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Nonlinear Singular Elliptic Equations of *p*-Laplace Type with Superlinear Growth in the Gradient

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Abstract. We consider a singular nonlinear elliptic Dirichlet problems with lower-order terms, where the combined effects of a superlinear growth in the gradient and a singular term allow us to establish some existence and regularity results. The model problem is

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \mu|u|^{p-1}u = b(x)\frac{|\nabla u|^q}{u^\theta} + \frac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, $\mu \geq 0$, $0 < \theta \leq 1$, $0 \leq \gamma < 1$ and f is a nonnegative function that belong to some Lebesgue space.

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1. Introduction

Singular equations as an important research topic in partial differential equations have been widely applied to describe and investigate various natural phenomena and applications, for example, fluid mechanics, pseudo-plastic flow, chemical reactions (the resistivity of the material), nerve impulses (Fitzhugh–Nagumo problems), population dynamics (Lotka–Volterra systems), combustion, morphogenesis, genetics, etc. The main goal for the study of singular equations is usually to explore the existence, uniqueness, regularity, and asymptotic behavior of solutions, see, for instance, [3, 10–18, 20–28, 30, 31, 33, 35–39].

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e investigate the interaction between two regularizing terms in the following nonlinear elliptic equation

$$\begin{cases} -\operatorname{div} a(x,\nabla u) + \mu |u|^{p-1}u = b(x)\frac{|\nabla u|^q}{u^\theta} + \frac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N \geq 1)$, f is a nonnegative $L^m(\Omega)$ function with $m \geq 1$ and, given a real number p such that $2 \leq p < N$, we have that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that the following holds: there exist $\alpha, \beta \in \mathbb{R}^+$ such that

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0 \quad \text{for } a.e.\ x \in \Omega \text{ and } \forall \xi, \eta \in \mathbb{R}^N \ s.t.\ \xi \neq \eta \tag{1.2}$$

$$a(x,\xi) \cdot \xi \geqslant \alpha |\xi|^p, \tag{1.3}$$

for a.e. $x \in \Omega$ and $\forall \xi \in \mathbb{R}^N$

$$|a(x,\xi)| \leqslant \beta |\xi|^{p-1}, \tag{1.4}$$

for a.e. $x \in \Omega$ and $\forall \xi \in \mathbb{R}^N$ and we assume that

$$0 \leq \gamma < 1, \tag{1.5}$$

$$0 \leq b(x) \in L^\infty(\Omega), \tag{1.6}$$

$$0 < \theta \leq 1, \tag{1.7}$$

and

$$0 \leq \mu, \ \ p-1 \leq q < \frac{p(p+\beta)}{p+1}, \text{ with } \beta = \min(\theta, \gamma). \tag{1.8}$$

The assumptions on the function a imply that the differential operator A acting between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ and defined by

$$A(u) = -\operatorname{div}(a(x,\nabla u)),$$

is coercive, monotone, surjective and satisfies the maximum principle. The simplest case is the p-Laplacian, which corresponds to the choice $a(x,\xi) = |\xi|^{p-2}\xi$. In the literature we find several papers about elliptic problems with lower-order terms having a natural or quadratic growth with respect to the gradient (see [2,4,6,8,28], for example, and the references therein), that is, for problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = g(x,u)|\nabla u| + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In these works it is assumed that $M : \Omega \to \mathbb{R}^{N^2}$ is a bounded elliptic Carathéodory map, so that there exists $\alpha > 0$ such that $\alpha|\xi|^2 \leq M(x)\xi \cdot \xi$ for every $\xi \in \mathbb{R}^N$. Various assumptions are made on g. With no attempt of being exhaustive, we will describe some recent results where a singular g has been considered, namely $g(x,u) = b(x) \times 1/|u|^\theta$. The case where $0 < \theta \leq 1$, introduced in [9], has been studied positive source $f \in L^m(\Omega)$; if $1 < m < N/2$ there exists a strictly positive solution $u \in L^{m^{**}}(\Omega)$; if

$m > N/2$, then the solution u belong to $L^\infty(\Omega)$. Furthermore, if $0 < \theta < 1/2$, and $r = Nm/[N(1-\theta) - m(1-2\theta)]$, then

$$\frac{|\nabla u|}{u^\theta} \quad \text{belong to} \begin{cases} L^r(\Omega) & \text{if } 1 < m < \frac{2N(1-\theta)}{N+2-4\theta} \\ L^2(\Omega) & \text{if } m \geq \frac{2N(1-\theta)}{N+2-4\theta}. \end{cases}$$

Later, in [29] it is proved the existence result of solutions for the nonlinear Dirichlet problem of the type

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + \gamma u^p = B\frac{|\nabla u|^q}{u^\theta} + f & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N, N > 2, M(x)$ is a uniformly elliptic and bounded matrix, $\gamma > 0, B > 0, 1 \leq q < 2, 0 < \theta \leq 1$ and the source f is a nonnegative (not identically zero) function belonging to $L^1(\Omega)$. Olivia [32] studied the existence and uniqueness of nonnegative solutions to a problem which is modeled by

$$\begin{cases} -\Delta_p u = u^{-\theta}|\nabla u|^p + f u^{-\gamma} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N(N \geq 2), \Delta_p$ is the p-Laplacian operator $(1 < p < N), f \in L^1(\Omega)$ is nonnegative and $\theta, \gamma \geq 0$.

The main novelty in the presence work is to show that the combined effects of a superlinear growth in the gradient and a singular term, lower-order term and the singular term has a "regularizing effect" in the sense that the problem (1.1) has a distributional solution for all $f \in L^m$ with $m \geq 1$.

The paper is organized as follows. In Sect. 2 we construct an approximate problem of (1.1), the existence of weak solution of the last one is proved by Schauder's fixed point Theorem. In Sect. 3.1 is devoted to prove to the existence and regularity results both in case $q = p-1$, $\mu = 0$ and $f \in L^m(\Omega)$ with $m > 1$. In the last subsection we deal with the case $p-1 < q < p$, $\mu > 0$ and $f \in L^1(\Omega)$, we prove the existence of solution of problem (1.1). Note that the presence of the lower-order term $\mu|u|^{p-1}u$ is crucial in the sense that it guarantees the existence of solution when the data f belong only in $L^1(\Omega)$.

