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Pell–Lucas Numbers as Sum of Same Power of Consecutive Pell Numbers

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Abstract. Our main objective in this paper is to find all Pell–Lucas numbers that are sum of same power of consecutive Pell numbers. So we find all the solutions of the Diophantine equation

$$P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x = Q_m$$

in positive integers m, n, k, x, where P_i is the ith term of the Pell sequence and Q_j is the jth term of the Pell–Lucas sequence.

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1. Introduction

Let $(P_n)_{n\geq 0}$ be the Pell sequence given by

$$P_0 = 0,\ P_1 = 1 \quad \text{and } P_{n+2} = 2P_{n+1} + P_n, \quad \text{for} \quad n \geq 0.$$

The initial terms of this sequence are

$$0,\ 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ \ldots.$$

Its companion sequence is the Pell–Lucas sequence $(Q_n)_{n\geq 0}$ given by

$$Q_0 = 2,\ Q_1 = 2 \quad \text{and } Q_{n+2} = 2Q_{n+1} + Q_n, \quad \text{for} \quad n \geq 0,$$

which initial terms are

$$2,\ 2,\ 6,\ 14,\ 34,\ 82,\ 198,\ 478,\ 1154, \ldots.$$

The Binet's formulas for Pell and Pell–Lucas numbers are as follows:

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \quad \text{and} \quad Q_n = \alpha^n + \beta^n, \tag{1.1}$$

where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ are the roots of the characteristic quadratic equation $x^2 - 2x - 1 = 0$. This easily implies that the inequalities

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1}, \quad \text{for } n \geq 1 \tag{1.2}$$

and

$$\alpha^{n-1} < Q_n < \alpha^{n+1}, \quad \text{for } n \geq 2. \tag{1.3}$$

It is easy to prove that

$$\frac{P_n}{P_{n+1}} \leq \frac{3}{7}, \quad \text{for } n \geq 2. \tag{1.4}$$

The sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ satisfy the following well-known properties (see [1, pp. 193–194]):

$$Q_n = 2(P_{n-1} + P_n), \tag{1.5}$$

$$\sum_{j=1}^{n} P_j = \frac{P_n + P_{n+1} - 1}{2}, \tag{1.6}$$

and

$$\sum_{j=1}^{n} P_j^2 = \frac{P_{2n} + P_{2n+1} + \tau}{8}, \tag{1.7}$$

where $\tau = -1$ if n is even and $\tau = 1$ otherwise.

In 2011, it has been proved that a term of the Pell sequence is never a perfect higher power of another term and that a sum of same powers of two consecutive terms cannot be a term apart from the family of identities $P_n^2 + P_{n+1}^2 = P_{2n+1}$ (see [5]). Earlier, in 2020, some of the authors of this paper gave a nice extension of this result, proving then that the Diophantine equation

$$P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x = P_m.$$

has only trivial solutions (see [3]).

In this paper, we look for all Pell–Lucas numbers that are sum of same power of consecutive Pell numbers. So we investigate the following Diophantine equation

$$P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x = Q_m. \tag{1.8}$$

It is obvious that $(m, n, k, x) = (1, 2, 1, 1)$ is a solution of equation (1.8) since $P_2 = Q_1$. Such a solution is called a trivial solution. We prove the following theorem:

Theorem 1.1. *The Diophantine Eq.* (1.8) *has only the trivial solution* $(m, n, k, x) = (1, 2, 1, 1)$ *in positive integers* (m, n, k, x).

We use Baker's method to prove our main result.

2. The Tools

2.1. Linear Forms in Logarithms

The proof of our main theorem uses lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker–Davenport reduction method. So, let us recall some results. For any non-zero algebraic number α of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a_0 \prod_{i=1}^{d} \left(X - \alpha^{(i)}\right)$ (with $a_0 > 0$), we denote by

$$h(\alpha) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\left(1, \left|\alpha^{(i)}\right|\right)\right)$$

the usual absolute logarithmic height of α.

