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Multiple Solutions for Generalized Biharmonic Equations with Two Singular Terms

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Abstract. In this article, we investigate more general nonlinear biharmonic equation

$$\Delta^2 u + V_\lambda(x)u = \mu f(x)u^{-\gamma} + g(x)u^{p-1} \text{ in } \mathbb{R}^N,$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $N \geq 1$, $\lambda > 0$ is a parameter, $0 < \gamma < 1$. Different from previous works on biharmonic problems, we suppose that $V(x) = \lambda a(x) - b(x)$ with $\lambda > 0$ and $b(x)$ could be singular at the origin. Under suitable conditions on $V_\lambda(x)$, $f(x)$ and $g(x)$, the multiplicity of solutions is obtained for $\lambda > 0$ sufficiently large and some new estimates will be established. Our analysis is based on the Nehari manifold as well as the fibering map.

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1. Introduction

The purpose of this paper is to consider the following biharmonic equation:

$$\begin{cases} \Delta^2 u + V_\lambda(x)u = \mu f(x)u^{-\gamma} + g(x)u^{p-1}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator with $N \geq 1$, and $0 < \gamma < 1$, $2 < p < 2^{**}(2^{**} = \frac{2N}{N-4})$. $\lambda,\ \mu > 0$ are parameters and the potential $V_\lambda(x) = \lambda a(x) - b(x)$. We assume that $a(x)$ and $b(x)$ satisfy the following conditions:

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(V_1) $a \in C(\mathbb{R}^N)$ and $a(x) \geq 0$ for all $x \in \mathbb{R}^N$ and there exists $a_0 > 0$ such that the set

$$\{a < a_0\} := \{x \in \mathbb{R}^N | a(x) < a_0\}$$

has finite positive Lebesgue measure for $N \geq 4$ and

$$|\{a < a_0\}| < S_\infty^{-2} \left(1 + \frac{A_0^2}{2}\right)^{-1} \text{ for } N \leq 3,$$

where $|\cdot|$ is the Lebesgue measure, S_∞ is the best Sobolev constant for the embedding of $H^2(\mathbb{R}^N)$ in $L^\infty(\mathbb{R}^N)$ with $N \leq 3$, and A_0 is defined in Lemma 2.1;
(V_2) $\Omega = \text{int}\{x \in \mathbb{R}^N : a(x) = 0\}$ is nonempty and has a smooth boundary with $\bar{\Omega} = \{x \in \mathbb{R}^N : a(x) = 0\}$;
(V_3) $b(x)$ is a measurable function on $\mathbb{R}^N$ and there exists $0 < b_0 < \bar{\gamma}$ such that $0 \leq b(x) \leq \frac{b_0}{|x|^4}$ for all $x \in \mathbb{R}^N$, where $\bar{\gamma} := \frac{N^2(N-4)^2}{16}$ is a critical Hardy-Sobolev constant.

The potential V_λ satisfies (V_1), (V_2) is called the steep well potential, which was first introduced by Bartsch and Wang [4] in the study of the nonlinear Schrödinger equations.

When Ω is a bounded domain of $\mathbb{R}^N$, the researchers mainly focused on the following Navier boundary value problem:

$$\begin{cases} \Delta^2 u + c\Delta u = f(x, u), \ x \in \Omega, \\ u = \Delta u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.2}$$

which arises in the study of traveling waves in suspension bridges, see [5,9,14] and the study of the static deflection of an elastic plate in a fluid. In the last decades, many authors have attached their attention to the existence and multiplicity of nontrivial solutions for biharmonic equations, we refer the readers to [2,6,10,12].

Recently, biharmonic equations on unbounded domain $\mathbb{R}^N$ have attracted a lot of attention. Especially, the researchers mainly investigated the following problems with the steep potential:

$$\begin{cases} \Delta^2 u - \Delta u + \lambda V(x) u = f(x, u) \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{cases} \tag{1.3}$$

With the aid of λ, they proved that the energy functional possesses the property of being locally compact, see [8,11,16,18] and their references therein. Especially, Ye and Tang [18] assumed that $f(x, u)$ was superlinear and subcritical at infinity, when λ was large enough, they obtained the existence and multiplicity of nontrivial solutions. Later, Zhang, Tang, Zhang and Luo [19] improved their results and obtained the existence of infinite nontrivial solutions when $\lambda > 0$ was large enough. Badiale, Greco and Rolando [3] obtained two nontrivial solutions for the case $f(x, u) = g(x, u) + \mu\xi(x)|u|^{p-2}u$ when $g(x, u)$, $\xi(x)$ satisfied some assumptions, λ was large enough and μ was small enough. Mao and Zhao [13] considered (1.3) with Kirchhoff terms and concave-convex nonlinearities, existence and multiplicity of solutions were proved using the variational method.

Very recently, replacing Laplacian with p-Laplacian in (1.3), Sun, Chu and Wu [15] studied the following biharmonic equation

$$\begin{cases} \Delta^2 u - \beta\Delta_p u + \lambda V(x)u = f(x,u) \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $N \geq 1$, $p \geq 2$ and $\beta > 0$ small enough or $\beta < 0$. Using the mountain pass theorem, and under some suitable assumptions on $V(x)$ and $f(x,u)$, they obtained the existence and multiplicity of nontrivial solutions for λ large enough. Later, Jiang and Zhai [7] supplemented their results, when $\beta \in \mathbb{R}$ and $\lambda V(x)$ was replaced by $V_\lambda(x)$, which was singular, the multiplicity of nontrivial solutions was obtained.

Motivated by the above papers, in the present paper, we consider a biharmonic problem with steep well potential and singular nonlinearity. To the best of knowledge, few works concerning this case up to now. To this end, we need some assumptions on $f(x)$ and $g(x)$ and make the following hypotheses:
(F) $f \in L^{\frac{p}{p+\gamma-1}}(\mathbb{R}^N)$ is a positive continuous function.
(G) $g \in L^\infty(\mathbb{R}^N)$ is a sign-changing function such that $|g^+|_\infty > 0$, where $g^+ = \max\{g(x), 0\}$.

Now, we state our main result.