Notations: For a given function $v\colon \Omega \to \mathbb{R}$, in what follows, we denote by $v^\pm = \max\{\pm v, 0\}$, i.e., $v^+ = \max\{v, 0\}$ and $v^- = -\min\{v, 0\}$. For a fixed $k > 0$, we introduce the truncation functions $T_k\colon \mathbb{R} \to \mathbb{R}$ and $G_k\colon \mathbb{R} \to \mathbb{R}$ defined by

$$T_k(s) := \max\{-k, \min\{s, k\}\} \text{ and } G_k(s) := (|s| - k)^+ \operatorname{sign}(s),$$

where $\operatorname{sign}(\cdot)$ is the sign function. It is not difficult to see that for each $k > 0$ the equality holds

$$s = T_k(s) + G_k(s) \text{ for all } s \in \mathbb{R}.$$

For convenience's sake, in the sequel, we denote by

$$\int_\Omega f(x)\,dx = \int_\Omega f \text{ and } |E| = \text{measure of } E.$$

2. A priori estimates

Since problem (1.1) contains singular terms, this cannot allow us to use the variational methods to obtain the existence result to problem (1.1). In order to bypass this obstacle, in the section, we will apply a standard approximation procedure to prove the existence of solutions to problem (1.1).

Let $n \in \mathbb{N}$ be arbitrary, let us consider the following approximated problem

$$\begin{cases} -\operatorname{div} a\left(x, \nabla u_n\right) + \mu |u_n|^{p-1} u_n \\ = b(x) \frac{|\nabla u_n|^q}{\left(1+\frac{1}{n}|\nabla u_n|^q\right)\left(\frac{1}{n}+u_n\right)^\theta} + \frac{f_n}{(\frac{1}{n}+u_n)^\gamma} & \text{in } \Omega \\ u_n \geq 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f_n = T_n(f)$. Then, we have the weak formulation of (2.1) as follows

$$\int_\Omega a(x, \nabla u_n)\nabla\varphi + \int_\Omega \mu |u_n|^{p-1} u_n \varphi$$
$$= \int_\Omega b(x) \frac{|\nabla u_n|^q}{\left(1+\frac{1}{n}|\nabla u_n|^q\right)\left(\frac{1}{n}+u_n\right)^\theta}\varphi + \int_\Omega \frac{f_n}{(\frac{1}{n}+u_n)^\gamma}\varphi, \tag{2.2}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. Now, we briefly sketch how to deduce the existence of a nonnegative solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of problem (2.1). For any nonnegative function $v \in L^p(\Omega)$ given, it follows from [6, Theorem 1] that the following nonlinear elliptic equation has a unique positive solution w

$$\begin{cases} -\operatorname{div} a\left(x, \nabla w\right) + \mu |w|^{p-1} w \\ = b(x) \frac{|\nabla w|^q}{\left(1+\frac{1}{n}|\nabla w|^q\right)\left(\frac{1}{n}+w\right)^\theta} + \frac{f_n}{(\frac{1}{n}+v)^\gamma} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

and there exists a constant $c_n > 0$ which is independent of v such that $\|w\|_{L^\infty(\Omega)} \leq c_n$. So, we denote by $T\colon L^p(\Omega) \to L^p(\Omega)$ the solution mapping of problem (2.3), namely, $T(v) = w$ for all $v \in L^p(\Omega)$, where w is the unique solution of problem (2.3) corresponding to v. It is obvious that if u is a fixed point of T, then u is also a solution of problem (2.1). Then, we are going to utilize Schauder fixed point theorem for examining the existence of a fixed point of T. Therefore, we will show that T is a completely continuous function (thus, T is continuous and compact), and maps a closed ball into itself. Taking w as a test function in (2.3), it yields

$$\alpha \int_\Omega |\nabla w|^p \leq c_p ||b||_{L^\infty(\Omega)} c(n,\theta,\gamma)|\Omega|^{p'} \left(\int_\Omega |\nabla w|^p \mathrm{d}x\right)^{\frac{1}{p}}. \tag{2.4}$$

Whereas, an application of the Poincaré inequality gives that

$$\|w\|_{L^p(\Omega)} \leq \left(\frac{c_p ||b||_{L^\infty(\Omega)} c(n,\theta,\gamma)|\Omega|^{p'}}{\alpha}\right)^{p'} = r,$$

where c_p is the Poincaré constant. This indicates that T maps the closed ball centered at the origin with the radius r into itself. To show that T is continuous, let $\{v_k\}$ be a sequence in the ball of radius r which converges to v in $L^p(\Omega)$ as $k \to \infty$ and let $w_k = T(v_k)$. Our goal is to prove that w_k converges to $w = T(v)$ in $L^p(\Omega)$ as $k \to \infty$. From (2.4), we can see that $\{w_k\}$ is bounded in $W_0^{1,p}(\Omega)$ with respect to k. Moreover, it follows from Lemma 2 of [5] that w_k is also bounded in $L^\infty(\Omega)$ with respect to k. The latter combined with Lemma 4 of [5] deduces that, passing to a subsequence if necessary, w_k converges to a function w in $W_0^{1,p}(\Omega)$ (indeed, this result could be obtained by using the pseudomonotonicity and (S_+)-property of $u \mapsto$-div$a(x,u)$). This is sufficient to pass to the limit as $k \to \infty$ for the weak formulation of the equation (2.3) with $w = w_k$ and $v = v_k$ that $w = T(v)$. For the compactness, it is sufficient to underline that if v_k is bounded in $L^p(\Omega)$ then one can recover that w_k is bounded in $W_0^{1,p}(\Omega)$ with respect to k thanks to (2.4). Taking into account the compactness of the embedding of $W_0^{1,p}(\Omega)$ to $L^p(\Omega)$, we obtain that, up to subsequences, $\{w_k\}$ converges to a function in $L^p(\Omega)$, namely, $T(\{v_n\})$ is relatively compact in $L^p(\Omega)$. So, T is compact.

Therefore, all conditions of Schauder fixed point Theorem are verified. We are now in a position to invoke this theorem to find that T has at least one fixed point, say u_n. It is obvious that u_n solves problem (2.1) too. Recall that the right-hand side of problem (2.1) is positive. This together with the hypotheses of a and maximum principle [34] implies that $u_n \geq 0$.

Lemma 2.1. *Let u_n be a solution to (2.1) then for every $\omega \subset\subset \Omega$ there exists a constant $c_\omega > 0$ which does not depend on n and such that*

$$u_n \geq c_\omega \text{ a.e. in } \omega \tag{2.5}$$

Proof. $\mu \geq 0$ and $f_n \geq 0$. Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the unique solution of the following elliptic equation (see [1, Lemma 2.1])

$$\begin{cases} -\operatorname{div}(a(x,\nabla v)) + \mu|v|^{p-1}v = \frac{f_n}{(v+\frac{1}{n})^\gamma} & \text{in } \Omega, \\ v \geq 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

