are non-empty. If $T : A \longrightarrow B$ is a linear isometry, then*

$$(Tf)(y) = a(y)f(\psi(y))$$

for all $f \in A$ *and all* $y \in Y_0 := \bigcup_{x\in Pk(A)} I_x$*, where* $a : Y_0 \longrightarrow \mathbb{K}$ *is a continuous unimodular function and* $\psi : Y_0 \longrightarrow Pk(A)$ *is a continuous surjective map.*

If, furthermore, T *is surjective, then* $Y_0 = Pk(B)$ *and* $\psi : Pk(B) \longrightarrow Pk(A)$ *is a homeomorphism.*

Proof. Let $T : A \longrightarrow B$ be a linear isometry. We prove the result through several claims.

Claim 1. Fix $x_0 \in Pk(A)$. If $f \in A$ satisfies $f(x_0) = 0$, then $(Tf)(y) = 0$ for all $y \in I_{x_0}$. Furthermore, $|f(x_0)| = |(Tf)(y)|$ for all $f \in A$ and all $y \in I_{x_0}$.

Suppose, contrary to what we claim, that there is a function $f \in A$ with $f(x_0) = 0$, such that $(Tf)(y_0) \neq 0$ for some $y_0 \in I_{x_0}$. With no loss of generality, we can assume that $\|f\|_\infty = 1$ and $(Tf)(y_0) = k > 0$.

Since x_0 is a peak point for A, there exists $g \in A$, such that $|g(x_0)| = 1 = \|g\|_\infty$ and $|g(x)| < 1$ for all $x \in X\backslash\{x_0\}$. In addition, since $y_0 \in I_{x_0}$, we can suppose that $(Tg)(y_0) = 1$. Let $U = \{x \in X : |f(x)| \geq k/2\}$. As U is clearly compact and $x_0 \notin U$, then we can consider $s = \sup\{|g(x)| : x \in U\} < 1$.

Next, choose $M > 0$, such that $1 + Ms < k + M$. If $x \in U$, then $|(f + Mg)(x)| \leq 1 + Ms < k + M$. On the other hand, if $x \notin U$, then $|(f + Mg)(x)| < k/2 + M$. As a consequence, $\|(f + Mg)\|_\infty < k + M$, but

$$k + M = (Tf)(y_0) + (MTg)(y_0) \leq \|T(f + Mg)\|_\infty,$$

which is a contradiction with the isometric property of T.

Let us suppose that there exists $f \in A$, such that $|f(x_0)| \neq |(Tf)(y_0)|$ for some $y_0 \in I_{x_0}$. Then, since x_0 is a peak point, we can define a function $m(x) = f(x) - f(x_0)l(x)$ in A where $l \in A$ with $\|l\|_\infty = 1 = l(x_0)$. Hence, $m(x_0) = 0$, but $(Tm)(y_0) = (Tf)(y_0) - f(x_0)(Tl)(y_0) \neq 0$ which contradicts the above paragraph.

Claim 2. Define $Y_0 := \bigcup_{x\in Pk(A)} I_x$ and a surjective map $\psi : Y_0 \longrightarrow Pk(A)$ as $\psi(y) := x$ if $y \in I_x$. Then, ψ is continuous.

It is apparent that ψ is well defined by the strongly separating property of A.

Let (y_α) be a net in Y_0 converging to y_0 and let $(\psi(y_\beta))$ be a subnet of $(\psi(y_\alpha))$ converging to some $x_1 \in X$. Also, take $x_0 \in Pk(A)$, such that $y_0 \in I_{x_0}$. If we suppose that $x_0 \neq x_1$, then we can find $g \in A$, such that $|g(x_1)| \neq |g(x_0)| = 1$. By the continuity of Tg, we can assume that

$$||(Tg)(y_\beta)| - |(Tg)(y_0)|| < \frac{|1 - |g(x_1)||}{2}$$

for each β. Hence, by Claim 1, we have

$$||g(\psi(y_\beta))| - 1| < \frac{|1 - |g(x_1)||}{2},$$

which would contradict the continuity of g.

Claim 3. For all $f \in A$ and all $y \in Y_0$, we can write $(Tf)(y) = a(y)f(\psi(y))$, where $a : Y_0 \longrightarrow \mathbb{K}$ is a continuous unimodular function.

Let $a : Y_0 \longrightarrow \mathbb{K}$ be a function defined as $a(y) := (Tf_0)(y)$ where $f_0 \in A$, such that $f_0(\psi(y)) = 1$. It is apparent, from Claim 1, that a is well defined and unimodular. Furthermore, for given $f \in A$, let $g := f - f(\psi(y))f_0$, where f_0 is any function in A, such that $f_0(\psi(y)) = 1$. Hence, since $g(\psi(y)) = 0$, by Claim 1, we infer that $(Tg)(y) = 0$, and consequently, $(Tf)(y) = a(y)f(\psi(y))$.

Claim 4. Assume that T is bijective. Then, $Y_0 = Pk(B)$ and $\psi : Pk(B) \longrightarrow Pk(A)$ is a homeomorphism.

Let us first check that if x_0 is a peak point for A, then $y_0 \in I_{x_0}$ is a peak point for B. That is, $Y_0 \subseteq Pk(B)$.

Let V be a neighborhood of y_0, which is the only element of I_{x_0} due to the strongly separating property of $T(A) = B$. Then

$$\bigcap_{f \in C_{x_0}} L(f) \subset V,$$

which is to say that

$$\left(\bigcap_{f \in C_{x_0}} L(f) \right) \cap (X \backslash V) = \emptyset.$$

Thus, since $X \backslash V$ is compact, we can find finitely many functions $\{f_1, \ldots, f_n\} \subset C_{x_0}$, such that

$$\bigcap_{i=1}^{n} L(f_i) \subset V.$$

Assume, multiplying by a constant is necessary, that $f_i(x_0) = 1$, $i = 1, \ldots, n$. Then, we can define a function $f := \sum_{i=1}^{n} f_i$ for which $f(x_0) = n$ and $|(Tf)(y)| < \|Tf\|_\infty = n$ for all $y \notin V$. Consequently, y_0 is a strong boundary point for B. Since Y is first countable and B is closed, y_0 is a peak point for B (see Preliminaries).