Theorem 1.1. *Let $0 < \gamma < 1$ and $2 < p < 2^{**}$. Suppose that f, g and V_λ satisfy (F), (G) and $(V_1) - (V_3)$, then there exist $\lambda^* > 0$ and $\mu^* > 0$ such that problem (1.1) has at least two solutions for all $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$.*

Remark 1.2. From the condition (V_3), it is easy to obtain that the function $b(x)$ could be singular at the origin. Moreover, the improved Hardy–Sobolev inequality (see Lemma 1.1 in [17]) gives

$$\int_{\mathbb{R}^N} b(x)u^2 dx \leq b_0 \int_{\mathbb{R}^N} \frac{u^2}{|x|^4} dx \leq \frac{b_0}{\bar{\gamma}} \int_{\mathbb{R}^N} |\Delta u|^2 dx.$$

2. Preliminaries

Let

$$X = \left\{ u \in H^2(\mathbb{R}^N) | \int_{\mathbb{R}^N} (|\Delta u|^2 + a(x)u^2) dx < +\infty \right\}$$

be equipped with the inner product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a(x)uv) dx, \ \|u\| = \langle u, u \rangle^{(1/2)}.$$

For $\lambda > 0$, we also need the inner product and norm

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} (\Delta u \Delta v + \lambda a(x)uv) dx, \ \|u\|_\lambda = \langle u, u \rangle_\lambda^{(1/2)}.$$

It is clear that $\|u\| \leq \|u\|_\lambda$ for $\lambda \geq 1$. For simplicity, we let

$$\|u\|^2_{\lambda,V} := \int_{\mathbb{R}^N} \left(|\Delta u|^2 dx + V_\lambda u^2 \right) dx,$$

then by Remark 1.2, one has

$$\|u\|_\lambda^2 \geq \|u\|_{\lambda,V}^2 \geq \frac{\mu_0-1}{\mu_0}\|u\|_\lambda^2,\ \lambda>0, \tag{2.1}$$

where $\mu_0=\frac{\bar{\gamma}}{b_0}>1$. Hence, $\|u\|_{\lambda,V}$ and $\|u\|_\lambda$ are equivalent in X_λ, where

$$X_\lambda=\left\{u\in H^2(\mathbb{R}^N)\Big|\int_{\mathbb{R}^N}(|\Delta u|^2+\lambda a(x)u^2)dx<+\infty\right\}.$$

Lemma 2.1 ([15]). *Under assumptions $(V_1),(V_2)$, the continuous embedding $X_\lambda\hookrightarrow L^r(\mathbb{R}^N)$ is compact for $2\leq r<2^{**}$, and there holds $\int_{\mathbb{R}^N}|u|^r dx\leq \Theta_r\|u\|_\lambda^r$ for $\lambda\geq\lambda_*$, where*

$$\Theta_r:=\begin{cases} S_\infty^{-(r-2)}\left[(1+\frac{A_0^2}{2})^{-1}-S_\infty^2|\{a<a_0\}|\right]^{-r/2} & \text{if } N\leq 3,\\ S_r^{-r}\left(1+\frac{A_0^2}{2}\right)^{r/2} & \text{if } N=4,\\ C_0^{N(r-2)/4}\left(1+\frac{A_0^2}{2}\right)^{r/2} & \text{if } N>4,\end{cases}$$

and

$$\lambda_*:=\begin{cases}\frac{1}{a_0} & \text{if } N\leq 3,\\ \frac{2(1+B_0^4|\{a<a_0\}|)}{a_0} & \text{if } N=4,\\ \frac{1+C_0^2|\{a<a_0\}|^{N/4}}{a_0} & \text{if } N>4,\end{cases}$$

where A_0, B_0, C_0 are positive constants, and S_r is the best Sobolev constant for the embedding of $H^2(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ for $2\leq r<2^{**}$.

In this paper, we make use of the following notations: the L^r-norm $(1\leq r\leq+\infty)$ by $|\cdot|_r$. C denotes various positive constants, which may vary from line to line. By (V_1), (V_2), the Hölder inequality and the Sobolev inequality, we have

$$\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx\leq |f|_{\frac{p}{p+\gamma-1}}\Theta_p^{\frac{1-\gamma}{p}}\|u\|_\lambda^{1-\gamma}. \tag{2.2}$$

The energy functional corresponding to (1.1) given by

$$I_{\lambda,\mu}(u)=\frac{1}{2}\|u\|_\lambda^2-\frac{1}{2}\int_{\mathbb{R}^N}b(x)u^2dx-\frac{\mu}{1-\gamma}\int_{\mathbb{R}^N}f|u|^{1-\gamma}dx \\ -\frac{1}{p}\int_{\mathbb{R}^N}g|u|^p dx,\ \text{for } u\in X_\lambda. \tag{2.3}$$

It is clear that $I_{\lambda,\mu}$ is a C^1 functional. Since $I_{\lambda,\mu}$ is not bounded below on X_λ, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu}=\{u\in X_\lambda\backslash\{0\}:\langle I'_{\lambda,\mu}(u),u\rangle=0\}.$$

We analyze $\mathcal{N}_{\lambda,\mu}$ in terms of the stationary points of fibering maps $N_u:(0,+\infty)\to\mathbb{R}$ given by

$$N_u(t)=I_{\lambda,\mu}(tu),\ t>0.$$

Then for each $u \in \mathcal{N}_{\lambda,\mu}$, we have

$$N_u'(t) = t\|u\|_{\lambda,V}^2 - \mu t^{-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - t^{p-1}\int_{\mathbb{R}^N} g|u|^p dx,$$
$$N_u''(t) = \|u\|_{\lambda,V}^2 + \mu\gamma t^{-\gamma-1}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - (p-1)t^{p-2}\int_{\mathbb{R}^N} g|u|^p dx.$$

It is easy to see that

$$tN_u'(t) = t^2\|u\|_{\lambda,V}^2 - \mu t^{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - t^p\int_{\mathbb{R}^N} g|u|^p dx,$$

and for $u \in X_\lambda\backslash\{0\}$ and $t > 0$, then $tu \in \mathcal{N}_{\lambda,\mu}$ if and only if $N_u'(t) = 0$, that is, the critical points of $N_u(t)$ correspond to the points on the Nehari manifold. In particular, $u \in \mathcal{N}_{\lambda,\mu}$ if and only if $N_u'(1) = 0$. Then we define

$$\mathcal{N}_{\lambda,\mu}^+ = \{u \in \mathcal{N}_{\lambda,\mu} : N_u''(1) > 0\},$$
$$\mathcal{N}_{\lambda,\mu}^0 = \{u \in \mathcal{N}_{\lambda,\mu} : N_u''(1) = 0\},$$
$$\mathcal{N}_{\lambda,\mu}^- = \{u \in \mathcal{N}_{\lambda,\mu} : N_u''(1) < 0\}.$$