But, [1, Lemma 2.2] points out that for any $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that

$$v \geq c_\omega \text{ a.e. in } \omega. \tag{2.7}$$

We take $(v - u_n)^+$ as a test function in (2.1) and (2.6), respectively. Rearranging the resulting equalities, we have

$$\begin{aligned} &\int_\Omega (a(x,\nabla v) - a(x,\nabla u_n)) \nabla (v-u_n)^+ + \int_\Omega \left(|v|^{p-1}v - |u_n|^{p-1}u_n\right)(v-u_n)^+ \\ &= \int_\Omega f_n \frac{\left(u_n + \frac{1}{n}\right)^\gamma - \left(v + \frac{1}{n}\right)^\gamma}{\left(u_n + \frac{1}{n}\right)^\gamma \left(v + \frac{1}{n}\right)^\gamma} (v-u_n)^+ \\ &\quad - \int_\Omega b(x) \frac{|\nabla u_n|^q (v-u_n)^+}{\left(1 + \frac{1}{n}|\nabla u_n|^q\right)\left(\frac{1}{n} + u_n\right)^\theta} \leq 0. \end{aligned}$$

However, the monotonicity of the second term on the left-hand side to the above inequality concludes

$$\int_\Omega (a(x,\nabla v) - a(x,\nabla u_n)) \cdot \nabla (v-u_n)^+ \leq 0.$$

This means that $u_n \geq v$ almost everywhere in Ω. Consequently, the desired conclusion is a direct consequence of (2.7). $\square$

3. The Main Results and Their Proof

3.1. The Case $q = p - 1$, $\mu = 0$ and $f \in L^m(\Omega)$ with $m > 1$

In this subsection, we want to analyze the case $0 \leq \gamma < 1$, $\mu = 0$, $0 \leq f \in L^m(\Omega)(m > 1)$. We first give the definition of a distributional solution to problem (1.1)

Definition 3.1. Let f be a nonnegative (not identically zero) function in $L^m(\Omega)$ function, with $m > 1$. A positive and measurable function u is a distributional solution to problem (1.1) if $u \in W_0^{1,1}(\Omega)$, if $|a(x, \nabla u)|$, $\frac{|\nabla u|^{p-1}}{u^\theta}$ $\in L^1_{\text{loc}}(\Omega)$,

$$\forall \omega \subset\subset \Omega, \ \exists c_\omega > 0 : u \geq c_\omega \text{ in } \omega \tag{3.1}$$

and if

$$\int_\Omega a(x, \nabla u)\nabla\varphi = \int_\Omega b(x)\frac{|\nabla u|^{p-1}}{u^\theta}\varphi + \int_\Omega \frac{f(x)}{u^\gamma}\varphi, \quad \forall\varphi \in C_c^1(\Omega). \tag{3.2}$$

The main results of this subsection are as follows

Theorem 3.2. *Assume* (1.3),(1.4) *and* (1.5)*. Then, if* $m_1 = \frac{mN(p-1+\gamma)}{N-pm}$ *and* $\tilde{m} = \frac{Nm(p-1+\gamma)}{N+m(1-\gamma)}$ *there exists a distributional solution* u *of* (1.1)

$$u \in \begin{cases} L^{m_1}(\Omega) & \text{if } 1 < m < N/p, \\ L^\infty(\Omega) & \text{if } m > N/p, \end{cases}$$

$$|\nabla u| \in \begin{cases} L^{\tilde{m}}(\Omega) & \text{if } 1 < m < pN/[N(p-1+\gamma) + p(1-\gamma)], \\ L^p(\Omega) & \text{if } m \geq pN/[N(p-1+\gamma) + p(1-\gamma)], \end{cases}$$

and if $r = \frac{\tilde{m}}{p-1}$, *we have*

$$\frac{|\nabla u|^{p-1}}{u^\theta} \in \begin{cases} L^r_{\text{loc}}(\Omega) & \text{if } 1 < m < pN/[N(p-1+\gamma) + p(1-\gamma)], \\ L^{p'}_{\text{loc}}(\Omega) & \text{if } m \geq pN/[N(p-1+\gamma) + p(1-\gamma)]. \end{cases}$$

Furthermore, if $0 < \theta < (p-1)(1-\gamma)/p$ *and* $r = Nm(p-1+\gamma)/[N(p-1-\theta) - m[(p-1)(1-\gamma) - p\theta]$, *then*

$$\frac{|\nabla u|^{p-1}}{u^\theta} \in \begin{cases} L^r(\Omega) \ \text{if } 1 < m < \frac{p(p-1)N(1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta}, \\ L^{p'}(\Omega) \ \text{if } m \geq \frac{p(p-1)N(1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta}. \end{cases}$$

Remark 3.3. In the case where the lower-order term does not exist (i.e., $b(x) = 0$), the results of previous theorem coincide with regularity results obtained in ([19, Theorem 4.4]).

Remark 3.4. If $p = 2$ and $\gamma = 0$; the result of Theorem 3.2 coincides with regularity results of [9].

Now, we can prove the following existence and regularity result

Lemma 3.5. *Let u_n be a solution of problem (2.1) and suppose that (1.3)–(1.7) hold true, let f be a nonnegative function in $L^m(\Omega)$, with $1 < m < N/p$, $\sigma = \min(\tilde{m}, p)$, $r = \frac{\tilde{m}}{p-1}$. Then we have*

- *the sequence* $\{u_n\}$ *is bounded in* $L^{m_1}(\Omega) \cap W_0^{1,\sigma}(\Omega)$, (3.3)
- *the sequence* $\left\{\dfrac{|\nabla u_n|^{p-1}}{u_n^\theta}\right\}$ *is bounded in* $L^r_{\mathrm{loc}}(\Omega) \cap L^{p'}_{\mathrm{loc}}(\Omega)$, (3.4)

with $\tilde{m}$ and m_1 are defined in the Theorem 3.2.

Proof. Here, and in the following, we will denote by C the generic constant which is independent of $n \in \mathbb{N}$. Define, for $k > 0$ and $s > 0$

$$\eta_k(s) = \frac{1}{k} T_k(G_1(s)) = \begin{cases} 0 & \text{if } 0 \le s < 1, \\ \frac{s-1}{k} & \text{if } 1 \le s < 1+k, \\ 1 & \text{if } s \ge 1+k. \end{cases}$$

We choose $v_n = u_n^{p\lambda-(p-1)} \eta_k(u_n)$ as test function in the weak formulation of (2.2) (this choice is possible since every u_n belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$). Noting that since $f_n \le f$ and let $\lambda > 1/p'$, dropping a first nonnegative term, we obtain

$$\begin{aligned} &\alpha(p\lambda - (p-1)) \int_\Omega |\nabla u_n|^p u_n^{p\lambda-p} \eta_k(u_n) \\ &\le \int_\Omega b(x) \frac{|\nabla u_n|^{p-1} u_n^{p\lambda-(p-1)}}{\left(1 + \frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n} + u_n\right)^\theta} \eta_k(u_n) + \int_\Omega f_n u_n^{p\lambda-(p-1)-\gamma} \eta_k(u_n) \\ &\le \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_n|^{p-1} u_n^{(p-1)(\lambda-1)} \eta_k(u_n) u_n^{\lambda-\theta} + \int_\Omega f u_n^{p\lambda-(p-1)-\gamma} \eta_k(u_n). \end{aligned}$$