To prove the converse, consider the inverse of T, which is an isometry from B onto A. Then, by the above claims, we can define a non-empty set

$$X_0 := \bigcup_{y \in Pk(B)} I_y,$$

such that $X_0 \subseteq Pk(A)$ and a continuous surjective map $\varphi : X_0 \longrightarrow Pk(B)$, such that

$$|(T^{-1}g)(x_0)| = |g(\varphi(x_0))|$$

for all $g \in B$ and $x_0 \in X_0$. Consequently

$$|f(x_0)| = |(Tf)(\varphi(x_0))|$$

for all $f \in A$, which is to say that $\psi(\varphi(x_0)) = x_0$. Thus, one can infer that $\psi : Pk(B) \longrightarrow Pk(A)$ is a homeomorphism. □

We recall that a linear map T defined between two normed spaces A and B is said to be a *local isometry* if, for every $f \in A$, there exists a surjective linear isometry $T_f : A \longrightarrow B$, such that $Tf = T_f f$.

Theorem 3.2. *Let X and Y be first countable compact Hausdorff spaces. Let A and B be closed strongly separating subspaces of $C(X)$ and $C(Y)$, respectively, such that $Pk(A)$ and $Pk(B)$ are non-empty. If $T : A \longrightarrow B$ is a local isometry, then*

$$(Tf)(y) = a(y)f(\psi(y))$$

for all $f \in A$ and all $y \in Pk(T(A))$, where $a : Pk(T(A)) \longrightarrow \mathbb{K}$ is a continuous unimodular function and $\psi : Pk(T(A)) \longrightarrow Pk(A)$ is a homeomorphism.

Proof. Let $T : A \longrightarrow B$ be a local isometry. From Theorem 3.1, there exist a continuous unimodular function $a : Y_0 \longrightarrow \mathbb{K}$ and a continuous surjective map $\psi : Y_0 \longrightarrow Pk(A)$, such that

$$(Tf)(y) = a(y)f(\psi(y))$$

for all $f \in A$ and all $y \in Y_0 := \bigcup_{x\in Pk(A)} I_x$. We continue the proof through several claims.

Claim 1. If $x_0 \in Pk(A)$, then I_{x_0} is a singleton.

To this end, let us suppose that there exist y_1 and y_2 in I_{x_0} with $y_1 \neq y_2$. Since x_0 is a peak point, there exists $f \in A$, such that $f(x_0) = 1$ and $|f| < 1$ on $X\backslash\{x_0\}$.

As T is a local isometry, we can find a linear surjective isometry $T_f : A \longrightarrow B$, such that $Tf = T_f f$. By Theorem 3.1, we can write $(T_f f)(y) = a_f(y)f(\psi_f(y))$ for all $f \in A$ and all $y \in Pk(B)$, where $a_f : Pk(B) \longrightarrow \mathbb{K}$ is a continuous unimodular function and $\psi_f : Pk(B) \longrightarrow Pk(A)$ is a homeomorphism. Hence, $(Tf)(y_1) = (T_f f)(y_1)$ which yields

$$a(y_1)f(\psi(y_1)) = a(y_1)f(x_0) = a_f(y_1)f(\psi_f(y_1)),$$

and consequently

$$1 = |f(x_0)| = |f(\psi_f(y_1))|.$$

Similarly we infer $1 = |f(x_0)| = |f(\psi_f(y_2))|$, which means that $\psi_f(y_1) = x_0 = \psi_f(y_2)$, a contradiction, since ψ_f is injective.

Claim 2. Let x_0 be a peak point for A. If $y_0 \in I_{x_0}$, then y_0 is a peak point for $T(A)$. That is, $Y_0 \subseteq Pk(T(A))$.

Let V be a neighborhood of y_0, which is the only element of I_{x_0}. Then, $\bigcap_{f\in C_{x_0}} L(f) \subset V$, and so

$$\left(\bigcap_{f\in C_{x_0}} L(f)\right) \cap (X\backslash V) = \emptyset.$$

Thus, since $X\backslash V$ is compact, we can find finitely many functions $\{f_1, \ldots, f_n\} \subset C_{x_0}$, such that $\bigcap_{i=1}^n L(f_i) \subset V$. Assume, multiplying by a constant is necessary, that $f_i(x_0) = 1$, $i = 1, \ldots, n$. Then, we can define a function $f := \sum_{i=1}^n f_i$ for which $f(x_0) = n$ and $|(Tf)(y)| < \|Tf\|_\infty = n$ for all $y \notin V$. Consequently, y_0 is a strong boundary point for $T(A)$. Since Y is first countable and $T(A)$ is a closed linear subspace of B, y_0 is a peak point for $T(A)$ (see Preliminaries).

Claim 3. Let y_0 be a peak point for $T(A)$. Then, $y_0 \in I_{x_0}$ for some $x_0 \in Pk(A)$. That is, $Pk(T(A)) \subseteq Y_0$.

Since y_0 is a peak point for $T(A)$, there exists $f \in A$, such that $|(Tf)(y_0)| = 1$ and $|(Tf)(y)| = |f(\psi(y))| < 1$ for any $y \in Y_0\backslash\{y_0\}$. Hence, by Theorem 3.1, $1 = |(Tf)(y_0)| = |(T_f f)(y_0)| = |f(\psi_f(y_0))|$ and, besides, it turns out that $\psi_f(y_0)$ is a peak point for A. Hence, by Claim 1, we know that $I_{\psi_f(y_0)}$ is a singleton. If we suppose that $\{y_0\} \neq \{y_1\} = I_{\psi_f(y_0)}$, then, by Claim 1 in Theorem 3.1, we infer $1 > |(Tf)(y_1)| = |f(\psi_f(y_0))| = 1$, a contradiction.

As a consequence of Claims 1, 2, and 3, we deduce that ψ is a continuous bijective map from $Pk(T(A))$ onto $Pk(A)$. Let us see that, indeed, such map is a homeomorphism.

Claim 4. $\psi : Pk(T(A)) \longrightarrow Pk(A)$ is a homeomorphism.

Since we already know that ψ is continuous, it suffices to check that $\psi^{-1} : Pk(A) \longrightarrow Pk(T(A))$ is continuous. To this end, fix x_0, a peak point for A, and let (x_α) be a net in $Pk(A)$ converging to x_0. Let us consider the net $(y_\alpha) := (\psi^{-1}(x_\alpha))$ in $Pk(T(A))$. Let (y_β) be a subnet of (y_α) converging to some $y_1 \in Y$ and assume that $\psi^{-1}(x_0) = y_0 \neq y_1$. Since y_0 is a peak point for $T(A)$, there exists $f \in A$, such that $|(Tf)(y_1)| < |(Tf)(y_0)| = 1$.

