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Zygmund-Type Integral Inequalities for Complex Polynomials

Abdullah Mir, N. A. Rather and Ishfaq Dar

Abstract. Let $\mathcal{P}_n$ be the class of all complex polynomials of degree at most n, and let $T : \mathcal{P}_n \to \mathcal{P}_n$ be a linear operator. We shall say that T is a B_n-operator if for every $P \in \mathcal{P}_n$ having all zeros in the closed unit disk, $T[P]$ has all its zeros in the closed unit disk. Recently, Rather et al. [On the zeros of certain composite polynomials and an operator preserving inequalities, Ramanujan J., **54** (2021), 605–617] introduced and considered the generalized B_n-operator $N_m : \mathcal{P}_n \to \mathcal{P}_n$, defined by $N_m[P](z) := \sum_{k=0}^{m} \lambda_k \left(\frac{nz}{2}\right)^k \frac{P^{(k)}(z)}{k!}$, $m \leq n$, where $\lambda_k \in \mathbb{C}$, $k = 0, 1, 2, \ldots, m$, are such that the polynomial $\phi(z) := \sum_{k=0}^{m} \binom{n}{k} \lambda_k z^k$ has all its zeros in the half plane $Re(z) \leq \frac{n}{4}$. They established several sharp Bernstein-type inequalities for this operator giving extensions and generalizations of various classical polynomial inequalities. In this paper, we establish sharp integral-norm estimates of Zygmund type for this operator and develop a unified method for getting various Bernstein-type inequalities and other related inequalities in the supremum-norm as special cases.

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1. Introduction

The study of inequalities and relating the norm between polynomials over some compact disk is a classical topic in analysis. Bernstein-type inequalities are known on various regions of the complex plane in different norms and for different classes of functions. In the past few years, a series of papers related to polynomial inequalities in both directions have appeared in the literature and significant advances have been achieved. Here, we study some of the new inequalities centered around Bernstein-type inequalities that relate the integral-norm of the generalized B_n-operator and the polynomial under some conditions.

Birkhäuser

By $\mathcal{P}_n$, we denote the class of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n. For brevity, we define the Hardy space norm for $P \in \mathcal{P}_n$ by

$$\|P\|_{H^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty,$$

and the Mahler measure by

$$\|P\|_{H^0} = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| d\theta \right).$$

It is well known that $\lim_{p \to 0+} \|P\|_{H^p} = \|P\|_{H^0}$. Also, note that the supremum-norm of the space H^∞ satisfies $\|P\|_{H^\infty} := \lim_{p\to\infty} \|P\|_{H^p} = \max_{|z|=1} |P(z)|$.

If $P \in \mathcal{P}_n$ and $\sigma(z) = Rz$, then

$$\|P'\|_{H^p} \leq n\|P\|_{H^p}, \quad 0 \leq p \leq \infty \tag{1.1}$$

and for $R > 1$,

$$\|P \circ \sigma\|_{H^p} \leq R^n \|P\|_{H^p}, \quad p > 0. \tag{1.2}$$

Inequality (1.1) was originally proved by Bernstein [4] (see also [14,18]) for $p = \infty$ and extended to Hardy space norm by Zygmund [21] for $p \geq 1$. His proof uses Riesz's interpolation formula by means of Minkowski's inequality, it was not clear, whether the restriction on p was indeed essential. This question remained open for a long time in spite of partial answer by Maté and Nevai [13]. Later, Arestov [2] proved that the inequality (1.1) is true for $0 < p < 1$ as well. For $p = 0$, (1.1) is a consequence of a remarkable inequality of de-Bruijn and Springer [7]. On the other hand, the inequality (1.2) is a simple consequence of a result of Hardy [10] and for $p = \infty$, it is a simple deduction from the maximum modulus principle [17, p.346].

For the class of polynomials $P \in \mathcal{P}_n$ having no zeros in $|z| < 1$, the inequalities (1.1) and (1.2) can be improved as follows:

$$\|P'\|_{H^p} \leq \frac{n\|P\|_{H^p}}{\|1+z\|_{H^p}}, \quad 0 \leq p \leq \infty, \tag{1.3}$$

and

$$\|P \circ \sigma\|_{H^p} \leq \frac{\|R^n z + 1\|_{H^p}}{\|1+z\|_{H^p}} \|P\|_{H^p}, \quad R > 1,\ 0 \leq p \leq \infty. \tag{1.4}$$

For $p = \infty$, the inequality (1.3) was conjectured by Erdös and later proved by Lax [12], while de-Bruijn [6] proved (1.3) for $p \geq 1$, and Rahman and Schmeisser [19] validated it for $0 \leq p < 1$ as well. Whereas, the inequality (1.4) was proved by Boas and Rahman [5] for $p \geq 1$, and Rahman and Schmeisser [19] have shown that (1.4) holds for $0 \leq p < 1$ as well. The case $p = \infty$ of inequality (1.4) is due to Ankeny and Rivlin [1].

As a compact generalization of inequalities (1.1) and (1.3) for the case $p = \infty$, Jain [11] proved that if $P \in \mathcal{P}_n$, then for every $\beta \in \mathbb{C}$ with $|\beta| \le 1$,

$$\left\| zP' + \beta \frac{n}{2} P \right\|_{H^\infty} \le n \left| 1 + \frac{\beta}{2} \right| \|P\|_{H^\infty}, \tag{1.5}$$

and if $P(z) \ne 0$ in $|z| < 1$, then

$$\left\| zP' + \beta \frac{n}{2} P \right\|_{H^\infty} \le \frac{n}{2} \left\{ \left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right\} \|P\|_{H^\infty}. \tag{1.6}$$

Recently, Rather et al. [20] studied the comparative position of the zeros of a polynomial which is derived by the 'composition' of two polynomials and obtained the following result:

Theorem A. *If all the zeros of polynomial $f(z)$ of degree n lie in $|z| \le r$ and if all the zeros of the polynomial*

$$g(z) = \lambda_0 + \binom{n}{1} \lambda_1 z + \cdots + \binom{n}{m} \lambda_m z^m, \qquad m \le n,$$

lie in $|z| \le s|z - \sigma|$, $s > 0$, then the polynomial

$$h(z) = \lambda_0 f(z) + \lambda_1 f'(z) \frac{(\sigma z)}{1!} + \cdots + \lambda_m f^{(m)}(z) \frac{(\sigma z)^m}{m!}$$

has all its zeros in $|z| \le r \max(1, s)$.

As an application of the above result, they [20] introduced a linear operator $N_m : \mathcal{P}_n \to \mathcal{P}_n$, defined by

$$N_m[P](z) := \sum_{k=0}^{m} \lambda_k \left(\frac{nz}{2} \right)^k \frac{P^{(k)}(z)}{k!}, \tag{1.7}$$

where $\lambda_k \in \mathbb{C}$, $k = 0, 1, 2, \ldots, m$, are such that the polynomial

$$\phi(z) := \sum_{k=0}^{m} \binom{n}{k} \lambda_k z^k, \quad m \le n,$$

has all its zeros in the half plane $Re(z) \le \frac{n}{4}$, and established various new Bernstein-type polynomial inequalities in the supremum-norm. More precisely they proved:

Theorem B. *If $P \in \mathcal{P}_n$, then*

$$|N[P](z)| \le |N[\psi_n](z)| \, \|P\|_{H^\infty}, \qquad \text{for} \quad |z| \ge 1, \tag{1.8}$$

where $\psi_n(z) = z^n$. The result is sharp and equality in (1.8) holds for $P(z) = e^{i\alpha} M z^n$, $\alpha \in \mathbb{R}$.