Let $\varepsilon > 0$ be such that $0 < \varepsilon\|b\|_{L^\infty(\Omega)} < \alpha(p\lambda - (p-1))$. By Young inequality with ε, we deduce that

$$\begin{aligned} &\left[\alpha(p\lambda - (p-1)) - \|b\|_{L^\infty(\Omega)}\right] \int_\Omega |\nabla u_n|^p u_n^{p\lambda-p} \eta_k(u_n) \\ &\le C\|b\|^p_{L^\infty(\Omega)} \int_\Omega \eta_k(u_n) u_n^{p(\lambda-\theta)} + \int_\Omega f u_n^{p\lambda-(p-1)-\gamma} \eta_k(u_n). \end{aligned}$$

Letting k tend to zero, and Lebesgue Theorem in the right-hand side using and Fatou Lemma in the left-hand side, we get

$$C \int_{\{u_n \ge 1\}} |\nabla u_n|^p u_n^{p\lambda-p} \le \int_{\{u_n \ge 1\}} u_n^{p(\lambda-\theta)} + \int_{\{u_n \ge 1\}} f u_n^{p\lambda-(p-1)-\gamma} \tag{3.5}$$

We now remark that for every $t \ge 1$ and $\delta > 0$, there exists $C_\delta > 0$ such that

$$t^{p(\lambda-\theta)} \le \delta t^{p\lambda} + C_\delta. \tag{3.6}$$

The inequality is trivially true if $\theta \ge \lambda$, while is a consequence of Young inequality if $\lambda > \theta$. Recall that the estimate (3.5), we have

$$\int_{\{u_n \ge 1\}} |\nabla u_n|^p u_n^{p\lambda-p} \le \delta \int_{\{u_n \ge 1\}} u_n^{p\lambda} + |\Omega| C_\delta + \int_{\{u_n \ge 1\}} f u_n^{p\lambda-(p-1)-\gamma}. \tag{3.7}$$

Taking into account that $0 \le u_n = T_1(u_n) + G_1(u_n) \le 1 + G_1(u_n)$, and using Poincaré inequality, we conclude that

$$\begin{aligned} C\int_\Omega \left|\nabla G_1(u_n)^\lambda\right|^p &\le \delta \int_\Omega G_1(u_n)^{p\lambda} + C + \int_\Omega f G_1(u_n)^{p\lambda-(p-1)-\gamma} \\ &\le \frac{\delta}{\lambda_1}\int_\Omega \left|\nabla G_1(u_n)^\lambda\right|^p + C + \int_\Omega f G_1(u_n)^{p\lambda-(p-1)-\gamma}, \end{aligned}$$

where λ_1 is the Poincaré constant for Ω (i.e., the first eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions). Choosing δ small enough, we thus have

$$\int_\Omega |\nabla G_1(u_n)^\lambda|^p \le C + C\int_\Omega f G_1(u_n)^{p\lambda-(p-1)-\gamma}.$$

Following the same technique as in [6], choosing $\lambda = \frac{m_1}{p^*}$, it is easy to see that if $\lambda = \frac{m(N-p)[p-1+\gamma]}{p(N-pm)} > \frac{(N-p)[p-1+\gamma]}{p(N-p)} = \frac{p-1+\gamma}{p}$ if only if $m > 1$. Note that with such a choice, we have that $\lambda p^* = m_1$, and $(p\lambda-(p-1)-\gamma)m' = \lambda p^* = m_1 = \frac{Nm[p-1+\gamma]}{N-pm}$. Therefore, using Sobolev and Hölder inequalities, we get

$$\begin{aligned} \mathcal{S}\left(\int_\Omega G_1(u_n)^{m_1}\right)^{\frac{p}{p^*}} &\le \int_\Omega \left|\nabla G_1(u_n)^\lambda\right|^p \le C + C\int_\Omega f G_1(u_n)^{p\lambda-(p-1)-\gamma} \\ &\le C + C\|f\|_{L^m(\Omega)}\left(\int_\Omega G_1(u_n)^{m_1}\right)^{\frac{1}{m'}}, \end{aligned}$$

where $\mathcal{S}$ is the Sobolev constant, thanks to the assumption $m < N/p$, we have $p/p^* > 1/m'$, putting to gather all the previous estimates we conclude that

$$\|G_1(u_n)\|_{L^{m_1}(\Omega)} \le C\|f\|_{L^m(\Omega)}. \tag{3.8}$$

Note that from the boundedness of $\{G_1(u_n)\}$ in $L^{m_1}(\Omega)$ it trivially follows the boundedness of $\{u_n\}$ in $L^{m_1}(\Omega)$ since, as before, $0 \le u_n \le 1 + G_1(u_n)$. Now we point out that $m \ge \frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$, since $\lambda \ge 1$. Therefore from (3.7) and (3.8) (note that the right-hand side is bounded), we have that

$$\int_\Omega |\nabla G_1(u_n)|^p \le \int_{\{u_n\ge 1\}} |\nabla u_n|^p\, u_n^{p\lambda-p} \le C,$$

we deduce that the sequence $\{G_1(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$. If on the other hand $1 < m < \frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$, then $\lambda < 1$ and we have to proceed differently. Let now σ be such that the use of by Hölder inequality, $\sigma < p$ we obtain

$$\begin{aligned} \int_\Omega |\nabla G_1(u_n)|^\sigma &= \int_\Omega \frac{|\nabla G_1(u_n)|^\sigma}{u_n^{\sigma(1-\lambda)}} u_n^{\sigma(1-\lambda)} \\ &\le \left(\int_{\{u_n\ge 1\}} |\nabla u_n|^p\, u_n^{p\lambda-p}\right)^{\frac{\sigma}{p}} \left(\int_{\{u_n\ge 1\}} u_n^{\frac{p\sigma(1-\lambda)}{p-\sigma}}\right)^{\frac{p-\sigma}{p}}. \end{aligned}$$

Imposing $\sigma = \frac{Nm(p+\gamma-1)}{N-m(1-\gamma)} (= \tilde{m})$, we obtain $\frac{p\sigma(1-\lambda)}{p-\sigma} = m_1$, so that the above inequality becomes, thanks to (3.7) and (3.8)

$$\int_\Omega |\nabla G_1(u_n)|^{\tilde{m}} \leq C.$$

Summing up, we have therefore proved that the sequence:

$$\{G_1(u_n)\} \text{ is bounded in } L^{m_1}(\Omega) \cap W_0^{1,\sigma}(\Omega), \sigma = \min(\tilde{m}, p). \tag{3.9}$$