Take a subnet (x_γ) of (x_β), such that

$$||f(x_\gamma)| - |f(x_0)|| \leq \frac{1 - |(Tf)(y_1)|}{2}.$$

Hence

$$||(Tf)(y_\gamma)| - 1| \leq \frac{1 - |(Tf)(y_1)|}{2},$$

which implies that $(Tf)(y_\gamma)$ cannot converge to $(Tf)(y_1)$, a contradiction with the continuity of Tf. □

Let us recall here that a topological space is said to be *incompressible* [13] if it admits no homeomorphism onto a proper subset of itself. Closed manifolds without boundaries are examples of incompressible spaces [11, Corollary 5.1.19].

Corollary 3.3. *Let X be a first countable compact incompressible space and let A be a completely regular subspace of $C(X)$. If $T : A \longrightarrow A$ is a local isometry, then*

$$(Tf)(y) = a(y)f(\psi(y))$$

for all $f \in A$ and all $y \in X$, where $a : X \longrightarrow \mathbb{K}$ is a continuous unimodular function and $\psi : X \longrightarrow X$ is a homeomorphism.

Proof. Since A is a completely regular subspace, $Pk(A) = X$. As a consequence of Theorem 3.2, there are a homeomorphism ψ between X and its subset $Pk(T(A))$ and a continuous unimodular function $a : Pk(T(A)) \longrightarrow \mathbb{K}$, such that $(Tf)(y) = a(y)f(\psi(y))$ for all $f \in A$ and all $y \in Pk(T(A))$. Since X is incompressible, we infer that $Pk(T(A)) = X$ and we are done. □

Theorem 3.4. *Let X be a first countable compact Hausdorff space and let A a be closed strongly separating subspace of $C(X)$ with $Pk(A) \neq \emptyset$. Assume there exists a function $f_0 \in A$, such that $|f_0|$ is injective.*

If there exists $T : A \longrightarrow A$ a local isometry, then

$$(Tf)(y) = a(y)f(\psi(y))$$

for all $f \in A$ and all $y \in Pk(A)$, where $a : Pk(A) \longrightarrow \mathbb{K}$ is a continuous unimodular function and ψ is a selfhomeomorphism of $Pk(A)$. If, furthermore, $Pk(A)$ is a boundary for A, then T is surjective.

Proof. Suppose that $T : A \longrightarrow A$ is a local isometry. By Theorem 3.2, $(Tf)(y) = a(y)f(\psi(y))$ for all $f \in A$ and $y \in Pk(T(A))$, where $a : Pk(T(A)) \longrightarrow \mathbb{K}$ is a continuous unimodular function and $\psi : Pk(T(A)) \longrightarrow Pk(A)$ is a homeomorphism.

Since T is a local isometry, there exists a surjective linear isometry T_{f_0}, such that $Tf_0 = T_{f_0}f_0$. Hence, by Theorem 3.1, $(Tf_0)(y) = (T_{f_0}f_0)(y) = a_{f_0}(y)f_0(\psi_{f_0}(y))$ for all $y \in Pk(A)$, where $a_{f_0} : Pk(A) \longrightarrow \mathbb{K}$ is a continuous unimodular function and $\psi_{f_0} : Pk(A) \longrightarrow Pk(A)$ is a homeomorphism. Then, for each $y \in Pk(T(A))$, we have

$$a_{f_0}(y)\, f_0(\psi_{f_0}(y)) = Tf_0(y) = a(y)f_0(\psi(y)),$$

which implies that $f_0(\psi_{f_0}(y)) = f_0(\psi(y))$, since f_0 is a positive function and $|a_{f_0}(y)| = |a(y)| = 1$. Thus, $\psi_{f_0}(y) = \psi(y)$ because of the injectivity of f_0. Now, again from the above relation, it follows that $a(y) = a_{f_0}(y)$ for all $y \in Pk(T(A))$. Now, we prove that $Pk(T(A)) = Pk(A)$. Contrary to what we claim, assume that $y_0 \in Pk(A)\backslash Pk(T(A))$. Since ψ is surjective, there is a point $y \in Pk(T(A))$, such that $\psi_{f_0}(y_0) = \psi(y)$. From the above part, $\psi_{f_0}(y) = \psi(y)$, and so, $\psi_{f_0}(y) = \psi_{f_0}(y_0)$ which contradicts the injectivity of ψ_{f_0}. Therefore, $Pk(T(A)) = Pk(A)$.

From the above discussion, one can deduce that $Tf = a_{f_0}f \circ \psi_{f_0}$ on $Pk(A)$ for each $f \in A$. Since $Pk(A)$ is a boundary for A, it follows that $Tf = T_{f_0}f$ $(f \in A)$, and consequently, T is surjective.

□

It seems difficult to obtain a general result for the algebraic reflexivity of the isometry groups of subspaces of continuous functions. However, if we restrict to certain important subspaces, we can apply the above results to obtain such algebraic reflexivity.

Let X be a compact subset of $\mathbb{R}$ with at least two points and E be a real or complex normed space. A function $f : X \longrightarrow E$ is said to be *absolutely continuous* on X if given $\epsilon > 0$, there exists a $\delta > 0$, such that

$$\sum_{i=1}^{n} \|f(b_i) - f(a_i)\| < \epsilon,$$

for every finite family of non-overlapping open intervals $\{(a_i, b_i) : i = 1, \ldots, n\}$ whose extreme points belong to X with $\sum_{i=1}^{n}(b_i - a_i) < \delta$. We denote by $AC(X, E)$ the space of all absolutely continuous E-valued functions on X.

When $E = \mathbb{C}$, we write $AC(X)$ instead of $AC(X, \mathbb{C})$. Furthermore, note that the total variation of each absolutely continuous function f is finite, that is

$$\mathcal{V}(f) = \sup \left\{ \sum_{i=1}^{n} \|f(x_i) - f(x_{i-1})\| : n \in \mathbb{N}, x_0, x_1, \ldots, x_n \in X, \right.$$
$$\left. x_0 < x_1 < \cdots < x_n \right\} < \infty.$$

Corollary 3.5. *Let X and Y be compact subsets of $\mathbb{R}$ with at least two points, and let $T : AC(X) \longrightarrow AC(Y)$ be a local isometry with respect to the norm $\|\cdot\|_\infty + \mathcal{V}(\cdot)$. Then, T is a surjective linear isometry.*

Proof. According to [21, Theorem 2.10], each surjective linear isometry $\mathcal{T} : AC(X) \longrightarrow AC(Y)$ is of the form $\mathcal{T}f = \tau f \circ \varphi$ for all $f \in AC(X)$, where τ is a unimodular scalar and $\varphi : Y \longrightarrow X$ is an absolutely continuous homeomorphism. Therefore, since T is a local isometry, it is an isometry with respect to $\|.\|_\infty$.