Next, they [20] (see also [16]) established the following result for the class of polynomials having no zeros inside the unit circle $|z| = 1$.

Theorem C. *If $P \in \mathcal{P}_n$ and $P(z) \ne 0$ in $|z| < 1$, then*

$$|N[P](z)| \le \frac{1}{2} \Big(|N[\psi_n](z)| + |\lambda_0| \Big) \|P\|_{H^\infty}, \qquad \text{for} \quad |z| \ge 1. \tag{1.9}$$

The result is best possible and equality in (1.9) holds for $P(z) = az^n + b$, $|a| = |b| \ne 0$.

A polynomial $P \in \mathcal{P}_n$ is said be self-inversive if $P(z) = P^\star(z)$ where $P^\star(z) := z^n\overline{P(1/\overline{z})}$. In the same paper, Rather et al. [20] proved that the inequality (1.9) also holds for self-inversive polynomials. These types of inequalities for constrained polynomials are nowadays a widely studied topic, and for the latest development in this direction, we refer the interested reader to ([9,15,16]). In this manuscript, we are interested to establish the integral-norm estimates of the above inequalities and their various refinements, for which we introduce the following notations:

$$\psi_n(z) = z^n \qquad \text{and} \qquad \Lambda := N_m[\psi_n](1) = N_m[z^n](1) = \sum_{k=0}^{m} \lambda_k \binom{n}{k} \left(\frac{n}{2}\right)^k. \tag{1.10}$$

2. Main Results

In this paper, we establish certain estimates in H^p-norm for $|N_m[P \circ \sigma] - \alpha N_m[P]|$ which among other things shows that the operator N_m preserves Zygmund-type polynomial inequalities. In this direction, we first present the following result, which extends Theorem B to H^p-norm.

Theorem 2.1. *If $P \in \mathcal{P}_n$ then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $0 \leq p < \infty$ and $R \geq 1$,*

$$\|N_m[P \circ \sigma](z) - \alpha N_m[P](z)\|_{H^p} \leq |R^n - \alpha|\,|N_m[z^n](1)|\,\|P(z)\|_{H^p}, \tag{2.1}$$

where $N_m[z^n](1)$ is given by (1.10). The result is best possible and equality in (2.1) holds for $P(z) = cz^n,\ c \neq 0$.

If we choose $\alpha = 0$ in (2.1), we get the following result.

Corollary 2.1. *If $P \in \mathcal{P}_n$ then for $0 \leq p < \infty$ and $R > 1$,*

$$\|N_m[P \circ \sigma](z)\|_{H^p} \leq R^n\,|N_m[z^n](1)|\,\|P(z)\|_{H^p}\,, \tag{2.2}$$

where $N_m[z^n](1)$ is given by (1.10). The result is sharp and equality in (2.2) holds for $P(z) = z^n$.

Taking $\lambda_i = 0\ \forall\ i < m$ and $\lambda_m \neq 0$ in Corollary2.1, it follows that if $P \in \mathcal{P}_n$, then we get for $R \geq 1$ and $0 \leq p < \infty$,

$$\|P^{(m)} \circ \varphi\|_{H^p} \ \leq \frac{n!\ R^{n-m}}{(n-m)!}\|P\|_{H^p}, \qquad m \leq n,$$

which includes inequalities (1.1) and (1.2) as a special cases. Taking $R = 1$ in Corollary 2.1, we get the following result:

Corollary 2.2. *If $P \in \mathcal{P}_n$ and $\psi_n(z) = z^n$, then for every $0 \leq p < \infty$,*

$$\|N_m[P]\|_{H^p} \leq |N_m[\psi_n](1)|\,\|P\|_{H^p}.$$

The result is sharp, as shown by $P(z) = az^n,\ a \neq 0$.

If we let $p \to \infty$, then Corollary 2.2 reduces to Theorem B. Taking

$$\lambda_0 = \frac{n!}{(n-m)!}\frac{\beta}{2^m},$$
$$\lambda_k = 0, \quad \text{for } k = 1, 2, \ldots, m-1,$$

$$\text{and} \qquad \lambda_m = \frac{2^m m!}{n^m},$$

in (1.7) then $\phi(z) = \frac{n!}{(n-m)!2^m}\left\{\beta + \left(\frac{4z}{n}\right)^m\right\}$ has all its zeros in the half plane $Re(z) \leq \frac{n}{4}$ for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$. Thus, from Corollary 2.2, we obtain the following extension of (1.5) in H^p-norm in a more generalized form.

Corollary 2.3. *If $P \in \mathcal{P}_n$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $0 \leq p < \infty$,*

$$\left\| z^m P^{(m)} + \frac{n!}{(n-m)!}\frac{\beta}{2^m}P \right\|_{H^p} \leq \frac{n!}{(n-m)!}\left|1 + \frac{\beta}{2^m}\right| \|P\|_{H^p}, m = 1, 2, \ldots, n. \quad (2.3)$$

If we let $p \to \infty$ in (2.3), we recover (1.5) when $m = 1$. The following estimate for the two leading coefficients of a polynomial readily follows from the above corollary by taking $\beta = 0$ and $m = n - 1$.

Corollary 2.4. *If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then for $0 \leq p < \infty$,*

$$\left\| a_n z + \frac{a_{n-1}}{n} \right\|_{H^p} \leq \|P\|_{H^p}.$$

Theorem 2.1 can be sharpened if we restrict ourselves to the class of polynomials $P \in \mathcal{P}_n$ which does not vanish in $|z| < 1$. In this direction, we next establish the following result, which extends Theorem C to H^p-norm.

Theorem 2.2. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $0 \leq p < \infty$ and $R > 1$,*

$$\|N_m[P \circ \sigma](z) - \alpha N_m[P](z)\|_{H^p} \leq \frac{\|(R^n - \alpha)N_m[z^n](1)z + (1-\alpha)\lambda_0\|_{H^p}}{\|1 + z\|_{H^p}} \|P(z)\|_{H^p}, \quad (2.4)$$

where $N_m[z^n](1)$ is defined by (1.10). The result is best possible and equality in (2.4) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

For $\alpha = 0$, Theorem 2.2 reduces to the following result.

Corollary 2.5. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $0 \leq p < \infty$ and $R > 1$,*

$$\|N_m[P \circ \sigma](z)\|_{H^p} \leq \frac{\|R^n N_m[z^n](1)z + \lambda_0\|_{H^p}}{\|1 + z\|_{H^p}} \|P(z)\|_{H^p}, \quad (2.5)$$

where $N_m[z^n](1)$ is defined by (1.10). The result is sharp as shown by $P(z) = az^n + b$, $|a| = |b| = 1$.