On the other hand, taking $T_1(u_n)$ as test function in (2.1) , we have

$$\begin{aligned} \alpha \int_\Omega |\nabla T_1(u_n)|^p &\leq \|b\|_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla T_1(u_n)|^{p-1}}{\left(\frac{1}{n}+u_n\right)^\theta} T_1(u_n) \\ &\quad + \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla G_1(u_n)|^{p-1} + \int_\Omega f \\ &\leq \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla T_1(u_n)|^{p-1} \\ &\quad + \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla G_1(u_n)|^{p-1} + \int_\Omega f, \end{aligned}$$

which implies (thanks to (3.9)) that the sequence $\{T_1(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$. This estimate and the estimate (3.9) give (3.3). First case: The proof of (3.4) is then a simple consequence of (2.5) and (3.3), if $w \subset\subset \Omega$, then

$$\int_w \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right)^{p'} \leq \frac{1}{c_w^{p'\theta}} \int_\Omega |\nabla u_n|^p \leq C. \tag{3.10}$$

In the second case, we take $r = \frac{\tilde{m}}{p-1}$, then by (2.5) and (3.3), we have

$$\int_w \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right)^{r} \leq \frac{1}{c_w^{r\theta}} \int_\Omega |\nabla u_n|^{\tilde{m}} \leq C. \tag{3.11}$$

Using (3.10) and (3.11), we deduce that (3.4) holds true. □

Lemma 3.6. *Let u_n be a solution of (2.1) under assumptions(1.3)–(1.7) and let f be a nonnegative function in $L^m(\Omega)$. Then, if $m > N/p$*

- *the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$,* (3.12)
- *the sequence $\left\{\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right\}$ is bounded in $L^{p'}_{\text{loc}}(\Omega)$.* (3.13)

Proof. We take $v_n = G_k(u_n)$ as test function in (2.1). Using (1.3), (1.4) and (1.5), we obtain

$$\begin{aligned} \alpha \int_{\{u_n \geq k\}} |\nabla u_n|^p &\leq \|b\|_{L^\infty(\Omega)} \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^{p-1} G_k(u_n)}{u_n^\theta} + \int_{\{u_n \geq k\}} \frac{f G_k(u_n)}{(u_n+\frac{1}{n})^\gamma} \\ &\leq \frac{1}{k^\theta} \|b\|_{L^\infty(\Omega)} \int_{\{u_n \geq k\}} |\nabla u_n|^{p-1} G_k(u_n) + \int_{\{u_n \geq k\}} \frac{f G_k(u_n)}{(u_n+\frac{1}{n})^\gamma}. \end{aligned}$$

Noting that $u_n + \frac{1}{n} \geq k \geq 1$ on the set $A_{n,k}$, where $G_k(u_n)$, we have

$$\alpha \int_{\{u_n \geq k\}} |\nabla u_n|^p \leq \frac{1}{k^\theta} \|b\|_{L^\infty(\Omega)} \int_{\{u_n \geq k\}} |\nabla u_n|^{p-1} G_k(u_n) + \int_{\{u_n \geq k\}} f G_k(u_n).$$

and by Young and Poincaré inequalities, we have that

$$\begin{aligned}\int_{\{u_n \geq k\}} |\nabla u_n|^{p-1} G_k(u_n) &\leq \frac{1}{p'} \int_{\{u_n \geq k\}} |\nabla u_n|^p + \frac{1}{p} \int_{\{u_n \geq k\}} G_k(u_n)^p \\ &\leq \frac{1 + \lambda_1(p-1)}{p\lambda_1} \int_{\{u_n \geq k\}} |\nabla u_n|^p.\end{aligned}$$

Therefore,

$$\left(\alpha - \frac{1}{k^\theta} \frac{\|b\|_{L^\infty(\Omega)}^{(1+\lambda_1(p-1))}}{p\lambda_1} \right) \int_{\{u_n \geq k\}} |\nabla u_n|^p \leq \int_{\{u_n \geq k\}} f G_k(u_n).$$

Next, we can take $k > k_0$, with

$$k_0 = \left(\frac{\|b\|_{L^\infty(\Omega)} (1 + \lambda_1(p-1))}{\alpha\lambda_1} \right)^{\frac{1}{\theta}}, \tag{3.14}$$

we have

$$\frac{\alpha}{p'} \int_{\{u_n \geq k\}} |\nabla u_n|^p \leq \int_{\{u_n \geq k\}} f G_k(u_n).$$

From this point outwards, we can proceed as in the proof of [8, Theorem 1.1], to prove that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, as desired and the proof of (3.13) is essentially the same technique used in (3.10). □

If $0 < \theta < (1-\gamma)/p'$, the estimates on the right-hand side $\frac{|\nabla u_n|^{p-1}}{u_n^\theta}$ are not only local but also global.

Lemma 3.7. *Let u_n be a solution of (2.1), let us assume that (1.3)–(1.6) and $0 < \theta < (1-\gamma)/p'$, hold true and that f be a nonnegative function in $L^m(\Omega)$, with*

$$m \geq \frac{p(p-1)N(1-\theta)}{N(p-1)(p-1+\gamma) + p(p-1)(1-\gamma) - p^2\theta}, \tag{3.15}$$

then,

$$\text{the sequence} \left\{ \frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right\} \text{ is bounded in } L^{p'}(\Omega). \tag{3.16}$$

Proof. We fix $\lambda > (p-1+\gamma)/p$, let $0 < \varepsilon < 1/n$, and choose $v_n = (u_n + \varepsilon)^{p\lambda-(p-1)} - \varepsilon^{p\lambda-(p-1)}$ as test function in (2.1) this choice is possible since every u_n belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We obtain, dropping some negative terms

$$\begin{aligned}
&\alpha(p\lambda-(p-1))\int_\Omega |\nabla u_n|^p (u_n+\varepsilon)^{p\lambda-p}\\
&\quad\le \int_\Omega b(x)\frac{|\nabla u_n|^{p-1}(u_n+\varepsilon)^{p\lambda-(p-1)}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^\theta}+\int_\Omega f_n (u_n+\varepsilon)^{p\lambda-(p-1)-\gamma}\\
&\quad\le \|b\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_n|^{p-1}(u_n+\varepsilon)^{(p-1)(\lambda-1)+(\lambda-\theta)}+\int_\Omega f(u_n+\varepsilon)^{p\lambda-(p-1)-\gamma}.
\end{aligned}$$

In view of the latter estimate we have used that $0\le f_n\le f$. We can apply Young inequality, we thus obtain

$$\begin{aligned}
&c\alpha(p\lambda-(p-1))/p\int_\Omega |\nabla u_n|^p (u_n+\varepsilon)^{p\lambda-p}\\
&\quad\le C\int_\Omega (u_n+\varepsilon)^{p(\lambda-\theta)}+C\int_\Omega f(u_n+\varepsilon)^{p\lambda-(p-1)-\gamma}.
\end{aligned}$$