Let α_1, α_2 be two non-zero algebraic numbers, and multiplicatively independent. We consider the linear form

$$\Lambda := b_2 \log \alpha_2 - b_1 \log \alpha_1, \tag{2.1}$$

where b_1 and b_2 are positive integers. Without loss of generality, we suppose that $|\alpha_1|$ and $|\alpha_2|$ are ≥ 1. Put

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}].$$

Let B_1, B_2 be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\}, \qquad \text{for} \quad i = 1, 2,$$

and put

$$b' := \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.$$

We note that $\Lambda \neq 0$ because α_1 and α_2 are multiplicatively independent. The following result is due to Laurent, Mignotte and Nesterenko [2, Corollary 2, p. 288].

Theorem 2.1. (Laurent, Mignotte, Nesterenko) *With the above notations, assuming that α_1, α_2 are real and positive,*

$$\log |\Lambda| > -24.34 D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2. \tag{2.2}$$

2.2. Continued Fraction

In this subsection, we will present a property of continued fractions used in this paper to reduce the upper bounds on x or m of the Diophantine Eq. (1.8). We begin by recalling the following classical result in the theory of Diophantine approximation, which is the well-known Legendre criterion (see Theorem 8.2.4 in [4]).

Lemma 2.2. (i) *Let τ be real number and u, v integers such that*

$$\left|\tau - \frac{u}{v}\right| < \frac{1}{2v^2}. \tag{2.3}$$

Then, $u/v = p_k/q_k$ is a convergent of τ. Furthermore,

$$\left|\tau - \frac{u}{v}\right| \geq \frac{1}{(a_{k+1}+2)v^2}. \tag{2.4}$$

(ii) *If u, v are integers with $v \geq 1$ and*

$$|v\tau - u| < |q_k\tau - p_k|,$$

then $v \geq q_{k+1}$.

Finally, we recall the following lemma (see Theorem 8.2.4 and top of p. 263 in [4]):

Lemma 2.3. *Let p_i/q_i be the convergents of the continued fraction $[a_0, a_1, \ldots]$ of the irrational number γ. Let M be a positive integer and put $a_M := \max\{a_i |\ 0 \leq i \leq N+1\}$ where $N \in \mathbb{N}$ is such that $q_N \leq M < q_{N+1}$. If $u, v \in \mathbb{Z}$ with $u > 0$, then*

$$|u\gamma - v| > \frac{1}{(a_M+2)u}, \quad \textit{for all } u < M.$$

3. Setup

In this section, we will study the cases of $k \in \{1, 2\}$ and $x \in \{1, 2\}$ and we will give an inequality for m in terms of n, k, and x.

3.1. The Small Values of k

It is convenient to rule out the small values of k. We will later rule out the case of small values of x in Sect. 7.

Note that it is well-known and it is easy to prove that P_l is even if and only if l is even (see Lemma 2 in [6]). So, for every positive integer n, $P_n + P_{n-1}$ is odd. From this, we deduce from (1.5) that for any positive integer m, $v_2(Q_m) = 1$ (where $v_2(Q_m)$ denotes the 2-adic value of Q_m), which implies that the equation $Q_m = a^x$ has no solutions in positive integers with $x \geq 2$. In particular the Diophantine equation $Q_m = P_n^x$ (which corresponds to our main equation for $k = 1$) is not possible if $x \geq 2$. So, the case $k = 1$ leads to solve the Diophantine equation $Q_m = P_n$. Observe that

$$P_n = \frac{Q_n + Q_{n-1}}{4}, \quad \text{for} \quad n \geq 1$$

and that

$$Q_{n-2} < \frac{Q_n + Q_{n-1}}{4} < Q_{n-1}, \quad \text{for} \quad n \geq 3,$$

which implies that the Diophantine equation $Q_m = P_n$ has no solution with $n \geq 3$. So, we check that the only solution in positive integers of the equation $Q_m = P_n$ is $(m, n) = (1, 2)$, so we obtained that the only solution of Eq. (1.8) with $k = 1$ is $(m, n, k, x) = (1, 2, 1, 1)$, as given in Theorem 1.1.

For $k = 2$, our main equation is $P_n^x + P_{n+1}^x = Q_m$, which has no solution in positive integers since for each positive integers n and x, $P_n^x + P_{n+1}^x$ is odd while Q_m is even for each positive integer m. So, we assume from now on that $k \geq 3$.