For $m = 0$, inequality (2.5) reduces to inequality (1.4). On taking $R = 1$ and letting $p \to \infty$, Corollary 2.5 reduces to Theorem C. Again, if we choose $\lambda_0 = \frac{n!}{(n-m)!}\frac{\beta}{2^m}$, where $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $\lambda_k = 0$ for $k = 1, 2, \ldots, m-1$ and $\lambda_m = \frac{2^m m!}{n^m}$, in (2.5), then similarly as in the case of Corollary 2.3, we get the following result, which includes H^p-norm extension of inequality (1.6) as a special case.

Corollary 2.6. *If $P \in \mathcal{P}_n$, has no zero in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $0 \leq p < \infty$,*

$$\left\| z^m P^{(m)} + \frac{n!}{(n-m)!}\frac{\beta}{2^m} P \right\|_{H^p} \leq \frac{n!}{(n-m)!} \left\| \left(1+\frac{\beta}{2^m}\right) z + \frac{\beta}{2^m} \right\|_{H^p} \|P\|_{H^p}, m = 1, 2, \ldots, n.$$

The result is best possible and the extremal polynomial is $P(z) = z^n + 1$.

By choosing $\lambda_i = 0 \; \forall \; i < m$ and $\lambda_m \neq 0$ in Corollary 2.5, we obtain the following result which includes inequality (1.3) is a special case.

Corollary 2.7. *If* $P \in \mathcal{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then*

$$\|P^{(m)} \circ \varphi\|_{H^p} \leq \frac{n! \; R^{n-m}}{(n-m)! \; \|1+z\|_{H^p}} \|P(z)|_{H^p}, \qquad 1 \leq m \leq n.$$

Finally, we establish the following result for self-inversive polynomials.

Theorem 2.3. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *is a self-inversive polynomial, then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \leq 1$, $0 \leq p < \infty$ *and* $R > 1$,

$$\|N_m[P \circ \sigma](z) - \alpha N_m[P](z)\|_{H^p} \leq \frac{\|(R^n - \alpha) N_m[z^n](1) z + (1-\alpha)\lambda_0\|_{H^p}}{\|1+z\|_{H^p}} \|P(z)\|_{H^p}, \tag{2.6}$$

where $N_m[z^n](1)$ *is given by* (1.10). *The result is sharp and the extremal polynomial is* $P(z) = c(az^n + \overline{a})$, $ac \neq 0$.

Setting $\alpha = 0$ in the above theorem, we get the following result.

Corollary 2.8. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *is a self-inversive polynomial, then for* $0 \leq p < \infty$ *and* $R > 1$,

$$\|N_m[P \circ \sigma](z)\|_{H^p} \leq \frac{\|R^n N_m[z^n](1) z + \lambda_0\|_{H^p}}{\|1+z\|_{H^p}} \|P(z)\|_{H^p}, \tag{2.7}$$

where $N_m[z^n](1)$ *is given by* (1.10). *The result is sharp.*

The following result is an immediate consequence of Corollary 2.8.

Corollary 2.9. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *is a self-inversive polynomial, then for* $0 \leq p < \infty$ *and* $R > 1$,

$$\|N_m[P \circ \sigma](z)\|_{H^p} \leq \frac{R^n \, |N_m[z^n](1)| + |\lambda_0|}{\|1+z\|_{H^p}} \|P(z)\|_{H^p}, \tag{2.8}$$

where $N_m[z^n](1)$ *is given by* (1.10).

Remark 2.1. As the conclusion of Theorem 2.3 is the same as that of Theorem 2.2, consequently Corollaries 2.6 and 2.7 hold for self-inversive polynomials as well. Various results concerning self-inversive polynomials due to Aziz and Rather [3], and Dewan and Govil [8] can also be obtained from Theorem 2.3.

3. Auxiliary Results

For the proofs of these theorems, we need the following lemmas. The first lemma follows by taking $r = s = 1$ and $\sigma = \frac{n}{2}$ in Theorem A.

Lemma 3.1. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \leq 1$, then all the zeros of $N_m[P](z)$ also lie in $|z| \leq 1$.*

Lemma 3.2. *If $P \in \mathcal{P}_n$ and $P(z)$ have all its zeros in $|z| \leq 1$ then for every $R > 1$, and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+1}{2}\right)^n |P(z)|.$$

Proof. Since all the zeros of $P(z)$ lie in $|z| \leq 1$, we write

$$P(z) = C \prod_{j=1}^{n} \left(z - r_j e^{i\theta_j}\right),$$

where $r_j \leq 1$. Now for $0 \leq \theta < 2\pi$, $R > 1$, we have

$$\begin{aligned} \left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\right| &= \left\{\frac{R^2 + r_j^2 - 2Rr_j \cos(\theta - \theta_j)}{1 + r_j^2 - 2r_j \cos(\theta - \theta_j)}\right\}^{1/2} \\ &\geq \left\{\frac{R + r_j}{1 + r_j}\right\} \\ &\geq \left\{\frac{R+1}{2}\right\}, \text{for } j = 1, 2, \cdots, n. \end{aligned}$$

Hence,

$$\begin{aligned} \left|\frac{P(Re^{i\theta})}{P(e^{i\theta})}\right| &= \prod_{j=1}^{n} \left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\right| \\ &\geq \prod_{j=1}^{n} \left(\frac{R+1}{2}\right) \\ &= \left(\frac{R+1}{2}\right)^n, \end{aligned}$$

for $0 \leq \theta < 2\pi$. This implies for $|z| = 1$ and $R > 1$,

$$|P(Rz)| \geq \left(\frac{R+1}{2}\right)^n |P(z)|,$$

which completes the proof of Lemma 3.2. □

Lemma 3.3. *If $P \in \mathcal{P}_n$ and $P(z)$ has no zero in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > 1$ and $|z| \geq 1$,*

$$|N_m[P \circ \sigma](z) - \alpha N_m[P](z)| \leq |N_m[P^\star \circ \sigma](z) - \alpha N_m[P^\star](z)|, \quad (3.1)$$

where $P^\star(z) = z^n \overline{P(1/\overline{z})}$.

Proof. Since the polynomial $P(z)$ has all its zeros in $|z| \geq 1$, therefore, for every real or complex number λ with $|\lambda| > 1$, the polynomial $f(z) = P(z) - \lambda P^\star(z)$, where $P^\star(z) = z^n\overline{P(1/\overline{z})}$, has all zeros in $|z| \leq 1$. Applying Lemma 3.2 to the polynomial $f(z)$, we obtain for every $R > 1$ and $0 \leq \theta < 2\pi$,

$$|f(Re^{i\theta})| \geq \left(\frac{R+1}{2}\right)^n |f(e^{i\theta})|. \tag{3.2}$$