Next, we claim that $T : AC(X) \longrightarrow AC(Y)$ can be extended to an isometry $\widetilde{T} : C(X) \longrightarrow C(Y)$. To this end, for any $f \in C(X)$, we can take a sequence $\{f_n\}$ in $AC(X)$, such that $\|f_n - f\|_\infty \to 0$ because of the sup-norm density of $AC(X)$ in $C(X)$. Hence, taking into account that T is an isometry with respect to $\|.\|_\infty$, we get $\|Tf_n - T_m f\|_\infty = \|f_n - f_m\|_\infty \to 0$, and so, $\{Tf_n\}$ is a Cauchy sequence in $C(Y)$. We put $\widetilde{T}f = \lim Tf_n$. Now, it is easily checked that the definition of $\widetilde{T}f$ is independent of the choice of the sequence $\{f_n\}$.

Then, since $T1$ is a unimodular constant function, say λ, from Theorem 3.1, it follows that there are a subset Y_0 of Y and a continuous surjection $\psi : Y_0 \to X$, such that

$$(Tf)(y) = \lambda f(\psi(y)) \qquad (f \in AC(X), y \in Y_0).$$

Obviously, $AC(X)$ has a positive injective function, for example, $f_0(x) = x - a + 1$, where $a = \min X$. Then, similarly to the proof of Theorem 3.4, one can see that $Tf = T_{f_0} f = \lambda_{f_0} f \circ \psi_{f_0}$, which implies that $T = T_{f_0}$ is surjective. □

Let X be a compact subset of $\mathbb{R}$, such that X coincides with the closure of its interior. For any $n \in \mathbb{N}$, let $C^{(n)}(X)$ be the Banach algebra of all n-times continuously differentiable complex-valued functions f on X, with the norm $\|f\|_C = \max_{x \in X}(\sum_{k=0}^{n}(|f^{(k)}(x)|/k!))$. In the following, we show that the isometry groups of $C^{(n)}(X)$-spaces are algebraically reflexive (see [16, Corollary 1] and also [20, page 409]).

Corollary 3.6. *Let X and Y be compact subsets of $\mathbb{R}$, such that X and Y coincide with the closures of their interiors. If $T\colon C^{(n)}(X) \to C^{(n)}(Y)$ is local isometry with respect to the norm $\|\cdot\|_C$, then T is a surjective linear isometry.*

Proof. From [24, Theorem 4.4], $\mathcal{T}\colon C^{(n)}(X) \to C^{(n)}(Y)$ is a surjective linear isometry if and only if there exist a function $a\colon Y \to \mathbb{C}$ with $|a(y)| = 1$ and

$a'(y) = 0$ for all $y \in Y$, and a homeomorphism $\psi\colon Y \to X$ with $|\psi'(y)| = 1$ and $\psi''(y) = 0$ for all $y \in Y$, such that

$$(\mathcal{T}f)(y) = a(y)f(\psi(y)) \qquad \Big(y \in Y,\ f \in C^{(n)}(X)\Big).$$

This especially implies that T is an isometry with respect to $\|.\|_\infty$. Now, taking into account that $\overline{C^{(n)}(X)}^{\|\cdot\|_\infty} = C(X)$ and $C^{(n)}(X)$ has positive injective functions, by an argument similar to the previous result, we can infer that T is surjective. □

4. Local Isometries on Subspaces of Vector-Valued Continuous Functions

Let X and Y be first countable compact Hausdorff spaces and let E and F be strictly convex normed spaces.

Definition 4.1. Let $\mathfrak{A}$ be a linear subspace of $C(X, E)$ and let T be a linear isometry of $\mathfrak{A}$ into $C(Y, F)$. If $e \in S_E$, where S_E is the unit sphere of E, and $x \in X$ with $\mathcal{F}(x, e) := \{f \in \mathfrak{A} : \|f\|_\infty = 1 \quad \text{and} \quad f(x) = e\} \neq \emptyset$, then we set

$$I(x, e) := \{y \in Y : \|(Tf)(y)\| = 1 \text{ for all } f \in \mathcal{F}(x, e)\}.$$

Moreover, put $I(x) := \bigcup_{e \in S_E} I(x, e)$.

The proof of the following lemma is standard (see, e.g., [12, Lemma 1]).

Lemma 4.2. *With the same hypothesis as in Definition 4.1, $I(x, e)$ is non-empty.*

Definition 4.3. Let A be a regular closed linear subspace of $C(X)$ with $Pk(A) \neq \emptyset$. We will denote by $A(X, E)$ any linear subspace of $C(X, E)$ which contains the set $\{f \cdot e : f \in A, e \in S_E\}$.

Let us recall that that A is *regular* if it separates any closed subset C of X from any $x \notin C$ in the sense that there is a function $f \in A$ with $f(x) = 1$ and $f = 0$ on C.

Theorem 4.4. (i) *Let T be a linear isometry of $A(X, E)$ into $C(Y, F)$. Then, there exist a continuous mapping ψ from $Y_0 := \bigcup_{x \in Pk(A)} I(x)$ onto $Pk(A)$, and a bounded linear map $\omega(y)$ from E into F and $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in Y_0$ and all $f \in A(X, E)$.*

(ii) *Let T be a linear isometry of $A(X, E)$ onto such a subspace $B(X, E)$ of $C(Y, F)$, where A and B are regular closed subspaces of $C(X)$ and $C(Y)$, respectively. Then, there exist a homeomorphism ψ of $Pk(B)$ onto $Pk(A)$, and a linear isometry $\omega(y)$ of E into F and $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in Pk(B)$ and all $f \in A(X, E)$.*

Proof. (i) **Claim 1.** Let $y \in I(x)$ for some $x \in Pk(A)$. If we take $f \in A(X, E)$, such that $f(x) = 0$, then $(Tf)(y) = 0$.

Take $x_0 \in Pk(A)$. From the definition of peak point, we know that there is $f_1 \in A$ with $1 = f_1(x_0) = \|f_1\|_\infty$ and $|f_1| < 1$ on $X\backslash\{x_0\}$. Hence, by Lemma 4.2, $I(x_0)$ is non-empty.