The existence of solutions to (1.1) can be studied by considering the existence of minimizers to $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$. Furthermore, for each $u \in \mathcal{N}_{\lambda,\mu}$, we know that

$$\begin{aligned} N_u''(1) &= \|u\|_{\lambda,V}^2 + \mu\gamma\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - (p-1)\int_{\mathbb{R}^N} g|u|^p dx \\ &= (1+\gamma)\|u\|_{\lambda,V}^2 - (p+\gamma-1)\int_{\mathbb{R}^N} g|u|^p dx \\ &= (2-p)\|u\|_{\lambda,V}^2 + \mu(p+\gamma-1)\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx. \end{aligned} \tag{2.4}$$

Lemma 2.2. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. For $u \in \mathcal{N}_{\lambda,\mu}$, we have

$$\|u\|_{\lambda,V}^2 - \mu\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - \int_{\mathbb{R}^N} g|u|^p dx = 0.$$

Therefore, by (2.1), (2.2), (2.3) and Lemma 2.1,

$$\begin{aligned} I_{\lambda,\mu}(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{\lambda,V}^2 - \frac{\mu(p+\gamma-1)}{p(1-\gamma)}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx \\ &\geq \frac{(p-2)(\mu_0-1)}{2p\mu_0}\|u\|_\lambda^2 - \frac{\mu(p+\gamma-1)}{p(1-\gamma)}|f|_{\frac{p}{p+\gamma-1}}\Theta_p^{\frac{1-\gamma}{p}}\|u\|_\lambda^{1-\gamma}. \end{aligned}$$

For $0 < \gamma < 1$, thus we get the conclusion. □

Before the following lemma, we define

$$\mu^* = \frac{(\mu_0-1)(p-2)}{\mu_0(p+\gamma-1)|f|_{\frac{p}{p+\gamma-1}}\Theta_p^{\frac{1-\gamma}{p}}} \times \left(\frac{(\mu_0-1)(1+\gamma)}{\mu_0(p+\gamma-1)|g^+|_\infty\Theta_p}\right)^{\frac{1+\gamma}{p-2}}.$$

Lemma 2.3. *Suppose that (F), (G), $(V_1)-(V_3)$ are satisfied. Then the set $\mathcal{N}_{\lambda,\mu}^0$ is empty for $(\lambda,\mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$.*

Proof. If $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$, by (2.4), we have

$$(1+\gamma)\|u\|^2_{\lambda,V} - (p+\gamma-1)\int_{\mathbb{R}^N} g|u|^p dx = 0$$

and

$$(2-p)\|u\|^2_{\lambda,V} + \mu(p+\gamma-1)\int_{\mathbb{R}^N} f|u|^{1-\gamma} dx = 0.$$

By (2.1), (2.2) and Lemma 2.1, we get that

$$\frac{\mu_0-1}{\mu_0}\|u\|^2_\lambda \leq \frac{p+\gamma-1}{1+\gamma}\int_{\mathbb{R}^N} g|u|^p dx \leq \frac{p+\gamma-1}{1+\gamma}|g^+|_\infty \Theta_p \|u\|^p_\lambda$$

and

$$\frac{\mu_0-1}{\mu_0}\|u\|^2_\lambda \leq \frac{\mu(p+\gamma-1)}{p-2}\int_{\mathbb{R}^N} f|u|^{1-\gamma} dx \leq \frac{\mu(p+\gamma-1)}{p-2}|f|_{\frac{p}{p+\gamma-1}} \Theta_p^{\frac{1-\gamma}{p}} \|u\|^{1-\gamma}_\lambda.$$

Then we get

$$\|u\|_\lambda \geq \left(\frac{(\mu_0-1)(1+\gamma)}{\mu_0(p+\gamma-1)|g^+|_\infty \Theta_p}\right)^{\frac{1}{p-2}}$$

and

$$\|u\|_\lambda \leq \left(\frac{\mu_0\mu(p+\gamma-1)}{(\mu_0-1)(p-2)}|f|_{\frac{p}{p+\gamma-1}} \Theta_p^{\frac{1-\gamma}{p}}\right)^{\frac{1}{1+\gamma}}.$$

Hence, we obtain $\mu \geq \mu^*$, which is impossible. Thus we get the conclusion. □

Lemma 2.4. *Suppose that* (F), (G), $(V_1)-(V_3)$ *are satisfied. Then*
(i) if $\int_{\mathbb{R}^N} g|u|^p dx \leq 0$, *then there is a unique* $0 < t^+ < t_{\max}$, *such that* $t^+u \in \mathcal{N}^+_{\lambda,\mu}$ *and*

$$I_{\lambda,\mu}(t^+u) = \inf_{t>0} I_{\lambda,\mu}(tu);$$

(ii) if $\int_{\mathbb{R}^N} g|u|^p dx > 0$, *then there are unique* t^+ *and* t^- *with* $t^- > t_{\max} > t^+ > 0$, *such that* $t^-u \in \mathcal{N}^-_{\lambda,\mu}$, $t^+u \in \mathcal{N}^+_{\lambda,\mu}$ *and*

$$I_{\lambda,\mu}(t^+u) = \inf_{0\leq 0\leq t_{\max}} I_{\lambda,\mu}(tu),\ I_{\lambda,\mu}(t^-u) = \sup_{t\geq t_{\max}} I_{\lambda,\mu}(tu).$$

Proof. Fix $u \in X_\lambda\backslash\{0\}$ with $\int_{\mathbb{R}^N} f|u|^{1-\gamma} dx > 0$. Note that

$$N'_u(t) = t\|u\|^2_{\lambda,V} - \mu t^{-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma} dx - t^{p-1}\int_{\mathbb{R}^N} g|u|^p dx.$$

For $t > 0$, we define

$$H(t) := t^{2-p}\|u\|^2_{\lambda,V} - \mu t^{1-\gamma-p}\int_{\mathbb{R}^N} f|u|^{1-\gamma} dx.$$