Since $f(Re^{i\theta}) \neq 0$ for every $R > 1$, $0 \leq \theta < 2\pi$ and $R+1 > 2$, it follows from (3.2) that

$$|f(Re^{i\theta})| > \left(\frac{R+1}{2}\right)^n |f(Re^{i\theta})| \geq |f(e^{i\theta})|,$$

for every $R > 1$ and $0 \leq \theta < 2\pi$. This gives

$$|f(z)| < |f(Rz)| \text{ for } |z| = 1, \text{ and } R > 1.$$

Using Rouché's theorem and noting that all the zeros of $f(Rz)$ lie in $|z| \leq 1/R < 1$, we conclude that the polynomial

$$T(z) = f(Rz) - \alpha f(z) = \{P(Rz) - \alpha P(z)\} - \lambda\{P^\star(Rz) - \alpha P^\star(z)\}$$

has all its zeros in $|z| < 1$ for every real or complex α with $|\alpha| \geq 1$ and $R > 1$. Applying Lemma 3.1 to polynomial $T(z)$ and noting that N_m is a linear operator, it follows that all the zeros of polynomial

$$\begin{aligned} N_m[T](z) &= N_m[f \circ \sigma](z) - \alpha N_m[f](z) \\ &= \{N_m[P \circ \sigma](z) - \alpha N_m[P](z)\} - \lambda\{N_m[P^\star \circ \sigma](z) - \alpha N_m[P^\star](z)\} \end{aligned}$$

lie in $|z| < 1$ where $\sigma(z) = Rz$. This implies

$$|N_m[P \circ \sigma](z) - \alpha N_m[P](z)| \leq |N_m[P^\star \circ \sigma](z) - \alpha N_m[P^\star](z)| \tag{3.3}$$

for $|z| \geq 1$ and $R > 1$. If inequality (3.3) is not true, then there exists a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|N_m[P \circ \sigma](z_0) - \alpha N_m[P](z_0)| > |N_m[P^\star \circ \sigma](z_0) - \alpha N_m[P^\star](z_0)|. \tag{3.4}$$

But all the zeros of $P^\star(Rz)$ lie in $|z| < 1/R < 1$; therefore, it follows (as in case of $f(z)$) that all the zeros of $P^\star(Rz) - \alpha P^\star(z)$ lie in $|z| < 1$. Hence, by Lemma 3.1, we have

$$N_m[P^\star \circ \sigma](z_0) - \alpha N_m[P^\star](z_0) \neq 0.$$

We take

$$\lambda = \frac{N_m[P \circ \sigma](z_0) - \alpha N_m[P](z_0)}{N_m[P^\star \circ \sigma](z_0) - \alpha N_m[P^\star](z_0)},$$

then λ is well-defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $N_m[T](z_0) = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $N_m[T](z)$ lie in $|z| < 1$. Thus (3.3) holds true for $|\alpha| \leq 1$ and $R > 1$. □

Next we describe a result of Arestov [2].

For $\delta = (\delta_0, \delta_1, \cdots, \delta_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$, we define

$$\Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j.$$

The operator Λ_δ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,

(ii) $P(z)$ has all its zeros in$\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov [2, Theorem 4] may now be stated as follows.

Lemma 3.4. *Let $\phi(x) = \psi(logx)$ where ψ is a convex non decreasing function on $\mathbb{R}$. Then, for all $P \in \mathcal{P}_n$ and each admissible operator Λ_δ,*

$$\int_0^{2\pi} \phi(|\Lambda_\delta P(e^{i\theta})|)d\theta \leq \int_0^{2\pi} \phi(C(\delta, n)|P(e^{i\theta})|)d\theta,$$

where $C(\delta, n) = max(|\delta_0|, |\delta_n|)$.

In particular, Lemma 3.4 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$ and with $\phi : x \to \log x$ as well. Therefore, we have for $0 \leq p < \infty$,

$$\left\{\int_0^{2\pi} (|\Lambda_\delta P(e^{i\theta})|^p)d\theta\right\}^{1/p} \leq C(\delta, n)\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{3.5}$$

We use (3.5) to prove the following interesting result.

Lemma 3.5. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every $p > 0$, $R > 1$ and for γ real, $0 \leq \gamma < 2\pi$,*

$$\begin{aligned}\int_0^{2\pi} \Big|\Big\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\Big\} e^{i\gamma} \\ + \big\{N_m[P^\star \circ \sigma]^\star(e^{i\theta}) - \overline{\alpha} N_m[P^\star]^\star(e^{i\theta})\big\}\Big|^p d\theta \\ \leq \Big|(R^n - \alpha)\varLambda e^{i\gamma} + (1 - \overline{\alpha})\overline{\lambda_0}\Big|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta,\end{aligned} \tag{3.6}$$

where $N_m[P^\star \circ \sigma]^\star(z) := (N_m[P^\star \circ \sigma](z))^\star$ and $\varLambda$ is defined by (1.10).

Proof. Recall that $P \in \mathcal{P}_n$ has all zeros in $|z| \geq 1$ and $P^*(z) = z^n \overline{P(1/\overline{z})}$, by Lemma 3.3, we have

$$|N_m[P \circ \sigma](z) - \alpha N_m[P](z)| \leq |N_m[P^* \circ \sigma](z) - \alpha N_m[P^*](z)| \quad \text{for} \quad |z| = 1. \tag{3.7}$$

Also, $P^* \circ \sigma(z) - \alpha P^*(z) = P^*(Rz) - \alpha P^*(z) = R^n z^n \overline{P(1/R\overline{z})} - \alpha z^n \overline{P(1/\overline{z})}$, this implies,

$$\begin{aligned}N_m[P^* \circ \sigma - \alpha P^*](z) &= N_m[P^* \circ \sigma](z) - \alpha N_m[P^*](z) \\ &= \lambda_0 \left[R^n z^n \overline{P(1/R\overline{z})} - \alpha \left(z^n \overline{P(1/\overline{z})}\right)\right] \\ &\quad + \lambda_1 \left(\frac{nz}{2}\right)\left[\binom{n}{1} R^n z^{n-1} \overline{P(1/R\overline{z})} - \binom{n}{0} R^{n-1} z^{n-2} \overline{P'(1/R\overline{z})}\right.\end{aligned}$$

$$-\alpha\left(\binom{n}{1}z^{n-1}\overline{P\big(1/\overline{z}\big)}-\binom{n}{0}z^{n-2}\overline{P'\big(1/\overline{z}\big)}\right)\Bigg]$$
$$+\ldots+\frac{\lambda_m}{m!}\left(\frac{nz}{2}\right)^m\Bigg[\frac{m!}{0!}\binom{n}{m}R^n z^{n-m}\overline{P\big(1/R\overline{z}\big)}$$
$$-\frac{m!}{1!}\binom{n}{m-1}R^{n-1}z^{n-m-1}\overline{P'\big(1/R\overline{z}\big)}+\ldots$$
$$+(-1)^m\frac{m!}{m!}\binom{n}{0}R^{n-m}z^{n-2m}\overline{P^{(m)}\big(1/R\overline{z}\big)}$$
$$-\alpha\Bigg(\frac{m!}{0!}\binom{n}{m}z^{n-m}\overline{P\big(1/\overline{z}\big)}-\frac{m!}{1!}\binom{n}{m-1}z^{n-m-1}\overline{P'\big(1/\overline{z}\big)}+\ldots$$
$$+(-1)^m\frac{m!}{m!}\binom{n}{0}z^{n-2m}\overline{P^{(m)}\big(1/\overline{z}\big)}\Bigg)\Bigg].$$