Fix $e \in S_E$ and $y_0 \in I(x_0, e)$ and let $f_2 \in A(X, E)$, such that f_2 vanishes on some open neighborhood U of x_0. Let us check that $(Tf_2)(y_0) = 0$. Dividing f_2 by a constant, if necessary, we can assume both that $\|f_2\|_\infty < 1$ and $|f_1| < 1 - \|f_2\|_\infty$ on $X\backslash U$. Let us define the functions

$$g := f_2 + f_1 \cdot e$$

and

$$h := \frac{1}{2}(g + f_1 \cdot e).$$

It is obvious that $g(x_0) = h(x_0) = f_1(x_0) \cdot e$. Furthermore, $\|f_1 \cdot e\|_\infty = \|g\|_\infty = \|h\|_\infty = f_1(x_0) = 1$. Hence, as $y_0 \in I(x_0, e)$, we have $\|T(f_1 \cdot e)(y_0)\| = \|(Tg)(y_0)\| = \|(Th)(y_0)\| = f_1(x_0) = 1$. Since F is strictly convex, $T(f_1 \cdot e)(y_0)$, $(Tg)(y_0)$ and $(Th)(y_0)$ belong to S_E, and $(Th)(y_0)$ is on the segment which joins $T(f_1 \cdot e)(y_0)$ and $(Tg)(y_0)$), we infer that $T(f_1 \cdot e)(y_0) = (Tg)(y_0)$ and, as a consequence, $(Tf_2)(y_0) = 0$.

Let $\hat{T}\widehat{y_0} : A(X, E) \longrightarrow F$ and $\hat{x_0} : A(X, E) \longrightarrow E$ be the functionals defined by the requirement that $\hat{T}\widehat{y_0}(f) := (Tf)(y_0)$ and $\widehat{x_0}(f) := f(x_0)$, $f \in A(X, E)$. It is straightforward to check that the functions in $A(X, E)$ that vanish on a neighborhood of x_0 are dense in $\ker(\widehat{x_0})$, since A is regular. Furthermore, $\ker(\widehat{x_0})$ is closed since the functional $\widehat{x_0}$ is continuous. Consequently, the above paragraph yields the inclusion $\ker(\widehat{x_0}) \subseteq \ker(\hat{T}\widehat{y_0})$; this is, if $f(x_0) = 0$, then $(Tf)(y_0) = 0$, as was to be proved.

Claim 2. $I(x_1) \cap I(x_2) = \emptyset$ for $x_1, x_2 \in Pk(A)$.

Suppose that there are $x_1, x_2 \in Pk(A)$ and $y \in Y$, such that $y \in I(x_1) \cap I(x_2)$. Choose $f \in A$, such that $f(x_1) = 1$ and $f(x_2) = 0$. Since $(f \cdot e)(x_2) = 0$ for every $e \in E$, we have, by Claim 1, that $T(f \cdot e)(y) = 0$ for all $e \in E$.

On the other hand, there exists $e_1 \in S_E$, such that $y \in I(x_1, e_1)$ and, as x_1 is peak point for A, there is a function $0 \neq g \in A(X, E)$, such that $g(x_1) = \|g\|_\infty \cdot e_1$. By Claim 1 and since $(g - f \cdot g(x_1))(x_1) = 0$, we infer $(Tg)(y) = T(f \cdot g(x_1))(y)$. Besides, by the above paragraph, $(Tg)(y) = T(f \cdot g(x_1))(y) = 0$. However, from the definition of $I(x_1, e_1)$, we know that $\|(Tg)(y)\| = \|g\|_\infty \neq 0$, which is a contradiction.

Claim 3. Let $x \in Pk(A)$ and $e \in S_E$. If $f(x) = e$ for $f \in A(X, E)$, then $\|(Tf)(y)\| = \|e\| = 1$ for all $y \in I(x, e)$.

Since x is a peak point, there is a function $g \in A$ with $1 = g(x) = \|g\|_\infty$. Define a function h in $A(X, E)$ by $h := f - g \cdot e$. The clear fact that $h(x) = 0$ and Claim 1 yield $(Th)(y) = 0$. By the linearity of T, we have $(Tf)(y) = T(g \cdot e)(y)$. Finally, from the definition of $I(x, e)$, $\|(Tf)(y)\| = \|T(g \cdot e)(y)\| = \|g \cdot e\|_\infty = \|e\| = 1$.

Let us define a mapping ψ from $Y_0 := \bigcup_{x \in Pk(A)} I(x)$ onto $Pk(A)$ by $\psi(y) := x$, where $y \in I(x)$.

Let $y \in I(x)$ for some $x \in Pk(A)$ and let $g \in A$, such that $g(x) = 1 = \|g\|_\infty$. Then, we can define a linear map $\omega(y)$ from E into F as $\omega(y)(e) := T(g \cdot e)(y)$ for all $e \in E$. It is clear that the definition of ω does not depend on the choice of g by Claim 1. Moreover, from Claim 3, it is obvious that $\omega(y)$ is a bounded linear map with $\|\omega(y)\| = 1$.

Claim 4. $\psi : Y_0 \longrightarrow Pk(A)$ is a well-defined surjective continuous mapping and $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in Y_0$ and all $f \in A(X, E)$.

By Claim 2, ψ is a well-defined mapping. To obtain the multiplicative representation of T, let $x \in Pk(A)$ and $y \in I(x)$. Choose any function $\xi \in A$, such that $\xi(x) = 1 = \|\xi\|_\infty$. For every $f \in A(X, E)$, the function $f - \xi \cdot f(x)$ vanishes at x. Thus, by Claim 1, we infer that $(Tf)(y) = T(\xi \cdot f(x))(y) = \omega(y)(f(x))$ for every $f \in A(X, E)$.

To check the continuity of ψ, let (y_α) be a net convergent to y in Y_0. Assume, contrary to what we claim, that $(\psi(y_\alpha))$ does not converge to $\psi(y)$. By taking a subnet if necessary, we can consider that $(\psi(y_\alpha))$ converges to an x in the compact space X. Let U and V be disjoint neighborhoods of x and $\psi(y)$ in X, respectively. There exist an α_0, such that $\psi(y_\alpha) \in U$, for all $\alpha \geq \alpha_0$, and, since A is regular, a function $f \in A(X, E)$, such that $coz(f) \subset V$ and $\|(Tf)(y)\| \neq 0$, where $coz(f) = \{x \in X : f(x) \neq 0\}$. For $\alpha \geq \alpha_0$, $\psi(y_\alpha) \notin coz(f)$. Hence, by Claim 1, $(Tf)(y_\alpha) = 0$, for all $y_\alpha \geq \alpha_0$. Consequently $((Tf)(y_\alpha))$ does not converge to $(Tf)(y) \neq 0$, which contradicts the continuity of Tf.

ii) Assume now that T is onto.