Then for $t > 0$ and $tu \in \mathcal{N}_{\lambda,\mu}$ if and only if t is a solution for $H(t) = \int_{\mathbb{R}^N} g|u|^p dx$, and $H(t) \to -\infty$ as $t \to 0^+$, $H(t) \to 0$ as $t \to \infty$. Since

$$H'(t) = (2-p)t^{1-p}\|u\|^2_{\lambda,V} - \mu(1-\gamma-p)t^{-\gamma-p}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx,$$

then $H(t)$ possesses a unique maximum point

$$t_{\max} = \left(\frac{\mu(1-\gamma-p)\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx}{(2-p)\|u\|^2_{\lambda,V}}\right)^{\frac{1}{\gamma+1}},$$

and

$$\begin{aligned} H(t_{\max}) &= \left[\left(\frac{\mu(1-\gamma-p)}{2-p}\right)^{\frac{2-p}{\gamma+1}} - \mu\left(\frac{\mu(1-\gamma-p)}{(2-p)}\right)^{\frac{1-\gamma-p}{\gamma+1}}\right]\frac{(\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx)^{\frac{2-p}{\gamma+1}}}{\|u\|_{\lambda,V}^{\frac{2(1-\gamma-p)}{\gamma+1}}} \\ &\geq \mu^{\frac{2-p}{\gamma+1}}\|u\|^p_{\lambda,V}\frac{\gamma+1}{p-2}\left(\frac{1-\gamma-p}{2-p}\right)^{\frac{1-\gamma-p}{\gamma+1}}\left((\frac{\mu_0}{\mu_0-1})^{\frac{1-\gamma}{2}}|f|_{\frac{p}{p+\gamma-1}}\Theta_p{}^{\frac{1-\gamma}{p}}\right)^{\frac{2-p}{\gamma+1}}. \end{aligned} \tag{2.5}$$

Moreover, $H(t)$ is increasing on $(0, t_{\max})$ and decreasing on $(t_{\max}, \infty)$.

(i) if $\int_{\mathbb{R}^N} g|u|^p dx \leq 0$, then there is a unique $0 < t^+ < t_{\max}$, such that

$$H(t^+) = \int_{\mathbb{R}^N} g|u|^p dx, \;\; H'(t^+) > 0.$$

Thus, $t^+u \in \mathcal{N}_{\lambda,\mu}$ and one has

$$\begin{aligned} N''_{t^+u}(1) &= (2-p)(t^+)^2\|u\|^2_{\lambda,V} + \mu(p+\gamma-1)(t^+)^{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx \\ &= t^{1+p}H'(t^+) > 0. \end{aligned}$$

Then $t^+u \in \mathcal{N}^+_{\lambda,\mu}$. Since for $0 < t < t_{\max}$, one has

$$\frac{d}{dt}I_{\lambda,\mu}(tu) = t\|u\|^2_{\lambda,V} - \mu t^{-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - t^{p-1}\int_{\mathbb{R}^N} g|u|^p dx = 0$$

and

$$\frac{d^2}{dt^2}I_{\lambda,\mu}(tu) = (2-p)t^2\|u\|^2_{\lambda,V} + \mu(p+\gamma-1)t^{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx > 0$$

for $t = t^+$. Therefore, $I_{\lambda,\mu}(t^+u) = \inf_{t>0} I_{\lambda,\mu}(tu)$ holds.

(ii) if $\int_{\mathbb{R}^N} g|u|^p dx > 0$, by (2.2),(2.5) and $\mu \in (0, \mu^*)$, we have

$$\begin{aligned} 0 < \int_{\mathbb{R}^N} g|u|^p dx &\leq (\frac{\mu_0}{\mu_0-1})^{p/2}|g^+|_\infty\Theta^p_p\|u\|^p_{\lambda,V} \\ &= (\mu^*)^{\frac{2-p}{\gamma+1}}\|u\|^p_{\lambda,V}\frac{1+\gamma}{p+\gamma-1}\left(\frac{p-2}{p+\gamma-1}\right)^{\frac{p-2}{1+\gamma}}\left((\frac{\mu_0}{\mu_0-1})^{\frac{1-\gamma}{2}}|f|_{\frac{p}{p+\gamma-1}}\Theta_p{}^{\frac{1-\gamma}{p}}\right)^{\frac{2-p}{\gamma+1}} \\ &< H(t_{\max}). \end{aligned}$$

There are t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$H(t^+) = \int_{\mathbb{R}^N} g|u|^p dx = H(t^-)$$

and

$$H'(t^+) > 0 > H'(t^-).$$

As in (i), we have $t^+u \in \mathcal{N}^+_{\lambda,\mu}$, $t^-u \in \mathcal{N}^-_{\lambda,\mu}$, and $I_{\lambda,\mu}(t^-u) \geq I_{\lambda,\mu}(tu) \geq I_{\lambda,\mu}(t^+u)$ for each $t \in [t^+, t^-]$ and $I_{\lambda,\mu}(t^+u) = \inf_{0\leq 0\leq t_{\max}} I_{\lambda,\mu}(tu)$, $I_{\lambda,\mu}(t^-u) = \sup_{t\geq t_{\max}} I_{\lambda,\mu}(tu)$. Thus we get the conclusion. □

We remark that from Lemmas 2.3 and 2.4, one has $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$ for all $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$. Since $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$ are non-empty, thus, by Lemma 2.4, we may define

$$c^+_{\lambda,\mu} = \inf_{u\in\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(u),\ c^-_{\lambda,\mu} = \inf_{u\in\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(u)$$

Then we have the following results.

Lemma 2.5. *Suppose that the functions f, g and V satisfy the conditions (F), (G) and $(V_1) - (V_3)$. Then for $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$, there exists a positive constant C_0 such that $c^+_{\lambda,\mu} < 0 < C_0 < c^-_{\lambda,\mu}$.*

Proof. (i) Let $u \in \mathcal{N}^+_{\lambda,\mu} \subset \mathcal{N}_{\lambda,\mu}$, then we have

$$(1 + \gamma)\|u\|^2_{\lambda,V} - (p + \gamma - 1)\int_{\mathbb{R}^N} g|u|^p dx > 0.$$

It follows that

$$\begin{aligned} I_{\lambda,\mu}(u) &= \frac{1}{2}\|u\|^2_\lambda - \frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2 dx - \frac{\mu}{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - \frac{1}{p}\int_{\mathbb{R}^N} g|u|^p dx \\ &= -\frac{1+\gamma}{2(1-\gamma)}\|u\|^2_{\lambda,V} + \frac{p+\gamma-1}{p(1-\gamma)}\int_{\mathbb{R}^N} g|u|^p dx \\ &< -\tfrac{(p-2)(1+\gamma)}{2p(1-\gamma)}\|u\|^2_{\lambda,V} < 0. \end{aligned}$$

Therefore, $c^+_{\lambda,\mu} < 0$.