3.2. An Inequality for m in Terms of n, k, x

Recall that we are working on Eq. (1.8) and we are now assuming that $k \geq 3$ and $x \geq 1$. It is also easy to check by induction that

$$P_1 + \cdots + P_n < P_{n+1}, \quad \text{for } n \geq 1. \tag{3.1}$$

This and (1.2) give the inequalities

$$P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x > P_{n+k-1}^x \geq \alpha^{(n+k-3)x},$$

and

$$\begin{aligned} P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x &\leq (P_0 + P_1 + P_2 + \cdots + P_{n+k-1})^x \\ &< P_{n+k}^x \leq \alpha^{(n+k-1)x}. \end{aligned}$$

Thus, we obtain

$$\alpha^{(n+k-3)x} < P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x < \alpha^{(n+k-1)x}.$$

This and (1.3) give

$$(n+k-3)x \leq m \leq (n+k-1)x.$$

We record this as a lemma.

Lemma 3.1. *If (m, n, k, x) is any nontrivial solution of (1.8) in positive integers, then we have the inequalities*

$$(n+k-3)x \leq m \leq (n+k-1)x.$$

3.3. The Case of Small Values of x

As for k, we find it convenient to rule out the case of small values of x, namely the cases when $x \in \{1, 2\}$.

For $x = 1$, our Eq. (1.8) is

$$P_n + P_{n+1} + \cdots + P_{n+k-1} = Q_m,$$

which leads to

$$\begin{aligned} Q_m &= \sum_{j=1}^{n+k-1} P_j - \sum_{j=1}^{n-1} P_j = \frac{P_{n+k-1} + P_{n+k} - 1}{2} - \frac{P_{n-1} + P_n - 1}{2} \\ &= \frac{Q_{n+k} - Q_n}{4}, \end{aligned}$$

where we used (1.5). Since $x = 1$, from Lemma 3.1, we have that $n + k - 3 \leq m \leq n + k - 1$, i.e. $m \in \{n+k-3, n+k-2, n+k-1\}$.

Assume that $m = n + k - 3$. Then, we get that $Q_{n+k-3} = \frac{Q_{n+k} - Q_n}{4}$, i.e.

$$\begin{aligned} 4Q_{n+k-3} &= Q_{n+k} - Q_n = 2Q_{n+k-1} + Q_{n+k-2} - Q_n \\ &= 5Q_{n+k-2} + 2Q_{n+k-3} - Q_n, \end{aligned}$$

which is false as $4Q_{n+k-3} < 5Q_{n+k-2} + 2Q_{n+k-3} - Q_n$.

Assume that $m = n + k - 2$. Then, we get

$$4Q_{n+k-2} = 5Q_{n+k-2} + 2Q_{n+k-3} - Q_n,$$

which is false as $4Q_{n+k-2} < 5Q_{n+k-2} + 2Q_{n+k-3} - Q_n$.

For $m=n+k-1$, we get

$$4Q_{n+k-1}=2Q_{n+k-1}+Q_{n+k-2}-Q_n,$$

i.e. $2Q_{n+k-1}=Q_{n+k-2}-Q_n$, which is false since $2Q_{n+k-1}>Q_{n+k-2}-Q_n$.

For $x=2$, our Eq. (1.8) is

$$P_n^2+P_{n+1}^2+\cdots+P_{n+k-1}^2=Q_m.$$

But using (1.7), we have

$$\begin{aligned}Q_m &= \sum_{j=1}^{n+k-1}P_j^2-\sum_{j=1}^{n-1}P_j^2=\frac{P_{2n+2k-2}+P_{2n+2k-1}+\tau}{8}-\frac{P_{2n-2}+P_{2n-1}+\varepsilon}{8}\\ &=\frac{Q_{2n+2k-1}-Q_{2n-1}}{16}+\frac{\tau-\varepsilon}{8},\end{aligned}$$

where $\tau=-1$ if $n+k-1$ is even and $\tau=1$ otherwise, and on the other hand $\varepsilon=-1$ if $n-1$ is even and $\varepsilon=1$ otherwise. So, we get