Claim 5. Let $x \in Pk(A)$ and let $y \in I(x)$. Then, $x \in I(y)$.

Suppose that $x \notin I(y)$. Then, since $T^{-1} : B(X, E) \longrightarrow A(X, E)$ is a linear isometry, we can deduce that there exists $x' \in X$, $x' \neq x$, such that $x' \in I(y)$. Choose $f \in A(X, E)$, such that $f(x) = 0$. By Claim 1, we infer that both $(Tf)(y) = 0$ and $T^{-1}(Tf)(x') = f(x') = 0$. This means that x and x' cannot be separated with functions of $A(X, E)$, which contradicts the regularity of A.

Claim 6. $I(x)$ is a singleton for $x \in Pk(A)$.

Let us now suppose that $I(x)$ contains two elements, y and y'. By the above paragraph, $x \in I(y) \cap I(y')$. Since the range of T separates the points of Y, there is a function $f \in A(X, E)$, such that $(Tf)(y) = 1$ and $(Tf)(y') = 0$. From Claim 1, we have $T^{-1}(Tf)(x) = f(x) = 0$ and, hence, $(Tf)(y) = 0$. This contradiction shows that $I(x)$ is a singleton.

As a straightforward consequence of the above claims, we infer that $Y_0 = Pk(B)$ and that $\psi : Pk(B) \longrightarrow Pk(A)$ is a continuous bijection. Furthermore T^{-1} induces a continuous bijection of $Pk(A)$ onto $Pk(B)$ which can be easily checked to be the inverse of ψ, whence $Pk(A)$ and $Pk(B)$ are homeomorphic.

Finally, take $y_0 \in Pk(B)$ and let $x_0 \in Pk(A)$, such that $\{y_0\} = I_{x_0}$. To see that $\omega(y_0)$ is a linear isometry of E into F, choose $e_0 \in S_E$ and $\xi \in A$, such that $1 = \xi(x_0) = \|\xi\|_\infty$. It suffices to check that $\|\omega(y_0)(e_0)\| = 1$. Hence, since $I(x_0)$ is a singleton, $\{y_0\} = \bigcap_{e \in S_E} I(x_0, e)$. In particular, $y_0 \in I(x_0, e_0)$. Consequently

$$\|\omega(y_0)(e_0)\| := \|T(\xi \cdot e_0)(y_0)\| = \|\xi \cdot e_0\|_\infty = 1. \qquad \square$$

Theorem 4.5. *Let T be a local isometry of $A(X,E)$ into such a subspace $B(X,E)$ of $C(Y,F)$. Then, there exist ψ a homeomorphism of $Pk(T(A(X,E)))$ onto $Pk(A)$, $\omega(y)$ a linear isometry of E into F and $(Tf)(y)=\omega(y)(f(\psi(y)))$ for all $y \in Pk(T(A(X,E)))$ and all $f \in A(X,E)$.*

Proof. Since T is a local isometry, from Theorem 4.4(i), there exist a continuous mapping ψ from $Y_0 := \bigcup_{x\in Pk(A)} I(x)$ onto $Pk(A)$, a bounded linear map $\omega(y)$ from E into F and $(Tf)(y)=\omega(y)(f(\psi(y)))$ for all $y \in Y_0$ and all $f \in A(X,E)$. First, note that for any $y \in Y_0$, $\omega(y)$ is an isometry. To see, let $y \in Y_0$ and $e \in S_E$. Since T is a local isometry, we have $\|\omega(y)(e)\| = \|T(g\cdot e)(y)\| = \|T_{g\cdot e}(g\cdot e)(y)\| = \|\omega_{g\cdot e}(y)(e)\| = \|e\| = 1$, where g is a function in A, such that $g(x)=1=\|g\|_\infty$.

Claim 1. $I(x_0)$ is a singleton for $x_0 \in Pk(A)$.

To this end, let us suppose that there exist y_1 and y_2 in $I(x_0)$ with $y_1 \neq y_2$. Since x_0 is a peak point for A, there exists $f \in A(X,E)$, such that $\|f(x_0)\|=1$ and $\|f\|<1$ on $X\backslash\{x_0\}$.

As T is a local isometry, we can find a linear surjective isometry $T_f : A(X,E) \longrightarrow B(Y,F)$, such that $Tf = T_f f$. By Theorem 4.4(ii), we know that there exist a ψ_f a homeomorphism of $Pk(B)$ onto $Pk(A)$ and $\omega_f(y)$ a linear isometry of E into F, such that $(T_f f)(y) = \omega_f(y)(f(\psi_f(y)))$ for all $y \in Pk(B)$. Hence, by Theorem 4.4(i), $(T_f f)(y_1) = \omega_f(y_1)(f(\psi_f(y_1))) = \omega(y_1)(f(\psi(y_1))) = \omega(y_1)(f(x_0))$. Consequently, $\|\omega_f(y_1)(f(\psi_f(y_1)))\| = \|\omega(y_1)(f(\psi(y_1)))\| = \|\omega(y_1)(f(x_0))\| = 1$, which yields $\|f(\psi_f(y_1))\|=1$, since $\omega_f(y_1)$ is an isometry. Similarly, $\|f(\psi_f(y_2))\|=1$, which, by the choice of f, implies that $\psi_f(y_1)=\psi_f(y_2)$, a contradiction with the injectivity of ψ_f.

Claim 2. $Y_0 \subseteq Pk(T(A(X,E)))$.

Let x_0 be a peak point for A. Let V be a neighborhood of $y_0 \in I(x_0)$. That is, $y_0 \in I(x_0,e)$ for some $e \in S_E$. Hence, by Claim 1, y_0 is the only element in $I(x_0,e)$, and consequently

$$\{y\in Y : \|(Tf)(y)\| = \|f\|_\infty = 1 \text{ for all } f \in A(X,E), \\ \text{such that } f(x)=e\} \cap (X\backslash V) = \emptyset.$$

Thus, since $X\backslash V$ is compact, we can find finitely many functions $\{f_1,\ldots,f_n\}$ with $f_i(x_0)=e$, $i=1,\ldots,n$, such that

$$\{y\in Y : \|(Tf_i)(y)\| = \|f_i\|_\infty = 1 \text{ for all } f_i, \quad i=1,\ldots,n\} \subset V.$$

Then, we can define a function $f := \sum_{i=1}^n f_i$ for which $f(x_0)=n\cdot e$ and $|(Tf)(y)| < \|Tf\|_\infty = n$ for all $y \notin V$. Consequently, y_0 is a strong boundary point for $T(A(X,E))$. Since Y is first countable and $T(A(X,E))$ is a closed linear subspace of B, by an argument as in the Preliminaries, we can prove that y_0 is a peak point for $T(A(X,E))$.