Letting ε tend to zero, and using Lebesgue Theorem (in the right one, recall that u_n is in $L^\infty(\Omega)$) and Fatou Lemma (in the left-hand side), we arrive at

$$\int_\Omega |\nabla u_n|^p u_n^{p\lambda-p}\le C\int_\Omega u_n^{p(\lambda-\theta)}+C\int_\Omega f u_n^{p\lambda-(p-1)-\gamma},$$

since now our assumption is $0<\theta<(p-1+\gamma)/p$ and $\lambda>(p-1+\gamma)/p$, we have that $\lambda>\theta$; thus, using Young inequality we have that, for $\delta>0$

$$\begin{aligned}
\int_\Omega |\nabla u_n|^p u_n^{p\lambda-p}&\le \delta\int_\Omega u_n^{p\lambda}+|\Omega|C_\delta+C\int_\Omega f u_n^{p\lambda-(p-1)-\gamma}\\
&\le \frac{\delta}{\lambda_1}\int_\Omega |\nabla u_n|^p u_n^{p\lambda-p}+C+C\int_\Omega f u_n^{p\lambda-(p-1)-\gamma},
\end{aligned}$$

where in the last inequality we have used Poincaré inequality. Thus, if δ is small enough, we have

$$\int_\Omega |\nabla u_n|^p u_n^{p\lambda-p}\le C+C\int_\Omega f u_n^{p\lambda-(p-1)-\gamma}.$$

If $1<m<\frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$, the choice $\lambda(m)=\frac{m(N-p)(p-1+\gamma)}{p(N-pm)}$ implies $\frac{p-1+\gamma}{p}<\lambda(m)<1$ and (reasoning as in the proof of Lemma 3.5)

$$\int_\Omega \frac{|\nabla u_n|^p}{u_n^{p(1-\lambda(s))}}\le C\left(\|f\|_{L^m(\Omega)}\right). \tag{3.17}$$

Let $\bar m$ be a real number, such that

$$\bar m=\frac{pN(1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta},$$

we have that $\lambda(m)=1-\frac{\theta}{p-1}$, and so (3.17) becomes

$$\int_\Omega \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right)^{p'}\le C\|f\|_{L^{\bar m}(\Omega)}, \tag{3.18}$$

which is (3.16) if $m = \bar{m}$. Since Ω has finite measure, if $m > \bar{m}$ and if f belong to $L^m(\Omega)$, then it is also in $L^{\bar{m}}(\Omega)$, so that (3.18) still holds for these values of m. □

Lemma 3.8. *Let u_n be a solution of (2.1). Suppose that (1.3)–(1.6) and $0 < \theta < (1-\gamma)/p'$ hold true. Then if $r = \frac{Nm(p-1+\gamma)}{N(p-1-\theta)-m[(p-1)(1-\gamma)-p\theta]}$ and that $0 \leq f \in L^m(\Omega)$, with*

$$1 < m < \frac{pN(p-1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta}, \tag{3.19}$$

then,

$$\text{the sequence} \left\{ \frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right\} \text{ is bounded in } L^r(\Omega). \tag{3.20}$$

Proof. Let $\theta > 0$ and $N > p$, we have $m < \frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$.
Let $1 < r < p'$; then, we used Hölder inequality with exponents $\frac{p'}{r}$ and $\frac{p'}{p'-r}$, we obtain

$$\begin{aligned}\int_\Omega \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right)^r &= \int_\Omega \frac{|\nabla u_n|^{r(p-1)}}{u_n^{r(p-1)(1-\lambda(m))}} u_n^{r(p-1)(1-\lambda(m)-\frac{\theta}{p-1})} \\ &\leq \left(\int_\Omega \frac{|\nabla u_n|^p}{u_n^{p(1-\lambda(m))}} \right)^{\frac{r}{p'}} \left(\int_\Omega u_n^{\frac{pr(1-\lambda(m)-\frac{\theta}{p-1})}{p'-r}} \right)^{\frac{p'-r}{p'}}.\end{aligned}$$

Moreover, using (3.17) which is admissible since $m < \frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$, we thus obtain

$$\int_\Omega \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right)^r \leq C\|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{\frac{pr(1-\lambda(m)-\frac{\theta}{p-1})}{p'-r}} \right)^{\frac{p'-r}{p'}}. \tag{3.21}$$

Taking $r = r(m)$ such that $\frac{pr(m)(1-\lambda(m)-\frac{\theta}{p-1})}{p'-r(m)} = \frac{Nm(p-1+\gamma)}{N-pm}$, that is $r(m) = \frac{Nm(p-1+\gamma)}{N(p-1-\theta)-m[(p-1)(1-\gamma)-p\theta]}$; the assumptions on m, and the fact that $r(m)$ is increasing, imply that
$1 < \frac{N(p-1+\gamma)}{N(1-\theta)-(1-\gamma-p\theta)} < r(m) < r\left(\frac{pN(p-1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta}\right) = p'$, hence by (3.21) we derive that

$$\int_\Omega \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right)^r \leq C\|f\|_{L^m(\Omega)},$$

as desired. □

Now, we are going to prove Theorem 3.2.

Proof of Theorem 3.2. Thanks to (3.3) (or (3.12)), the sequence $\{u_n\}$ of solutions of (2.1) is bounded in $W_0^{1,\sigma}(\Omega)$, with $\sigma = \min(\tilde{m}, p)$. Thus, up to subsequences, u_n weakly converges to some function u in $W_0^{1,\sigma}(\Omega)$, with σ as above and therefore u satisfies the boundary condition. However, due to the nonlinear nature of the lower-order term, the weak convergence of u_n is not enough to pass to the limit in the distributional formulation of (2.1). In order to proceed, we use the fact that, thanks to (3.4) (or (3.13)), we have that the right-hand side

$$b(x)\frac{|\nabla u_n|^{p-1}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^{\theta}}\quad \text{is bounded in (at least) } L^1_{\text{loc}}(\Omega).$$

Therefore, thanks to Remark 2.2 after Theorem 2.1 of [7] (see also [1] and [32]), we have that $\nabla u_n(x)$ almost everywhere converges to $\nabla u(x)$ in Ω; this implies that

$$\lim_{n\to+\infty}\frac{|\nabla u_n|^{p-1}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^{\theta}}=\frac{|\nabla u|^{p-1}}{u^{\theta}}\quad \text{almost everywhere in } \Omega.$$

This almost everywhere convergence, and the local boundedness of the sequence in $L^r(\Omega)$, with $r=\frac{\tilde{m}}{p-1}$ or $r=p'$, yield that

$$\lim_{n\to+\infty}\frac{|\nabla u_n|^{p-1}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^{\theta}}=\frac{|\nabla u|^{p-1}}{u^{\theta}}\quad \text{locally weakly in } L^r(\Omega).$$