$$\begin{aligned}16Q_m &= Q_{2n+2k-1}-Q_{2n-1}+2(\tau-\varepsilon)\\ &=2Q_{2n+2k-2}+Q_{2n+2k-3}-Q_{2n-1}+2(\tau-\varepsilon)\\ &=5Q_{2n+2k-3}+2Q_{2n+2k-4}-Q_{2n-1}+2(\tau-\varepsilon)\\ &=12Q_{2n+2k-4}+5Q_{2n+2k-5}-Q_{2n-1}+2(\tau-\varepsilon).\end{aligned}\tag{3.2}$$

Since $x=2$, from Lemma 3.1, we have $2n+2k-6\le m\le 2n+2k-2$, i.e. $m\in\{2n+2k-6,2n+2k-5,2n+2k-4,2n+2k-3,2n+2k-2\}$. We will check for each possibility of m, whether (3.2) is satisfied or not.

If $m\in\{2n+2k-4,2n+2k-3,2n+2k-2\}$, then we have

$$16Q_m\ge 16Q_{2n+2k-4}>12Q_{2n+2k-4}+5Q_{2n+2k-5}-Q_{2n-1}+2(\tau-\varepsilon),$$

so that $16Q_m\ne 12Q_{2n+2k-4}+5Q_{2n+2k-5}-Q_{2n-1}+2(\tau-\varepsilon)$.

For $m\in\{2n+2k-6,\ 2n+2k-5\}$, we obtain

$$\begin{aligned}16Q_m &\le 16Q_{2n+2k-5}\\ &<12Q_{2n+2k-4}+5Q_{2n+2k-5}-Q_{2n-1}+2(\tau-\varepsilon).\end{aligned}$$

Hence, Eq. (1.8) has no solutions with $x=2$. So, we assume from now on that $x\ge 3$ and $k\ge 3$.

4. Bounds on x, m in Terms of $n+k$

Recall that $k\ge 3$ and $x\ge 3$, so $n+k\ge 4$ and $m\ge 6$. Now, using (1.1), we express equation (1.8) in the following form:

$$\alpha^m-P_{n+k-1}^x=P_n^x+P_{n+1}^x+\cdots+P_{n+k-2}^x+\beta^m,$$

which leads to

$$\left|\alpha^m-P_{n+k-1}^x\right|\le P_n^x+P_{n+1}^x+\cdots+P_{n+k-2}^x+|\beta|^m.$$

Using inequality (3.1), one gets

$$P_n^x+P_{n+1}^x+\cdots+P_{n+k-3}^x\le (P_n+P_{n+1}+\cdots+P_{n+k-3})^x<P_{n+k-2}^x,$$

so that $P_n^x + P_{n+1}^x + \cdots + P_{n+k-2}^x < 2P_{n+k-2}^x$. Then, we have

$$\left|\alpha^m - P_{n+k-1}^x\right| < 2P_{n+k-2}^x + |\beta|^m < 3P_{n+k-2}^x,$$

since $P_{n+k-2}^x > 1$, while $|\beta|^m < 1$. If we divide both sides of the above inequality by P_{n+k-1}^x and use the inequality (1.4), we obtain

$$\left|\alpha^m P_{n+k-1}^{-x} - 1\right| < 3\left(\frac{P_{n+k-2}}{P_{n+k-1}}\right)^x < \frac{3}{2.3^x} < \frac{3}{2.3^3} < \frac{1}{2}.$$

Let us now define the following linear form in two logarithms:

$$\Lambda_1 := m\log\alpha - x\log P_{n+k-1}. \tag{4.1}$$

Using the fact that $|\Lambda_1| < 2\left|e^{\Lambda_1} - 1\right|$ whenever $\left|e^{\Lambda_1} - 1\right| < \frac{1}{2}$, we get

$$|\Lambda_1| < \frac{6}{2.3^x}. \tag{4.2}$$

With the goal to get a lower bound for Λ_1, we apply Theorem 2.1 by fixing

$$\alpha_1 := \alpha, \quad \alpha_2 := P_{n+k-1}, \quad b_1 := m \quad \text{and} \quad b_2 := x.$$