Claim 3. $Pk(T(A(X,E))) \subseteq Y_0$.

Since y_0 is a peak point for $T(A(X,E))$, there exists $f\in A(X,E)$, such that $\|(Tf)(y_0)\|=1$ and $\|(Tf)(y)\|<1$ for any $y \in Y\backslash\{y_0\}$. Hence, by Theorem 4.4(ii), $\|(Tf)(y_0)\| = \|(T_f f)(y_0)\| = \|f(h_f(y_0))\| = 1$ and, besides, $h_f(y_0)$ is a peak point for $A(X,E)$. Hence, by Claim 1, we know that $I_{h_f(y_0)}$

is a singleton. If we suppose that $\{y_0\} \neq \{y_1\} = I_{h_f(y_0)}$, then, by Claim 1 in Theorem 4.4(ii), we infer $1 > \|(Tf)(y_1)\| = \|f(h_f(y_0))\| = 1$, a contradiction.

Finally, as in Claim 4 of the proof of Theorem 3.2, we can prove that $h : Pk(T(A(X,E))) \longrightarrow Pk(A)$ is a homeomorphism. □

Theorem 4.6. *Let X be a first countable compact Hausdorff space and let A be a regular closed subspace of $C(X)$. Assume there exists a function $f_0 \in A$, such that $|f_0|$ is injective.*

If there exists $T : A(X,E) \longrightarrow A(X,E)$ a local isometry, then there exist a homeomorphism ψ of $Pk(A)$ onto itself and $\omega(y)$ a linear isometry of E into E for each $y \in Pk(A)$, such that $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in Pk(A)$ and all $f \in A(X,E)$. If, furthermore, $Pk(A)$ is a boundary for A, then T is surjective.

Proof. By Theorem 4.5, there exists ψ a homeomorphism of $Pk(T(A(X,E)))$ onto $Pk(A)$, $\omega(y)$ is a linear isometry of E into E and and $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in Pk(A)$ and all $f \in A(X,E)$. Then, $(Tf_0 \cdot e)(y) = \omega(y)(f_0 \cdot e(\psi(y)))$ for all $y \in Pk(T(A(X,E)))$ and for a fixed $e \in S_E$.

Since T is a local isometry, there exists a linear surjective isometry $T_{f_0 \cdot e}$, such that $(Tf_0 \cdot e)(y) = (T_{f_0 \cdot e} f_0 \cdot e)(y)$. Hence, by Theorem 4.4, $(Tf_0 \cdot e)(y) = (T_{f_0 \cdot e} f_0 \cdot e)(y) = \omega_{f_0 \cdot e}(y)(f_0 \cdot e(\psi_{f_0 \cdot e}(y)))$ for all $y \in Pk(A)$, $\omega_{f_0 \cdot e}(y)$ is an isometry for all $y \in Pk(A)$, and $\psi_{f_0 \cdot e} : Pk(A) \longrightarrow Pk(A)$ is a homeomorphism.

Hence, $\omega(y)(f_0 \cdot e(\psi(y))) = \omega_{f_0 \cdot e}(y)(f_0 \cdot e(\psi_{f_0 \cdot e}(y)))$ for all $y \in Pk(T(A(X,E)))$. Since $\omega(y)$ and $\omega_{f_0}(y)$ are isometries, we infer that $||f_0 \cdot e(\psi_{f_0 \cdot e}(y)))||$ $= ||f_0 \cdot e(\psi(y))||$ for all $y \in Pk(T(A(X,E)))$. Consequently, $|f_0(\psi_{f_0 \cdot e}(y))| = |f_0(\psi(y))|$ for all $y \in Pk(T(A(X,E)))$.

Since $|f_0|$ is injective, we infer that $\psi(y) = \psi_{f_0 \cdot e}(y)$ for all $y \in Pk(T(A(X,E)))$. Furthermore, similarly to the proof of Theorem 3.2, one can see that $Pk(T(A(X,E))) = Pk(A)$.

From the above discussion, it follows that $\omega(y) = \omega_{f_0 \cdot e}(y)$ for all $y \in Pk(A)$, whence $(Tf)(y) = \omega_{f_0 \cdot e}(y) f(\psi_{f_0 \cdot e}(y)) = (T_{f_0 \cdot e} f)(y)$ for all $y \in Pk(A)$. Now, if $Pk(A)$ is a boundary for A, then $Tf = T_{f_0 \cdot e} f$ on Y. Therefore, $T = T_{f_0 \cdot e}$ is surjective. □

Definition 4.7. A surjective linear isometry $T : AC(X,E) \longrightarrow AC(Y,F)$ with respect to the norm $\max(\|\cdot\|_\infty, \mathcal{V}(\cdot))$ is called a $\star$-isometry if, for each $y \in Y$, there exists a constant function e in $AC(X,E)$, such that $(Te)(y) \neq 0$ (see [1] and [15] for more details concerning this property).

A linear map $T : AC(X,E) \longrightarrow AC(Y,F)$ is called a $\star$-local isometry if, for every $f \in AC(X,E)$, there exists a $\star$-isometry $T_f : AC(X,E) \longrightarrow AC(Y,F)$, such that $Tf = T_f f$.

The following result shows that each $\star$-local isometry $T : AC(X,E) \longrightarrow AC(Y,F)$ is a linear surjective isometry, which yields [14, Theorem 2.1].