Next we note that, for all $0\le\gamma<1$ and $\varphi\in C_0^1(\Omega)$, if $\omega=\{x\in\Omega:|\varphi|>0\}$, we have

$$\left|\frac{f_n\varphi}{(u_n+1/n)^{\gamma}}\right|\leqslant\frac{\|\varphi\|_\infty f}{c_\omega^{\gamma}}\in L^1(\Omega)$$

and that, for $n\to\infty$

$$\frac{f_n\varphi}{(u_n+1/n)^{\gamma}}\longrightarrow\frac{f\varphi}{u^{\gamma}}\quad \text{a.e in } \Omega.$$

Here we use the convention that if $u=+\infty$, then $\frac{f\varphi}{u^{\gamma}}=0$. Therefore, by Lebesgue Theorem, it follows that

$$\lim_{n\to\infty}\int_\Omega\frac{f_n\varphi}{(u_n+1/n)^{\gamma}}=\int_\Omega\frac{f\varphi}{u^{\gamma}}. \tag{3.22}$$

Concerning the left hand side of (2.2), we can use the assumption (1.4) on a and the generalized Lebesgue Theorem, we can pass to the limit for $n\longrightarrow\infty$ obtaining

$$\lim_{n\to\infty}\int_\Omega a(x,\nabla u_n)\nabla\varphi=\int_\Omega a(x,\nabla u)\nabla\varphi.$$

We now take φ in $C_c^1(\Omega)$ as test function in (2.1), to have that

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi=\int_\Omega b(x)\frac{|\nabla u_n|^{p-1}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^{\theta}}\varphi+\int_\Omega\frac{f_n}{(\frac{1}{n}+u_n)^{\gamma}}\varphi.$$

Passing to the limit in n, we obtain

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi=\int_\Omega b(x)\frac{|\nabla u|^{p-1}}{u^{\theta}}\varphi+\int_\Omega\frac{f}{u^{\gamma}}\varphi,$$

for every φ in $C_c^1(\Omega)$, so that u is a solution in the sense of distributions. □

3.2. The Case $p-1 \le q < \frac{p(p+\beta)}{p+1}$, $\mu > 0$ and $0 \le f \in L^1(\Omega)$.

In this subsection, we treat the case where $0 \le f \in L^1(\Omega), \beta = \min(\theta,\gamma)$, $\mu > 0$ and $p-1 \le q < \frac{p(p+\beta)}{p+1}$. Here, we give our main existence result for this subsection

Theorem 3.9. *Assume that* (1.3)–(1.7) *hold true and let* f *be a nonnegative function in* $L^1(\Omega)$. *Then there exists a solution* u *for* (1.2), *in the sense that:* $u \in W_0^{1,r}(\Omega) \cap L^{p+\beta}(\Omega)$, *with* $\beta = \min(\theta,\gamma)$, $1 \le r < \frac{p(p+\beta)}{p+1}$, $\frac{|\nabla u|^q}{u^\theta} \in L^1_{loc}(\Omega)$

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 : u \ge c_\omega \ \text{in}\ \omega \tag{3.23}$$

and that

$$\int_\Omega a(x,\nabla u)\nabla\varphi + \mu\int_\Omega u^p\varphi = \int_\Omega b(x)\frac{|\nabla u|^q}{u^\theta}\varphi + \int_\Omega \frac{f}{u^\gamma}\varphi, \quad \forall \varphi \in C_c^1(\Omega).$$

The next Lemma will be used in the proof of Theorem 3.9, we state some a priori estimates on the solution u_n and on the lower-order term of the approximate problem (2.1).

Lemma 3.10. *Let* u_n *be a solution of* (2.1). *Suppose that* f *be a nonnegative function in* $L^1(\Omega)$ *and* (1.3)–(1.7) *hold true. Then the sequence* u_n *is bounded in* $W_0^{1,r}(\Omega) \cap L^{p+\beta}(\Omega)$, *with* $\beta = \min(\theta,\gamma)$, $1 \le r < \frac{p(p+\beta)}{p+1}$ *and* $\frac{|\nabla u_n|^q}{u_n^\theta}$ *is bounded in* $L^1_{loc}(\Omega)$.

Proof. In the case $\theta \ge \gamma$, let $\big(G_1(u_n)\big)^\gamma$ as test function in (2.1), using (1.3), (1.4) and the fact that $0 \le f_n \le f$, we thus have

$$\gamma\alpha\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + \int_{\{u_n\ge 1\}} u_n^{p+\gamma} \le ||b||_{L^\infty(\Omega)}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^q}{u_n^{\theta-\gamma}} + \int_\Omega f \tag{3.24}$$

and then, by Young inequality, we deduce that

$$\begin{aligned} ||b||_{L^\infty(\Omega)}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^q}{u_n^{\theta-\gamma}} &\le ||b||_{L^\infty(\Omega)}\int_{\{u_n\ge 1\}}|\nabla u_n|^q \\ &= ||b||_{L^\infty(\Omega)}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^q}{u_n^{\frac{q(1-\gamma)}{p}}}u_n^{\frac{q(1-\gamma)}{p}} \\ &\le \frac{\gamma\alpha}{p}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + C\int_{\{u_n\ge 1\}} u_n^{\frac{q(1-\gamma)}{p-q}}, \end{aligned}$$

which implies from (3.24) that

$$\frac{\gamma\alpha}{p'}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + \int_{\{u_n\ge 1\}} u_n^{p+\gamma} \le C\int_{\{u_n\ge 1\}} u_n^{\frac{q(1-\gamma)}{p-q}} + \int_{\{u_n\ge 1\}} f, \tag{3.25}$$

thanks to (3.25) we have

$$\frac{\gamma\alpha}{p'}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + \frac{1}{p}\int_{\{u_n\ge 1\}} u_n^{p+\gamma} \le C\int_{\{u_n\ge 1\}} u_n^{\frac{q(1-\gamma)}{p-q}} + C.$$

Since, $\frac{q(1-\gamma)}{p-q} < p+\gamma$ the above estimate implies that

$$\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} \le C \tag{3.26}$$

and

$$\int_{\{u_n\geq 1\}} u_n^{p+\gamma} \leq C. \tag{3.27}$$

Now we choose $\varepsilon < 1/n$ and use $(T_1(u_n)+\varepsilon)^\theta - \varepsilon^\theta$ as test function, dropping the positive term and using (1.3), (1.4) we obtain

$$\begin{aligned} \alpha\theta \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} &\leq ||b||_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla u_n|^q}{\left(u_n+\frac{1}{n}\right)^\theta} (T_1(u_n)+\varepsilon)^\theta \\ &\quad + \int_\Omega f_n (T_1(u_n)+\varepsilon)^{\theta-\gamma} \leq ||b||_{L^\infty(\Omega)} \int_{\{u_n\geq 1\}} |\nabla u_n|^q \\ &\quad + ||b||_{L^\infty(\Omega)} \int_{\{u_n<1\}} |\nabla u_n|^q + (1+\varepsilon)^{\theta-\gamma} \int_\Omega f. \end{aligned} \tag{3.28}$$