This gives

$$\begin{aligned}&N_m[P^*\circ\sigma]^*(z)-\overline{\alpha}N_m[P^*]^*(z)=(N_m[P^*\circ\sigma-\alpha P^*])^*(z)\\&=\left\{\overline{\lambda_0}+\overline{\lambda_1}\frac{n}{2}\binom{n}{1}+....+\overline{\lambda_m}\left(\frac{n}{2}\right)^m\binom{n}{m}\right\}\{R^nP(z/R)-\overline{\alpha}P(z)\}\\&\quad-\left\{\overline{\lambda_1}\frac{n}{2}+\overline{\lambda_2}\left(\frac{n}{2}\right)^2\binom{n}{1}+\ldots+\overline{\lambda_m}\left(\frac{n}{2}\right)^m\binom{n}{m-1}\right\}\{R^{n-1}zP'(z/R)-\overline{\alpha}zP'(z)\}\\&\quad+\ldots+(-1)^m\frac{\overline{\lambda_m}}{m!}\left(\frac{n}{2}\right)^m\binom{n}{0}\left\{R^{n-m}z^mP^{(m)}(z/R)-\overline{\alpha}z^mP^{(m)}(z)\right\}.\end{aligned}\tag{3.8}$$

Also, $|N_m[P^*\circ\sigma](z)-\alpha N_m[P^*](z)|=|N_m[P^*\circ\sigma]^*(z)-\overline{\alpha}N_m[P^*]^*(z)|$ for $|z|=1$; therefore, by using (3.7), we get

$$|N_m[P\circ\sigma](z)-\alpha N_m[P](z)|\leq|N_m[P^*\circ\sigma]^*(z)-\overline{\alpha}N_m[P^*]^*(z)|\qquad\text{for } |z|=1,\ R>1.$$

Since $P^*\circ\sigma-\alpha P^*\in\mathcal{P}_n$ has all zeros in $|z|<1$ then by Lemma 3.1, $N_m[P^*\circ\sigma-\alpha P^*]\in\mathcal{P}_n$ has all zeros in $|z|<1$. This implies the polynomial $N_m[P^*\circ\sigma-\alpha P^*]^*(z)=N_m[P^*\circ\sigma]^*(z)-\overline{\alpha}N_m[P^*]^*(z)\in\mathcal{P}_n$ has all zeros in $|z|>1$. Hence, by the maximum modulus principle,

$$|N_m[P\circ\sigma](z)-\alpha N_m[P](z)|\leq|N_m[P^*\circ\sigma]^*(z)-\overline{\alpha}N_m[P^*]^*(z)|\qquad\text{for } |z|<1.\tag{3.9}$$

A direct application of Rouché's theorem shows that if $P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_0$, then

$$\begin{aligned}&C_\delta P(z)=\{N_m[P\circ\sigma](z)-\alpha N_m[P](z)\}e^{i\gamma}+N_m[P^*\circ\sigma]^*(z)-\overline{\alpha}N_m[P^*]^*(z)\\&=\left\{(R^n-\alpha)\left(\lambda_0+\lambda_1\frac{n}{2}\binom{n}{1}+\ldots+\lambda_m\left(\frac{n}{2}\right)^m\binom{n}{m}\right)e^{i\gamma}+(1-\overline{\alpha})\overline{\lambda_0}\right\}a_nz^n\\&\quad+\ldots+\left\{(R^n-\overline{\alpha})\left(\overline{\lambda_0}+\overline{\lambda_1}\frac{n}{2}\binom{n}{1}+\ldots+\overline{\lambda_m}\left(\frac{n}{2}\right)^m\binom{n}{m}\right)+e^{i\gamma}(1-\alpha)\lambda_0\right\}a_0,\end{aligned}$$

has all zeros in $|z| \geq 1$ for every $\gamma \in \mathbb{R}$, that is C_δ is an admissible operator. Note that

$$\Lambda = \lambda_0 + \lambda_1 \frac{n}{2}\binom{n}{1} + \ldots + \lambda_m \left(\frac{n}{2}\right)^m \binom{n}{m} = N_m[\psi_n](1) \quad \text{where} \quad \psi_n(z) = z^n.$$

Applying Lemma 3.4 with $\phi(x) = x^p$, where $p > 0$, the desired result follows immediately for every $p > 0$. □

The next lemma shows that the condition on the zeros of $P(z)$ in Lemma 3.5 is not required.

Lemma 3.6. *If $P \in \mathcal{P}_n$, then for every $p > 0$, $R > 1$ and $\alpha \in \mathbb{R}$, $0 \leq \alpha < 2\pi$,*

$$\int_0^{2\pi} \left|\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\} e^{i\gamma} + \left\{N_m[P^* \circ \sigma]^*(e^{i\theta}) - \overline{\alpha} N_m[P^*]^*(e^{i\theta})\right\}\right|^p d\theta$$
$$\leq \left|(R^n - \alpha)\, \Lambda e^{i\gamma} + (1 - \overline{\alpha})\, \overline{\lambda_0}\right|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \tag{3.10}$$

Proof. If $P \in \mathcal{P}_n$, has all its zeros in $|z| \geq 1$, then the result follows by Lemma 3.5. Hence, we assume that $P(z)$ has at least one zero in $|z| < 1$, so we can write

$$P(z) = P_1(z)P_2(z) = a\prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(z - z_j), \quad 0 \leq k \leq n-1,\ a \neq 0,$$

where $z_1, z_2, ..., z_k$ lie in $|z| \geq 1$ and $z_{k+1}, z_{k+2}, ...z_n$ lie in $|z| < 1$. First we suppose that all the zeros of $P_1(z)$ lie in $|z| > 1$. Since all the zeros of $P_2(z)$ lie in $|z| < 1$, the polynomial $P_2^*(z) = z^{n-k}\overline{P_2(1/\overline{z})}$ has all zeros in $|z| > 1$ and $|P_2^*(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$F(z) = P_1(z)P_2^*(z) = a\prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(1 - z\overline{z_j}),$$

then all the zeros of $F(z)$ lie in $|z| > 1$ and for $|z| = 1$,

$$|F(z)| = |P_1(z)||P_2^*(z)| = |P_1(z)||P_2(z)| = |P(z)|. \tag{3.11}$$

By the help of Rouché's theorem, it follows that for every $\beta \in \mathbb{R}$ with $|\beta| > 1$, all the zeros of $G(z) = P(z) + \beta F(z)$ lie in $|z| > 1$, so that $T(z) = G(rz) \in \mathcal{P}_n$, with $r > 1$ has all zeros in $|z| \geq 1$. Proceeding similarly as in (3.8) and (3.9) with regard to polynomial $T(z)$, we get for $R > 1$ and $|z| < 1$,

$$|N_m[T \circ \sigma](z) - \alpha N_m[T](z)| < |N_m[T^* \circ \sigma]^*(z) - \overline{\alpha} N_m[T^*]^*(z)|$$
$$= \Bigg| \left\{\overline{\lambda_0} + \overline{\lambda_1}\left(\frac{n}{2}\right)\binom{n}{1} + \ldots \right.$$
$$\left. + \overline{\lambda_m}\left(\frac{n}{2}\right)^m \binom{n}{m}\right\}(R^n T(z/R) - \overline{\alpha} T(z))$$
$$- \left\{\overline{\lambda_1}\frac{n}{2} + \overline{\lambda_2}\left(\frac{n}{2}\right)^2 \binom{n}{1} + \ldots\right.$$