Corollary 4.8. *Let X and Y be compact subsets of $\mathbb{R}$ with at least two points, and let $T : AC(X,E) \longrightarrow AC(Y,F)$ be a $\star$-local isometry. Then, T is a linear surjective isometry and there exist a monotonic absolutely continuous* *homeomorphism* $\psi : Y \longrightarrow X$*, and a surjective linear isometry* $J : E \longrightarrow F$*, such that* $(Tf)(y) = J(f(\psi(y)))$ *for all* $f \in AC(X, E)$ *and* $y \in Y$*.*

Proof. From [15, Theorem 4.1], for each $\star$-isometry $\mathcal{T} : AC(X, E) \longrightarrow AC(Y, F)$, there exist a monotonic absolutely continuous homeomorphism $\varphi : Y \longrightarrow X$, and a surjective linear isometry $\mathtt{J} : E \longrightarrow F$, such that $(\mathcal{T}f)(y) = \mathtt{J}(f(\varphi(y)))$ for all $f \in AC(X, E)$ and $y \in Y$, which especially shows that T is an isometry with respect to $\|.\|_\infty$. Then, T is an isometry with respect to $\|.\|_\infty$, whence similarly to Corollary 3.5, it can be extended to an isometry $\widetilde{T} : C(X, E) \longrightarrow C(Y, F)$. Now, taking into account that $Pk(AC(X)) = X$, from Theorem 4.4(i), it follows that there exist a continuous mapping ψ from $Y_0 := \bigcup_{x \in X} I(x)$ onto X and a bounded linear map $\omega(y)$ from E into F for each $y \in Y_0$, such that $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in Y_0$ and all $f \in AC(X, E)$.

Since T is a $\star$-local isometry, Te is a constant function for each $e \in E$ [15, Lemma 3.14], whence $(Te)(y) = (Te)(y')$ for all $y, y' \in Y_0$. Then, we infer that $\omega(y) = \omega(y')$ for all $y, y' \in Y_0$. Put $J = \omega(y)$ for some $y \in Y_0$. Thus, $(Tf)(y) = J(f(\psi(y)))$ for all $f \in AC(X, E)$ and $y \in Y_0$. Let $\mathtt{e} \in S_E$ and define $f_0(x) = x - a + 1$, where $a = \min X$. Now, by an argument similar to Theorem 4.6, one can conclude that $Y_0 = Y$ and $T = T_{f_0 \cdot \mathtt{e}}$. Therefore, T is surjective, ψ is a monotonic absolutely continuous homeomorphism, and J is a surjective linear isometry. □

Author Contributions Juan J. Font and Maliheh Hosseini have contributed equally to this work. Both authors participated in the conception and design of the work equally. Both authors reviewed the manuscript.

Funding The work of Juan J. Font was supported by Spanish Government under Grant No. PID2019-106529GB-I00.

Data Availability There are no data associated with this paper.

Declarations

Conflict of Interest The authors have no competing interests to declare that are relevant to the content of this article.

Ethical Approval Not applicable.

References

[1] Araujo, J., Dubarbie, L.: Noncompactness and noncompleteness in isometries of Lipschitz spaces. J. Math. Anal. Appl. **377**, 15–29 (2011)

[2] Araujo, J., Font, J.J.: Linear isometries between subspaces of continuous functions. Trans. Am. Math. Soc. **349**, 413–428 (1997)

[3] Berens, H., Lorentz, G.G.: Geometric theory of Korovkin sets. J. Approx. Theory **15**(3), 161–189 (1975)

[4] Bishop, E.: A minimal boundary for function algebras. Pac. J. Math. **9**, 629–642 (1959)

[5] Botelho, F., Jamison, J.: Algebraic reflexivity of $C(X, E)$ and Cambern's theorem. Studia Math. **186**, 295–301 (2008)

[6] Browder, A.: Introduction to Function Algebras. W. A. Benjamin, New York (1969)

[7] Cabello-Sánchez, F.: Local isometries on spaces of continuous functions. Math. Z. **251**, 735–749 (2005)

[8] Cabello-Sánchez, F., Molnár, L.: Reflexivity of the isometry group of some classical spaces. Rev. Mat. Iberoam. **18**, 409–430 (2002)

[9] Conway, J.B.: A Course in Functional Analysis, 2nd edn. Springer, New York (1990)

[10] Dales, H.G.: Banach Algebras and Automatic Continuity. Clarendon Press, Oxford (2000)

[11] Engelking, R., Ryszard, K.: Topology: A Geometric Approach. Heldermann Verlag, Berlin (1992)

[12] Font, J.J.: Linear isometries between certain subspaces of continuous vector-valued functions. Ill. J. Math. **42**, 389–397 (1998)

[13] Ford, Jo., Heath, R.W.: Incompressible spaces. Houst. J. Math. **5**(2), 193–198 (1979)

[14] Hosseini, M.: Algebraic reflexivity of sets of bounded linear operators on absolutely continuous function spaces. Oper. Matrices **13**(3), 887–905 (2019)

[15] Hosseini, M.: Isometries on spaces of absolutely continuous vector-valued functions. J. Math. Anal. Appl. **463**(1), 386–397 (2018)

[16] Hosseini, M., Jiménez-Vargas, A.: Approximate local isometries of Banach algebras of differentiable functions. J. Math. Anal. Appl. **500**(1), 125092 (2021)

[17] Jarosz, K., Rao, T.S.S.R.K.: Local isometries of function spaces. Math. Z. **243**, 449–469 (2003)

[18] Molnár, L.: Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces, Lecture Notes in Mathematics. Springer, Berlin (2003)

[19] Molnár, L., Zalar, B.: Reflexivity of the group of surjective isometries. Proc. Edinb. Math. Soc. **42**, 17–36 (1999)

[20] Oi, S.: A spherical version of the Kowalski–Slodkowski theorem and its applications. J. Aust. Math. Soc. **111**, 386–411 (2021)

[21] Pathak, V.D.: Linear isometries of spaces of absolutely continuous functions. Can. J. Math. **XXXIV**, 298–306 (1982)

[22] Phelps, R.R.: Lectures on Choquet's theorem. Lecture Notes in Mathematics, vol. 1757, 2nd edn. Springer-Verlag, Berlin (2001)

[23] Rao, N.V., Roy, A.K.: On Šilov boundary for function spaces. Top. Appl. **193**, 175–181 (2015)

[24] Wang, R.: Isometries of $C_0^{(n)}(X)$. Hokkaido Math. J. **25**(3), 465–519 (1996)

Juan J. Font
Departamento de Matemáticas
Universitat Jaume I
Campus Riu Sec
12071 Castellón
Spain
e-mail: `font@mat.uji.es`

Maliheh Hosseini
Faculty of Mathematics
K. N. Toosi University of Technology
Tehran 16315-1618
Iran
e-mail: `m.hosseini@kntu.ac.ir`

Received: January 1, 2023.
Revised: January 3, 2023.
Accepted: March 29, 2023.