(ii) Let $u \in \mathcal{N}^-_{\lambda,\mu}$, then we have

$$(1 + \gamma)\|u\|^2_{\lambda,V} - (p + \gamma - 1)\int_{\mathbb{R}^N} g|u|^p dx < 0.$$

According to (2.1), we get

$$\frac{\mu_0 - 1}{\mu_0}\|u\|^2_\lambda \leq \|u\|^2_{\lambda,V} < \frac{p+\gamma-1}{1+\gamma}\int_{\mathbb{R}^N} g|u|^p dx \leq \frac{p+\gamma-1}{1+\gamma}|g^+|_\infty \Theta_p \|u\|^p_\lambda.$$

Therefore, we can show that

$$\|u\|_\lambda > \left(\frac{(\mu_0 - 1)(1+\gamma)}{\mu_0(p+\gamma-1)|g^+|_\infty}\Theta_p\right)^{\frac{1}{p-2}} := C.$$

Then, we know

$$\begin{aligned} I_{\lambda,\mu}(u) &\geq \frac{(p-2)(\mu_0-1)}{2p\mu_0}\|u\|^2_\lambda - \frac{\mu(p-1+\gamma)}{p(1-\gamma)}|f|_{\frac{p}{p-1+\gamma}}\Theta_p^{1-\gamma}\|u\|^{1-\gamma}_\lambda \\ &> C^{1-\gamma}\left[\tfrac{(p-2)(\mu_0-1)}{2p\mu_0}C^{1+\gamma} - \tfrac{\mu(p-1+\gamma)}{p(1-\gamma)}|f|_{\frac{p}{p-1+\gamma}}\Theta_p^{1-\gamma}\right] := C_0. \end{aligned}$$

Since $(\lambda, \mu) \in [\lambda_*, +\infty) \times (0, \mu^*)$, we can verify that $C_0 > 0$. Hence $I_{\lambda,\mu}(u) > C_0 > 0$ for all $u \in \mathcal{N}^-_{\lambda,\mu}$ and the proof is completed. □

Lemma 2.6. *Suppose that the functions f, g and V satisfy the conditions (F), (G) and $(V_1) - (V_3)$. Then $\mathcal{N}^-_{\lambda,\mu}$ is a closed subset in X_λ for $(\lambda, \mu) \in [\lambda_*, +\infty) \times (0, \mu^*)$.*

Proof. In order to prove that $\mathcal{N}^-_{\lambda,\mu}$ is a closed subset in X_λ, let us consider a sequence $\{u_n\} \subset \mathcal{N}^-_{\lambda,\mu}$ such that $u_n \to u$ in X_λ. It is obvious that $\langle I'_{\lambda,\mu}(u), u\rangle = 0$. By the proof of Lemma 2.5, we have

$$\|u\|_\lambda = \lim_{n\to\infty} \|u_n\|_\lambda \geq C > 0.$$

Thus, $u \in \mathcal{N}_{\lambda,\mu}$. By the definition of $\mathcal{N}^-_{\lambda,\mu}$, it holds

$$(1+\gamma)\|u_n\|^2_{\lambda,V} - (p+\gamma-1)\int_{\mathbb{R}^N} g|u_n|^p dx < 0.$$

Combining with Lemma 2.1, one has

$$(1+\gamma)\|u\|^2_{\lambda,V} - (p+\gamma-1)\int_{\mathbb{R}^N} g|u|^p dx \leq 0,$$

which implies that $u \in \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$. By Lemma 2.3, we know $\mathcal{N}^0_{\lambda,\mu} = \emptyset$. Therefore, $u \in \mathcal{N}^-_{\lambda,\mu}$. Thus, $\mathcal{N}^-_{\lambda,\mu}$ is a closed subset in X_λ. □

Lemma 2.7. *Suppose $u \in \mathcal{N}^+_{\lambda,\mu}$ and $v \in \mathcal{N}^-_{\lambda,\mu}$ are minimizers of $I_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$. Then for every nonnegative $w \in X_\lambda$, we have*
(i) there exists $\varepsilon_0 > 0$ such that $I_{\lambda,\mu}(u + \varepsilon w) \geq I_{\lambda,\mu}(u)$ for all $0 \leq \varepsilon \leq \varepsilon_0$.
(ii) $t_\varepsilon \to 1$ as $\varepsilon \to 0^+$, for $\varepsilon \geq 0$, where t_ε is the unique positive real number satisfying $t_\varepsilon(v + \varepsilon w) \in \mathcal{N}^-_{\lambda,\mu}$.

Proof. (i) Let $w \geq 0$ and for each $\varepsilon \geq 0$, set

$$\sigma(\varepsilon) = \|u+\varepsilon w\|^2_{\lambda,V} + \mu\gamma\int_{\mathbb{R}^N} f|u+\varepsilon w|^{1-\gamma}dx - (p-1)\int_{\mathbb{R}^N} g|u+\varepsilon w|^p dx.$$

Then by using continuity of σ and $\sigma(0) = N''_u(1) > 0$, there exists $\varepsilon_0 > 0$ such that $\sigma(\varepsilon) > 0$ for all $0 \leq \varepsilon \leq \varepsilon_0$. Similar to the proof of Lemma 2.4, for each $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that $s_\varepsilon(u+\varepsilon w) \in \mathcal{N}^+_{\lambda,\mu}$, such that $I_{\lambda,\mu}(s_\varepsilon(u+\varepsilon w)) = \inf_{t>0} I_{\lambda,\mu}(t(u+\varepsilon w))$, then for each $\varepsilon \in [0, \varepsilon_0]$, we have

$$I_{\lambda,\mu}(u+\varepsilon w) \geq I_{\lambda,\mu}(s_\varepsilon(u+\varepsilon w)) \geq I_{\lambda,\mu}(u).$$

(ii) For each $v \in \mathcal{N}^-_{\lambda,\mu}$, we define $J : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ by

$$J(t, l_1, l_2, l_3) = l_1 t - \mu l_2 t^{-\gamma} - l_3 t^{p-1},$$

for $(t, l_1, l_2, l_3) \in (0,\infty) \times \mathbb{R}^3$. Since $v \in \mathcal{N}^-_{\lambda,\mu}$, one obtains

$$\frac{\partial J}{\partial t}(1, \|v\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v|^p dx) = N''_v(1) < 0.$$

Moreover, for each $\varepsilon > 0$,

$$J(t_\varepsilon, \|v+\varepsilon w\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v+\varepsilon w|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v+\varepsilon w|^p dx) = 0.$$

We also have

$$J(1, \|v\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v|^p dx) = N'_v(1) = 0.$$

Applying the implicit function theorem, there exists an open neighbourhood $A \subset (0,\infty)$ and $B \subset \mathbb{R}^3$ containing 1 and $(\|v\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v|^p dx)$ respectively such that for all $J(t,y) = 0$ has a unique solution $t = j(y)$ with $j : B \to A$ being a smooth function. Then one has

$$(\|v + \varepsilon w\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v + \varepsilon w|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v + \varepsilon w|^p dx) \in B,$$

and

$$j(\|v + \varepsilon w\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v + \varepsilon w|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v + \varepsilon w|^p dx) = t_\varepsilon.$$