$$+\overline{\lambda_m}\left(\frac{n}{2}\right)^m\binom{n}{m-1}\Big\}\left(R^{n-1}zT'(z/R)-\overline{\alpha}zT'(z)\right)$$
$$+\ldots+(-1)^m\frac{\overline{\lambda_m}}{m!}\left(\frac{n}{2}\right)^m\binom{n}{0}\left(R^{n-m}z^mT^{(m)}(z/R)-\overline{\alpha}z^mT^{(m)}(z)\right)\Bigg|,$$

that is,

$$|N_m[T\circ\sigma](z)-\alpha N_m[T](z)|<\Bigg|\Big\{\overline{\lambda_0}+\overline{\lambda_1}\left(\frac{n}{2}\right)\binom{n}{1}+\ldots$$
$$+\overline{\lambda_m}\left(\frac{n}{2}\right)^m\binom{n}{m}\Big\}\left(R^nG(rz/R)-\overline{\alpha}G(rz)\right)$$
$$-\Big\{\overline{\lambda_1}\left(\frac{n}{2}\right)+\overline{\lambda_2}\left(\frac{n}{2}\right)^2\binom{n}{1}+\ldots$$
$$+\overline{\lambda_m}\left(\frac{n}{2}\right)^m\binom{n}{m-1}\Big\}\left(R^{n-1}(rz)G'\left(\frac{rz}{R}\right)-\overline{\alpha}(rz)G'\left(rz\right)\right)$$
$$+\ldots+(-1)^m\frac{\overline{\lambda_m}}{m!}\left(\frac{n}{2}\right)^m\binom{n}{0}\left(R^{n-m}(rz)^mG^{(m)}\left(\frac{rz}{R}\right)\right.$$
$$\left.-\overline{\alpha}(rz)^mG^{(m)}\left(rz\right)\right)\Bigg|$$

for $|z|<1$. Now if $z=e^{i\theta}/r,\ 0\leq\theta<2\pi$, then for $|z|=(1/r)<1$ as $r>1$, we get

$$|N_m[T\circ\sigma](e^{i\theta}/r)-\alpha N_m[T](e^{i\theta}/r)|<\Bigg|\Big\{\overline{\lambda_0}+\overline{\lambda_1}\left(\frac{n}{2}\right)\binom{n}{1}+\ldots$$
$$+\overline{\lambda_m}\left(\frac{n}{2}\right)^m\binom{n}{m}\Big\}\left(R^nG\left(\frac{e^{i\theta}}{R}\right)-\overline{\alpha}G\left(e^{i\theta}\right)\right)$$
$$-\Big\{\overline{\lambda_1}\left(\frac{n}{2}\right)+\overline{\lambda_2}\left(\frac{n}{2}\right)^2\binom{n}{1}+\ldots+\overline{\lambda_m}\left(\frac{n}{2}\right)^m\binom{n}{m-1}\Big\}\left(R^{n-1}e^{i\theta}G'\left(\frac{e^{i\theta}}{R}\right)\right.$$
$$\left.-\overline{\alpha}e^{i\theta}G'\left(e^{i\theta}\right)\right)$$
$$+\ldots+(-1)^m\frac{\overline{\lambda_m}}{m!}\left(\frac{n}{2}\right)^m\binom{n-m}{0}\left(R^{n-m}e^{mi\theta}G^{(m)}\left(\frac{e^{i\theta}}{R}\right)-\overline{\alpha}e^{mi\theta}G^{(m)}\left(e^{i\theta}\right)\right)\Bigg|,$$
$$=|N_m[G^*\circ\sigma]^*(e^{i\theta})-\overline{\alpha}N_m[G^*]^*(e^{i\theta})|.$$

Equivalently,

$$|N_m[G\circ\sigma](z)-\alpha N_m[G](z)|<|N_m[G^*\circ\sigma]^*(z)$$
$$-\overline{\alpha}N_m[G^*]^*(e^{i\theta})|\qquad\text{for } |z|=1,$$

where $\sigma(z)=Rz$.

Since $G\in\mathcal{P}_n$ has all zeros in $|z|>1$, it follows that $G^*\in\mathcal{P}_n$ has all zeros in $|z|<1$ and hence $G^*\circ\sigma-\alpha G^*\in\mathcal{P}_n$ has all zeros in $|z|<1$. By Lemma 3.1, it follows that all the zeros of $N_m[G^*\circ\sigma]^*-\overline{\alpha}N_m[G^*]^*\in\mathcal{P}_n$ lie in $|z|>1$. That is, $N_m[G^*\circ\sigma]^*-\overline{\alpha}N_m[G^*]^*\neq 0$ for $|z|\leq 1$.

An application of Rouché's theorem shows that for each $\alpha\in\mathbb{R}$, the polynomial

$$\ell(z)=\{N_m[G\circ\sigma](z)-\alpha N_m[G](z)\}\,e^{i\gamma}+N_m[G^*\circ\sigma]^*(z)$$

$$-\overline{\alpha}N_m[G^*]^*(z) \neq 0 \qquad \text{for } |z| \leq 1. \tag{3.12}$$

Since $G(z) = P(z) + \beta F(z)$ and operator N_m is linear; therefore, from (3.12), it follows that all the zeros of

$$\ell(z) = \{N_m[P \circ \sigma](z) - \alpha N_m[P](z)\} e^{i\gamma} + \{N_m[P^* \circ \sigma]^*(z) - \overline{\alpha}N_m[P^*]^*(z)\} \\ + \beta \{N_m[F \circ \sigma](z) - \alpha N_m[F](z)\} e^{i\gamma} + \{N_m[F^* \circ \sigma]^*(z) - \overline{\alpha}N_m[F^*]^*(z)\}$$

lie in $|z| > 1$. This gives

$$|\{N_m[P \circ \sigma](z) - \alpha N_m[P](z)\} e^{i\gamma} + N_m[P^* \circ \sigma]^*(z) - \overline{\alpha}N_m[P^*]^*(z)| \\ \leq |\{N_m[F \circ \sigma](z) - \alpha N_m[F](z)\} e^{i\gamma} + N_m[F^* \circ \sigma]^*(z) - \overline{\alpha}N_m[F^*]^*(z)|,$$

for all $|z| \leq 1$, which in particular gives for each $p > 0$ and $\gamma \in \mathbb{R}$,

$$\int_0^{2\pi} |\{N_m[P \circ \sigma](z) - \alpha N_m[P](z)\} e^{i\gamma} + N_m[P^* \circ \sigma]^*(z) - \overline{\alpha}N_m[P^*]^*(z)|^p d\theta \\ \leq \int_0^{2\pi} |\{N_m[F \circ \sigma](z) - \alpha N_m[F](z)\} e^{i\gamma} + N_m[F^* \circ \sigma]^*(z) - \overline{\alpha}N_m[F^*]^*(z)|^p d\theta.$$