Using Young inequality together with (3.26) and (3.27) and the fact that $\frac{q(1-\gamma)}{p-q} < p+\gamma$ yields that

$$\begin{aligned} \int_{\{u_n\geq 1\}} |\nabla u_n|^q &= \int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^q}{u_n^{\frac{(1-\gamma)q}{p}}} u_n^{\frac{(1-\gamma)q}{p}} \leq C \int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^p}{u_n^{(1-\gamma)}} + C\int_\Omega u_n^{p+\gamma} \\ &\leq C. \end{aligned}$$

Then we deduce from (3.28) and the above estimate, using again young inequality, we obtain

$$\begin{aligned} &\alpha\theta \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} \\ &\leq ||b||_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla T_1(u_n)|^q (T_1(u_n)+\varepsilon)^{\frac{q}{p}(1-\theta)}}{(T_1(u_n)+\varepsilon)^{\frac{q}{p}(1-\theta)}} + (1+\varepsilon)^{\theta-\gamma}\int_\Omega f + C \\ &\leq \frac{\alpha\theta}{p} \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} + C(1+\varepsilon)^{\frac{q}{p-q}(1-\theta)} + (1+\varepsilon)^{\theta-\gamma}\int_\Omega f + C, \end{aligned} \tag{3.29}$$

it follows that

$$\int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} \leq C\left((1+\varepsilon)^{\frac{q}{p-q}(1-\theta)} + (1+\varepsilon)^{\theta-\gamma}\right).$$

Thus, we obtain

$$\begin{aligned} \int_\Omega |\nabla T_1(u_n)|^p &= \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} (T_1(u_n)+\varepsilon)^{1-\theta} \\ &\leq C(1+\varepsilon)^{1-\theta}\left((1+\varepsilon)^{\frac{q}{p-q}(1-\theta)} + (1+\varepsilon)^{\theta-\gamma}\right). \end{aligned}$$

Hence, taking ε tends to 0, we deduce that

$$\int_\Omega |\nabla T_1(u_n)|^p \leq C, \tag{3.30}$$

from (3.26) and (3.30) we conclude that

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1-\gamma}} \leq \int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + \int_\Omega |\nabla T_1(u_n)|^p \leq C. \tag{3.31}$$

Let $1 \leq r < p$, using the estimate (3.31) together with Hölder inequality we arrive at

$$\int_\Omega |\nabla u_n|^r \leq \int_\Omega \frac{|\nabla u_n|^r}{(1+u_n)^{\frac{r(1-\gamma)}{p}}}(1+u_n)^{\frac{r(1-\gamma)}{p}} \leq C\left(\int_\Omega (1+u_n)^{\frac{r(1-\gamma)}{p-r}}\right)^{1-\frac{r}{p}}, \tag{3.32}$$

starting from (3.32) and thanks to (3.27) noticing that $\frac{r(1-\gamma)}{p-r} \leq p+\gamma$ is equivalent to $r \leq \frac{p(p+\gamma)}{p+1}$, we Thus obtain

$$\int_\Omega |\nabla u_n|^r \leq C, \quad \forall 1 \leq r \leq \frac{p(p+\gamma)}{p+1} < p. \tag{3.33}$$

Thus, recalling (2.5), (1.5), estimate (3.33) and by means of Hölder inequality, it follows for every $\omega \subset\subset \Omega$ that

$$\int_\omega \frac{|\nabla u_n|^q}{u_n^\theta} \leq \frac{|\Omega|^{\frac{r-q}{r}}}{c_\omega^\theta}\|u_n\|^q_{W_0^{1,r}(\Omega)} \leq C. \tag{3.34}$$

In the case $\gamma \geq \theta$, we can obtaining the results, changing γ by θ in the exponents of the test functions and namely arguing exactly as above. Then Lemma 3.10 is completely proved.

We prove now the following convergence result.

Proposition 3.11. *Under assumption* (1.3), *we have*

$$u_n^p \to u^p \text{ strongly in } L^1(\Omega).$$

Proof. We take $T_1(u_n - T_h(u_n))$ as test function in (2.1) dropping the positive term, using (1.3), (1.4) and we then have

$$\begin{aligned}
&\alpha \int_{\{h \leq u_n \leq h+1\}} |\nabla u_n|^p + \mu \int_{\{u_n \geq h+1\}} u_n^p \\
&\leq ||b||_{L^\infty(\Omega)} \int_{\{h \leq u_n \leq h+1\}} |\nabla u_n|^q \\
&\quad + ||b||_{L^\infty(\Omega)} \int_{\{u_n > h+1\}} |\nabla u_n|^q + \frac{1}{h^\gamma}\int_{\{u_n \geq h\}} f,
\end{aligned}$$

which implies using (3.33), Young together with Hölder inequalities that

$$\begin{aligned}
&\frac{\alpha}{p}\int_{\{h \leq u_n \leq h+1\}} |\nabla u_n|^p + \mu \int_{\{u_n \geq h+1\}} u_n^p \\
&\leq C|u_n > h|^{1-\frac{q}{p}} + ||b||_{L^\infty(\Omega)}\|u_n\|^q_{W_0^{1,r}(\Omega)}|u_n > h|^{\frac{r-q}{r}} + \frac{1}{h^\gamma}\int_{\{u_n \geq h\}} f \\
&\leq C|u_n > h|^{1-\frac{q}{p}} + C|u_n > h|^{\frac{r-q}{r}} + \frac{1}{h^\gamma}\int_{\{u_n \geq h\}} f.
\end{aligned}$$

Letting $n \to +\infty$ and then $h \to +\infty$, we obtain

$$\int_{\{u_n \geq h+1\}} u_n^p \leq w(n,h), \tag{3.35}$$

where $w(n, h)$ tends to zero when $n \to +\infty$ and $h \to +\infty$. Let E be a measurable subset of Ω, we have

$$\int_E u_n^p \le \int_{\{u_n>h\}} u_n^p + h^p|E|.$$

Then, thanks to (3.35), we take the limit as $|E|$ tends to zero, h tends to infinity and since u_n^p converges to u^p almost everywhere, we easily conclude by Vitali's Theorem the proof of Proposition 3.11. □

Proof of Theorem 3.9. Using Proposition 3.11 and Lemma 3.10, we can obtain a solution passing to the limit, namely arguing exactly as in Theorem 3.2. □

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