Since

$$J(t_\varepsilon, \|v + \varepsilon w\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v + \varepsilon w|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v + \varepsilon w|^p dx) = 0.$$

Thus, by continuity of g, we get $t_\varepsilon \to 1$ as $\varepsilon \to 0^+$. □

Lemma 2.8. *Suppose $u \in \mathcal{N}^+_{\lambda,\mu}$ and $v \in \mathcal{N}^-_{\lambda,\mu}$ are minimizers of $I_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$. Then for every nonnegative $w \in X_\lambda$, we have*

$$\langle u, w\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} fu^{-\gamma}wdx - \int_{\mathbb{R}^N} gu^{p-1}wdx \geq 0,$$
$$\langle v, w\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} fv^{-\gamma}wdx - \int_{\mathbb{R}^N} gv^{p-1}wdx \geq 0.$$

Proof. Let $w \in X_\lambda$ be a nonnegative function, then by Lemma 2.7, for each $\varepsilon \in (0, \varepsilon_0)$, we have

$$\begin{aligned} 0 &\leq \frac{I_{\lambda,\mu}(u + \varepsilon w) - I_{\lambda,\mu}(u)}{\varepsilon} \\ &= \frac{1}{2\varepsilon}(\|u + \varepsilon w\|^2_{\lambda,V} - \|w\|^2_{\lambda,V}) - \frac{\mu}{(1-\gamma)} \int_{\mathbb{R}^N} f\frac{(u + \varepsilon w)^{1-\gamma} - u^{1-\gamma}}{\varepsilon}dx \\ &\quad -\frac{1}{p}\int_{\mathbb{R}^N} g\frac{(u + \varepsilon w)^p - u^p}{\varepsilon}dx. \end{aligned} \tag{2.6}$$

By (G) and the Lebesgue dominate convergence theorem, we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{p}\int_{\mathbb{R}^N} g\frac{(u + \varepsilon w)^p - u^p}{\varepsilon}dx = \int_{\mathbb{R}^N} gu^{p-1}wdx.$$

For $0 < \gamma < 1$ and f is a positive continuous function, we have

$$f((u + \varepsilon w)^{1-\gamma} - u^{1-\gamma}) \geq 0.$$

It follows from (2.6) that

$$\liminf_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} f\frac{(u + \varepsilon w)^{1-\gamma} - u^{1-\gamma}}{\varepsilon}dx < \infty.$$

Then, by (2.6) and Fatou's lemma, we get

$$\begin{aligned}\mu\int_{\mathbb{R}^N} fu^{-\gamma}wdx &\le \frac{\mu}{1-\gamma}\liminf_{\varepsilon\to 0^+}\int_{\mathbb{R}^N} f\frac{(u+\varepsilon w)^{1-\gamma}-u^{1-\gamma}}{\varepsilon}dx\\ &\le \langle u,w\rangle_{\lambda,V}-\int_{\mathbb{R}^N} gu^{p-1}wdx,\end{aligned}$$

consequently, for each nonnegative $w\in X_\lambda$, we have

$$\langle u,w\rangle_{\lambda,V}-\mu\int_{\mathbb{R}^N} fu^{-\gamma}wdx-\int_{\mathbb{R}^N} gu^{p-1}wdx\ge 0.$$

Next, we will show that these properties are also held for $v\in\mathcal{N}^-_{\lambda,\mu}$. For each $\varepsilon>0$, there exists $t_\varepsilon>0$ such that $t_\varepsilon(v+\varepsilon w)\in\mathcal{N}^-_{\lambda,\mu}$. By Lemma 2.7, for $\varepsilon>0$ small enough, we get

$$I_{\lambda,\mu}(t_\varepsilon(v+\varepsilon w))\ge I_{\lambda,\mu}(v),$$

which implies $I_{\lambda,\mu}(t_\varepsilon(v+\varepsilon w))-I_{\lambda,\mu}(v)\ge 0$. Thus, one obtains

$$\begin{aligned}\frac{\mu t_\varepsilon^{1-\gamma}}{(1-\gamma)}\int_{\mathbb{R}^N} f\frac{(v+\varepsilon w)^{1-\gamma}-v^{1-\gamma}}{\varepsilon}dx &\le \frac{t_\varepsilon^2}{2\varepsilon}(\|v+\varepsilon w\|^2_{\lambda,V}-\|v\|^2_{\lambda,V})\\ &\quad -\frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N} g\frac{(v+\varepsilon w)^p-v^p}{\varepsilon}dx.\end{aligned}$$

Using the similar argument as in the previous case, we have

$$\langle v,w\rangle_{\lambda,V}-\mu\int_{\mathbb{R}^N} fv^{-\gamma}wdx-\int_{\mathbb{R}^N} gv^{p-1}wdx\ge 0.$$

□

3. Proof of Theorem 1.1

Since $I_{\lambda,\mu}(u)=I_{\lambda,\mu}(|u|)$, we can assume that $u\ge 0$ for every $u\in X_\lambda$. To get the main result, it is necessary to prove the following lemmas.

Lemma 3.1. *Suppose that $0<\gamma<1$ and $2<p<2^{**}$, and the conditions (F), (G) and $(V_1)-(V_3)$ are satisfied. Then for $(\lambda,\mu)\in[\lambda^*,+\infty)\times(0,\mu^*)$, $I_{\lambda,\mu}$ has a minimizer u_0 in $\mathcal{N}^+_{\lambda,\mu}$ such that $I_{\lambda,\mu}(u_0)=c^+_{\lambda,\mu}$.*

Proof. By the Ekeland variational principle ([1]), there exists a minimizing sequence $\{u_n\}\subset\mathcal{N}^+_{\lambda,\mu}$ satisfying
(i) $c^+_{\lambda,\mu}<I_{\lambda,\mu}(u_n)<c^+_{\lambda,\mu}+\frac{1}{n}$,
(ii) $I_{\lambda,\mu}(u)\ge I_{\lambda,\mu}(u_n)-\frac{1}{n}\|u_n-u\|$.