By applying Lemma 3.5 to $F(z)$ and using (3.11), we obtain for each $p > 0$,

$$\int_0^{2\pi} |\{N_m[P \circ \sigma](z) - \alpha N_m[P](z)\} e^{i\gamma} + N_m[P^* \circ \sigma]^*(z) - \overline{\alpha}N_m[P^*]^*(z)|^p d\theta \\ \leq \left|(R^n - \alpha)\Lambda e^{i\gamma} + (1 - \overline{\alpha})\overline{\lambda_0}\right|^p \int_0^{2\pi} |F(e^{i\theta})|^p d\theta \\ = \left|(R^n - \alpha)\Lambda e^{i\gamma} + (1 - \overline{\alpha})\overline{\lambda_0}\right|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \tag{3.13}$$

Now, if $P_1(z)$ has a zero on $|z| = 1$, then applying (3.13) to the polynomial $E(z) = P_1(tz)P_2(z)$ where $t < 1$, we get for each $p > 0, R > 1$ and $\gamma \in \mathbb{R}$,

$$\int_0^{2\pi} |\{N_m[E \circ \sigma](z) - \alpha N_m[E](z)\} e^{i\gamma} + N_m[E^* \circ \sigma]^*(z) - \overline{\alpha}N_m[E^*]^*(z)|^p d\theta \\ \leq \left|(R^n - \alpha)\Lambda e^{i\gamma} + (1 - \overline{\alpha})\overline{\lambda_0}\right|^p \int_0^{2\pi} |E(e^{i\theta})|^p d\theta. \tag{3.14}$$

Letting $t \to 1$ in (3.14) and using continuity, the desired result follows immediately and this proves Lemma 3.6 completely. □

Lemma 3.7. *If $P \in \mathcal{P}_n$, then for every $p > 0$, $R > 1$ and $\gamma \in \mathbb{R}$, $0 \leq \gamma < 2\pi$,*

$$\int_0^{2\pi}\int_0^{2\pi} \left|\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\} + e^{i\gamma}\left\{N_m[P^* \circ \sigma](e^{i\theta}) \right.\right. \\ \left.\left. -\alpha N_m[P^*](e^{i\theta})\right\}\right|^p d\theta d\gamma$$

$$\leq \int_0^{2\pi} |R^n \Lambda e^{i\gamma} + \lambda_0|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

Proof. Since $N_m[P^* \circ \sigma]^*(z) - \overline{\alpha} N_m[P^*]^*(z)$ is conjugate of $N_m[P^* \circ \sigma](z) - \alpha N_m[P^*](z)$, then

$$|N_m[P^* \circ \sigma]^*(e^{i\theta}) - \overline{\alpha} N_m[P^*]^*(e^{i\theta})| = |\left\{N_m[P^* \circ \sigma](e^{i\theta}) - \alpha N_m[P^*](e^{i\theta})\right\}|, \qquad 0 \leq \theta < 2\pi,$$

and therefore, for each $p > 0$, $R > 1$ and $0 \leq \theta < 2\pi$ fixed, we have

$$\begin{aligned}
&\int_0^{2\pi} |\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\} \\
&\quad + e^{i\gamma} \left\{N_m[P^* \circ \sigma](e^{i\theta}) - \alpha N_m[P^*](e^{i\theta})\right\}|^p d\gamma \\
&= \int_0^{2\pi} ||\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\}| e^{i\gamma} + |\left\{N_m[P^* \circ \sigma](e^{i\theta}) - \alpha N_m[P^*](e^{i\theta})\right\}||^p d\gamma \\
&= \int_0^{2\pi} ||\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\}| \\
&\quad + e^{i\gamma} \left\{|N_m[P^* \circ \sigma]^*(e^{i\theta}) - \overline{\alpha} N_m[P^*]^*(e^{i\theta})|\right\}|^p d\gamma. \qquad (3.15)
\end{aligned}$$

Integrating both sides of (3.15) with respect to θ from 0 to 2π and using (3.10), we get

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} |\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\} e^{i\gamma} \\
&\quad + \left\{N_m[P^* \circ \sigma](e^{i\theta}) - \alpha N_m[P^*](e^{i\theta})\right\}|^p d\gamma d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi} ||\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\}| \\
&\quad + e^{i\gamma} |N_m[P^* \circ \sigma]^*(e^{i\theta}) - \overline{\alpha} N_m[P^*]^*(e^{i\theta})||^p d\gamma d\theta \\
&= \int_0^{2\pi} \left\{ \int_0^{2\pi} |\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\} e^{i\gamma} \right. \\
&\quad \left. + N_m[P^* \circ \sigma]^*(e^{i\theta}) - \overline{\alpha} N_m[P^*]^*(e^{i\theta})|^p d\theta \right\} d\gamma \\
&\leq \int_0^{2\pi} |(R^n - \alpha) \Lambda e^{i\gamma} + (1 - \overline{\alpha}) \overline{\lambda_0}|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \\
&= \int_0^{2\pi} |(R^n - \alpha) \Lambda e^{i\gamma} + (1 - \alpha) \lambda_0|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.
\end{aligned}$$

That completes the proof of Lemma 3.7. □

4. Proofs of Theorems

Proof of Theorem 2.1. By hypothesis $P \in \mathcal{P}_n$, we can write

$$P(z) = P_1(z)P_2(z) = a\prod_{j=1}^{k}(z - z_j)\prod_{j=k+1}^{n}(z - z_j),\ \ k \geq 1,\ \ a \neq 0,$$

where the zeros $z_1, z_2, \ldots, z_k$ of $P_1(z)$ lie in $|z| \leq 1$ and the zeros $z_{k+1}, z_{k+2}, \ldots, z_n$ of $P_2(z)$ lie in $|z| > 1$. First, we suppose that all the zeros of $P_1(z)$ lie in $|z| < 1$. Since all the zeros of $P_2(z)$ lie in $|z| > 1$, the polynomial $P_2^\star(z) = z^{n-k}\overline{P_2(1/\overline{z})}$ has all its zeroes in $|z| < 1$ and $|P_2^\star(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$M(z) = P_1(z)P_2^\star(z) = a\prod_{j=1}^{k}(z - z_j)\prod_{j=k+1}^{n}(1 - z\overline{z_j}),$$

then all the zeros of $M(z)$ lie in $|z| < 1$, and for $|z| = 1$,

$$|M(z)| = |P_1(z)|\,|P_2^\star(z)| = |P_1(z)|\,|P_2(z)| = |P(z)|\,. \tag{4.1}$$

Observe that $P(z)/M(z) \to 1/\prod_{j=k+1}^{n}(-\overline{z_j})$ when $z \to \infty$, so it is regular even at ∞ and thus from (4.1) and by the maximum modulus principle, it follows that

$$|P(z)| \leq |M(z)| \ \text{ for } \ |z| \geq 1.$$

Since $M(z) \neq 0$ for $|z| \geq 1$, a direct application of Rouché's theorem shows that the polynomial $H(z) = P(z) + \lambda M(z)$ has all its zeros in $|z| < 1$ for every λ with $|\lambda| > 1$. Applying Lemma 3.2 to the polynomial $H(z)$ and noting that the zeros of $H(Rz)$ lie in $|z| < 1/R < 1$, we deduce (as in the proof of Lemma 3.3) that for every real or complex α with $|\alpha| \leq 1$, all the zeros of polynomial