Moreover, by Lemma 2.2, one has $\{u_n\}$ is bounded in X_λ. Then there exists a subsequence of $\{u_n\}$(still denotes$\{u_n\}$) such that

$$\begin{aligned}&u_n\rightharpoonup u_0,\ \text{in } X_\lambda,\\ &u_n\to u_0,\ \text{in } L^p(\mathbb{R}^N),\ p\in[2,2^{**}),\end{aligned}$$

with $u_0\ge 0$. For $0<\gamma<1$, $f\in L^{\frac{p}{p+\gamma-1}}(\mathbb{R}^N)$ is a positive continuous function, by the Vitali convergence theorem, one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f|u_n|^{1-\gamma}dx=\int_{\mathbb{R}^N} f|u_0|^{1-\gamma}dx.$$

Step1: We prove that $u_n \to u_0$ in X_λ and $u_0 \in \mathcal{N}^+_{\lambda,\mu}$.

First, we show that $u_0 \neq 0$. Using the weak lower semi-continuity norm, we have

$$I_{\lambda,\mu}(u_0) \leq \liminf_{n\to\infty} I_{\lambda,\mu}(u_n) = c^+_{\lambda,\mu} < 0.$$

If $u_0 = 0$, then $I_{\lambda,\mu}(u_0) = 0$, which is a contradiction.

Next, we prove that $u_n \to u_0$ in X_λ. Suppose the contrary, by (2.1), one has

$$\|u_0\|^2_{\lambda,V} < \liminf_{n\to\infty} \|u_n\|^2_{\lambda,V}.$$

For $u_n \in \mathcal{N}^+_{\lambda,\mu}$, one has

$$\|u_0\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f|u_0|^{1-\gamma}dx - \int_{\mathbb{R}^N} g|u_0|^p dx < 0. \tag{3.1}$$

Now, we prove that for u_0, there exists $0 < t^+ \neq 1$ such that $t^+u_0 \in \mathcal{N}^+_{\lambda,\mu}$.

If $\int_{\mathbb{R}^N} g|u|^p dx \leq 0$, then by Lemma 2.4($i$), there exists $t^+ > 0$ such that $t^+u_0 \in \mathcal{N}^+_{\lambda,\mu}$ and $I'_{\lambda,\mu}(t^+u_0) = 0$. By (3.1), we obtain that $I'_{\lambda,\mu}(u_0) \neq 0$. Hence, $t^+ \neq 1$.

If $\int_{\mathbb{R}^N} g|u|^p dx > 0$, then by Lemma 2.4($ii$), there exists $0 < t^+ \neq 1$ such that $t^+u_0 \in \mathcal{N}^+_{\lambda,\mu}$.

Since t^+u_0 is a minimizer of $I_{\lambda,\mu}$ in X_λ, then

$$I_{\lambda,\mu}(t^+u_0) < I_{\lambda,\mu}(u_0) \leq \lim_{n\to\infty} I_{\lambda,\mu}(u_n) = c^+_{\lambda,\mu},$$

which contradicts $c^+_{\lambda,\mu} = \inf_{u\in\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(u)$. Then, we obtain $u_n \to u_0$ in X_λ.

Finally, we claim that $u_0 \in \mathcal{N}^+_{\lambda,\mu}$. Suppose the contrary, assume that $u_0 \in \mathcal{N}^-_{\lambda,\mu}$. It follows from (2.4) and $u_0 \in \mathcal{N}^-_{\lambda,\mu}$ that

$$\int_{\mathbb{R}^N} g|u_0|^p dx > 0.$$

Then, by Lemma 2.4(ii), there exist unique $t^+ > 0,\ t^- > 0$ with $t^- > t^+ > 0$, such that $t^+u_0 \in \mathcal{N}^+_{\lambda,\mu}$, $t^-u_0 \in \mathcal{N}^-_{\lambda,\mu}$ and

$$I_{\lambda,\mu}(t^+u_0) = \inf_{0\leq 0\leq t_{\max}} I_{\lambda,\mu}(tu_0),\ I_{\lambda,\mu}(t^-u_0) = \sup_{t\geq t_{\max}} I_{\lambda,\mu}(tu_0).$$

For $u_0 \in \mathcal{N}^-_{\lambda,\mu}$, it suffices to prove that

$$\frac{d}{dt}I_{\lambda,\mu}(u_0) = 0,\ \frac{d^2}{dt^2}I_{\lambda,\mu}(u_0) < 0.$$

This indicates $t^- = 1$. Also, since

$$\frac{d}{dt}I_{\lambda,\mu}(t^+u_0) = 0,\ \frac{d^2}{dt^2}I_{\lambda,\mu}(t^+u_0) > 0,$$

then there exists $t \in (t^+, 1]$, such that

$$c^+_{\lambda,\mu} \leq I_{\lambda,\mu}(t^+u_0) < I_{\lambda,\mu}(tu_0) \leq I_{\lambda,\mu}(u_0) = c^+_{\lambda,\mu},$$

this is a contradiction. Therefore, $u_0 \in \mathcal{N}^+_{\lambda,\mu}$.
Step2: u_0 is a solution of (1.1).

In the following, we show the solution u_0 is a weak solution of (1.1). Let $v \in X_\lambda$ and $\varepsilon > 0$. Set $\Omega_+ = \{x \in \mathbb{R}^N : u_0 + \varepsilon v \geq 0\}$ and $\Omega_- = \{x \in \mathbb{R}^N : u_0 + \varepsilon v < 0\}$, then by Lemma 2.8, we obtain that

$$\begin{aligned} 0 \leq & \int_{\Omega_+} (\Delta u_0 \Delta(u_0+\varepsilon v) + V_\lambda(x) u_0(u_0+\varepsilon v))\, dx - \mu \int_{\Omega_+} f u_0^{-\gamma}(u_0+\varepsilon v) dx \\ & - \int_{\Omega_+} g u_0^{p-1}(u_0+\varepsilon v) dx \\ = & \|u_0\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f u_0^{1-\gamma} dx - \int_{\mathbb{R}^N} g u_0^p dx \\ & + \varepsilon \left(\langle u_0, v\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} f u_0^{-\gamma} v dx - \int_{\mathbb{R}^N} g u_0^{p-1} v dx \right) \\ & - \Bigg(\int_{\Omega_-} (\Delta u_0 \Delta(u_0+\varepsilon v) + V_\lambda(x) u_0(u_0+\varepsilon v)) dx - \mu \int_{\Omega_-} f u_0^{-\gamma}(u_0+\varepsilon v) dx \\ & - \int_{\Omega_-} g u_0^{p-1}(u_0+\varepsilon v) dx \Bigg). \end{aligned}$$