$$\begin{aligned} G(z) &= H(Rz) - \alpha H(z) \\ &= \{P(Rz) - \alpha P(z)\} - \lambda\{M(Rz) - \alpha M(z)\} \end{aligned}$$

lie in $|z| < 1$. Applying Lemma 3.1 to $G(z)$ and noting that N_m is a linear operator, it follows that all the zeros of

$$N_m[G](z) = \{N_m[P\circ\sigma](z) - \alpha N_m[P](z)\} - \lambda\{N_m[M\circ\sigma](z) - \alpha N_m[M](z)\},$$

lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies

$$|N_m[P\circ\sigma](z) - \alpha N_m[P](z)| \leq |N_m[M\circ\sigma](z) - \alpha N_m[M](z)| \ \text{ for } \ |z| \geq 1,$$

which, in particular, gives for each $p > 0$, $R > 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} &\int_0^{2\pi}\left|N_m[P\circ\sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right|^p d\theta \\ &\quad\leq \int_0^{2\pi}\left|N_m[M\circ\sigma](e^{i\theta}) - \alpha N_m[M](e^{i\theta})\right|^p d\theta. \end{aligned} \tag{4.2}$$

Again (as in case of $H(z)$), $M(Rz) - \alpha M(z)$ has all its zeros in $|z| < 1$, thus by Lemma 3.1, $N_m[M\circ\sigma](z) - \alpha N_m[M](z)$ also has all its zeros in $|z| < 1$.

Therefore, if $E(z) = e_n z^n + \cdots + e_1 z + e_0$ has all its zeros in $|z| < 1$, then the operator Λ_δ defined by

$$\begin{aligned}\Lambda_\delta E(z) &= N_m[E \circ \sigma](z) - \alpha N_m[E](z)\\ &= (R^n - \alpha)\, \mathit{\Lambda} e_n z^n + \ldots + (R-\alpha)\left(\lambda_0 + \lambda_1 \frac{n}{2}\right) e_1 z + (1-\alpha)\,(\lambda_0) e_0,\end{aligned} \tag{4.3}$$

is admissible. Since $M(z) = b_n z^n + \cdots + b_0$, has all its zeros in $|z| < 1$, in view of (4.3) it follows by (3.5) of Lemma 3.4 that for each $p > 0$,

$$\begin{aligned}&\int_0^{2\pi} \left|N_m[M \circ \sigma](e^{i\theta}) - \alpha N_m[M](e^{i\theta})\right|^p d\theta\\ &\quad \le |R^n - \alpha|^p |N_m[z^n](1)|^p \int_0^{2\pi} \left|M(e^{i\theta})\right|^p d\theta.\end{aligned} \tag{4.4}$$

Combining the inequalities (4.2), (4.4) and noting that $|M(e^{i\theta})| = |P(e^{i\theta})|$, we obtain for each $p > 0$ and $R > 1$,

$$\begin{aligned}&\int_0^{2\pi} \left|N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right|^p d\theta\\ &\quad \le |R^n - \alpha|^p |N_m[z^n](1)|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta,\end{aligned} \tag{4.5}$$

In case $P_1(z)$ has a zero on $|z| = 1$, then inequality (4.5) follows by continuity. This proves Theorem 2.1 for $p > 0$. To obtain this result for $p = 0$, we simply make $p \to 0+$. $\square$

Proof of Theorem 2.2. By hypothesis $P(z)$ does not vanish in $|z| < 1$, $\sigma(z) = Rz$ and $R > 1$, therefore, for $0 \le \theta < 2\pi$, (3.1) holds. Also, for each $p > 0$ and γ real, (3.9) holds. Now it can be easily verified that for every real number γ and $s \ge 1$,

$$\left|s + e^{i\gamma}\right| \ge \left|1 + e^{i\gamma}\right|.$$

This implies for each $p > 0$,

$$\int_0^{2\pi} \left|s + e^{i\gamma}\right|^p d\gamma \ge \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^p d\gamma. \tag{4.6}$$

If $N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta}) \ne 0$, we take

$$s = \frac{N_m[P^\star \circ \sigma](e^{i\theta}) - \alpha N_m[P^\star](e^{i\theta})}{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})},$$

then by (3.1), $s \ge 1$ and we get with the help of (4.6), that

$$\begin{aligned}&\int_0^{2\pi} \left|\left\{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right\} + e^{i\gamma}\left\{N_m[P^\star \circ \sigma](e^{i\theta}) - \alpha N_m[P^\star](e^{i\theta})\right\}\right|^p d\gamma\\ &= \left|N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right|^p \int_0^{2\pi} \left|1 + e^{i\gamma}\frac{N_m[P^\star \circ \sigma](e^{i\theta}) - \alpha N_m[P^\star](e^{i\theta})}{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})}\right|^p d\gamma\\ &= \left|N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right|^p \int_0^{2\pi} \left|1 + e^{i\gamma}\left|\frac{N_m[P^\star \circ \sigma](e^{i\theta}) - \alpha N_m[P^\star](e^{i\theta})}{N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})}\right|\right|^p d\gamma\\ &\ge \left|N_m[P \circ \sigma](e^{i\theta}) - \alpha N_m[P](e^{i\theta})\right|^p \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^p d\gamma.\end{aligned}$$

For $N_m[P\circ\sigma](e^{i\theta})-\alpha N_m[P](e^{i\theta})=0$, this inequality is trivially true. Using this in (3.9), we conclude that for each $p>0$,

$$\int_0^{2\pi}\left|N_m[P\circ\sigma](e^{i\theta})-\alpha N_m[P](e^{i\theta})\right|^p d\theta\int_0^{2\pi}\left|1+e^{i\gamma}\right|^p d\gamma$$
$$\leq\int\limits_0^{2\pi}|\,(R^n-\alpha)\,\Lambda e^{i\gamma}+(1-\alpha)\,\lambda_0|^p d\gamma\int\limits_0^{2\pi}|P(e^{i\theta})|^p d\theta,$$

from which Theorem 2.2 follows for $p>0$. To establish this result for $p=0$, we simply let $p\to 0+$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Since $P(z)$ is a self-inversive polynomial, then we have for some ν, with $|\nu|=1$ $P(z)=\nu P^\star(z)$ for all $z\in\mathbb{C}$, where $P^\star(z)$ is the conjugate polynomial $P(z)$. This gives, for $0\leq\theta<2\pi$,

$$|N_m[P\circ\sigma](e^{i\theta})-\alpha N_m[P](e^{i\theta})|=|N_m[P^\star\circ\sigma](e^{i\theta})-\alpha N_m[P^\star](e^{i\theta})|.$$

Using this in place of (3.1) and proceeding similarly as in the proof of Theorem 2.2, we get the desired result for each $p>0$. The extension to $p=0$ is obtained by letting $p\to 0+$. □

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Abdullah Mir, N. A. Rather and Ishfaq Dar
Department of Mathematics
University of Kashmir
Srinagar 190006
India
e-mail: drabmir@yahoo.com

N. A. Rather
e-mail: dr.narather@gmail.com

Ishfaq Dar
e-mail: ishfaq619@gmail.com

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