Then, for the fact $u_0 \in \mathcal{N}^+_{\lambda,\mu}$ and $f(x)$ is a positive continuous function, we have

$$\begin{aligned} 0 \leq & \varepsilon \left(\langle u_0, v\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} f u_0^{-\gamma} v dx - \int_{\mathbb{R}^N} g u_0^{p-1} v dx \right) \\ & - \varepsilon \int_{\Omega_-} (\Delta u_0 \Delta v + V_\lambda(x) u_0 v)\, dx + \int_{\Omega_-} g u_0^{p-1}(u_0+\varepsilon v) dx. \end{aligned} \tag{3.2}$$

Since the measure of the domain of integration $\Omega_- = \{x \in \mathbb{R}^N : u_0 + \varepsilon v < 0\}$ tends to 0 as $\varepsilon \to 0^+$, it follows that

$$\left| \int_{\Omega_-} (\Delta u_0 \Delta v + V_\lambda(x) u_0 v)\, dx \right| \to 0.$$

Moreover, by (G) and Lemma 2.1, when $\varepsilon \to 0^+$, one has

$$\left| \int_{\Omega_-} g u_0^{p-1}(u_0+\varepsilon v) dx \right| \leq |g|_\infty \int_{\Omega_-} g|u_0|^p dx + \varepsilon |g|_\infty \left| \int_{\Omega_-} g|u_0|^{p-1} v dx \right| \to 0.$$

Dividing by ε and letting $\varepsilon \to 0$ in (3.2), one obtains

$$\langle u_0, v\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} f u_0^{-\gamma} v dx - \int_{\mathbb{R}^N} g u_0^{p-1} v dx \geq 0.$$

Since v is arbitrary, the inequality above holds for $-v$. Hence, for all $v \in X_\lambda$, one has

$$\langle u_0, v\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} f u_0^{-\gamma} v dx - \int_{\mathbb{R}^N} g u_0^{p-1} v dx = 0.$$

Then u_0 is a positive solution for (1.1). □

Lemma 3.2. *Suppose that* $0 < \gamma < 1$ *and* $2 < p < 2^{**}$, *and the conditions* (F), (G) *and* $(V_1) - (V_3)$ *are satisfied. Then for* $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$, $I_{\lambda,\mu}$ *has a minimizer* v_0 *in* $\mathcal{N}^-_{\lambda,\mu}$ *such that* $I_{\lambda,\mu}(v_0) = c^-_{\lambda,\mu}$.

Proof. On account of $I_{\lambda,\mu}$ is also coercive on $\mathcal{N}^-_{\lambda,\mu}$, we apply the Ekeland's variational principle to the minimization problem $c^-_{\lambda,\mu} = \inf_{u\in\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(u)$, there exists a minimizing sequence $\{v_n\} \subset \mathcal{N}^-_{\lambda,\mu}$ of $I_{\lambda,\mu}$ with the following properties

(*i*) $c^-_{\lambda,\mu} < I_{\lambda,\mu}(v_n) < c^-_{\lambda,\mu} + \frac{1}{n}$,

(*ii*) $I_{\lambda,\mu}(v) \geq I_{\lambda,\mu}(v_n) - \frac{1}{n}\|v_n - v\|$.

Moreover, $\{v_n\}$ is bounded in X_λ, then there exists a subsequence of $\{v_n\}$(still denotes$\{v_n\}$) such that

$$\begin{aligned} &v_n \rightharpoonup v_0, \text{ in } X_\lambda, \\ &v_n \to v_0, \text{ in } L^p(\mathbb{R}^N), \ p \in [2, 2^{**}), \end{aligned}$$

with $v_0 \geq 0$. Then we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f|v_n|^{1-\gamma}dx = \int_{\mathbb{R}^N} f|v_0|^{1-\gamma}dx$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g|v_n|^p dx = \int_{\mathbb{R}^N} g|v_0|^p dx.$$

We will show that $v_0 \neq 0$. If $v_0 = 0$, then v_n converges to 0 strongly in X_λ, which contradicts Lemma 2.5. Next, we prove that $v_n \to v_0$ in X_λ. If $v_n \not\to v_0$ in X_λ then

$$\begin{aligned} &\|v_0\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f|v_0|^{1-\gamma}dx - \int_{\mathbb{R}^N} g|v_0|^p dx \\ &\quad < \liminf_{n\to\infty} \left[\|v_n\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f|v_n|^{1-\gamma}dx - \int_{\mathbb{R}^N} g|v_n|^p dx\right] = 0. \end{aligned} \tag{3.3}$$

Since $\{v_n\} \subset \mathcal{N}^-_{\lambda,\mu}$, we deduce from (2.4) that

$$\mu(1+\gamma) \int_{\mathbb{R}^N} f|v_0|^{1-\gamma}dx + (2-p) \int_{\mathbb{R}^N} g|v_0|^p dx \leq 0.$$

Consequently, one has $\int_{\mathbb{R}^N} g|v_0|^p dx > 0$. Then by Lemma 2.5(*ii*), there exists a $t^- > 0$ such that $I'_{\lambda,\mu}(t^-v_0) = 0$ and $t^-v_0 \in \mathcal{N}^-_{\lambda,\mu}$. Note that $I'_{\lambda,\mu}(v_0) \neq 0$ by (3.3). Thus, $t^- \neq 1$. Since $t^-v_n \rightharpoonup t^-v_0$ and $t^-v_n \not\to t^-v_0$ in X_λ. Hence,

$$I_{\lambda,\mu}(t^-v_0) < \liminf_{n\to\infty} I_{\lambda,\mu}(t^-v_n).$$

Observe that $I_{\lambda,\mu}(tv_n)$ attains its maximum at $t = 1$. Thus, one obtains

$$I_{\lambda,\mu}(t^-v_0) < \liminf_{n\to\infty} I_{\lambda,\mu}(t^-v_n) \leq \lim_{n\to\infty} I_{\lambda,\mu}(v_n) = c^-_{\lambda,\mu},$$

which is absurd. Therefore, we obtain that $v_n \to v_0$ in X_λ. Since $\mathcal{N}^-_{\lambda,\mu}$ is closed by Lemma 2.6, it follows that $v_0 \in \mathcal{N}^-_{\lambda,\mu}$. By Lemmas 2.7 and 2.8, similar to Lemma 3.1, we deduce that v_0 is also a positive solution of (1.1).

□

Proof of Theorem 1.1. According to Lemmas 3.1 and 3.2, for $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$, we know that (1.1) admits at least two positive solutions $u_0 \in \mathcal{N}^+_{\lambda,\mu}$ and $v_0 \in \mathcal{N}^-_{\lambda,\mu}$. Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, the two solutions are different. This finishes the proof. □

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