With this data we have $\alpha_1, \alpha_2 \in \mathbb{Q}(\sqrt{2})$. Thus, we take $D := 2$. Since

$$h(\alpha_1) = \frac{\log\alpha}{2} \quad \text{and} \quad h(\alpha_2) = \log P_{n+k-1} \le (n+k-2)\log\alpha,$$

we take $\log B_1 := 1/2,\ \log B_2 := (n+k-2)\log\alpha$. Thus,

$$b' = \frac{m}{2(n+k-2)\log\alpha} + x \le \frac{(n+k-1)x}{2(n+k-2)\log\alpha} + x < 2x,$$

where we used the fact that $m \le (n+k-1)x$ (see Lemma 3.1), as well as the fact that $(n+k-1)/\left(2(n+k-2)\log\alpha\right) < 1$, for $n+k \ge 4$.

Before we can apply Theorem 2.1, we have to show that α_1 and α_2 are multiplicatively independent. Indeed, since $\alpha \in \mathcal{O}_{\mathbb{Q}(\sqrt{2})}$ whereas P_{n+k-1} does not, then α_1 and α_2 are multiplicatively independent. Thus, Theorem 2.1 implies

$$\begin{aligned}\log|\Lambda| &> -24.34 \times 2^4(1/2)((n+k-2)\log\alpha)\max\{\log(2x)+0.14, 10.5\}^2\\ &> -172(n+k-2)\max\{\log(2.4x), 10.5\}^2.\end{aligned}$$

Combining the above inequality with (4.2), we get

$$x\log(2.3) - \log 6 < 172(n+k-2)(\max\{\log(2.4x), 10.5\})^2.$$

If the maximum in the right above is 10.5, then $\log(2.4x) \le 10.5$, which leads to

$$x < 15{,}132. \tag{4.3}$$

Otherwise, we get

$$x\log(2.3) - \log 6 < 172(n+k-2)(\log(2.4x))^2.$$

So

$$x < 670(n+k-2)(\log x)^2,$$

where we used the fact that $\log(2.4x) < 1.8\log x$, for $x \geq 3$. Using the fact that for $A \geq 100$

$$\frac{y}{\log^2 y} < A \quad \text{yields} \quad y < 4A\log^2 A,$$

with $A := 670(n+k-2)$ and $y := x$, we get

$$\begin{aligned} x &< 4 \times 670(n+k-2)\left(\log(670(n+k-2))\right)^2 \\ &< 2680(n+k-2)\left(6.51+\log(n+k-2)\right)^2 \\ &< 2.9 \times 10^5(n+k)\log^2(n+k), \end{aligned} \tag{4.4}$$

where we used the fact that $6.51 + \log(n+k-2) < 10.4\log(n+k-2)$, for $n+k \geq 4$. Comparing (4.3) and (4.4), we conclude that inequality (4.4) always holds. Moreover, inequality (4.4) and Lemma 3.1 give

$$m \leq (n+k-1)x < 2.9 \times 10^5(n+k)^2\log^2(n+k).$$

We record this as a lemma.

Lemma 4.1. *If (m,n,k,x) is any nontrivial solution in positive integers of Eq. (1.8) with $x \geq 3$, $k \geq 3$ and $n+k \geq 4$, then we have the following inequalities:*

$$x < 2.9 \times 10^5(n+k)\log^2(n+k)$$

and

$$m < 2.9 \times 10^5(n+k)^2\log^2(n+k).$$

5. The Case of Small $n+k$

In this section, we will treat the case when $4 \leq n+k \leq 50$. In this case, by Lemma 4.1, we have

$$x < 2.9 \times 10^5 \times 50 \times \log^2 50 < 2.22 \times 10^8$$

and

$$m < 2.9 \times 10^5 \times 50^2 \times \log^2 50 < 1.11 \times 10^{10}.$$

On the other hand, by Lemma 3.1, we have $m \leq (n+k-1)x < 50x$. Then, from (4.1) and (4.2), it follows that

$$\begin{aligned} \left| m\left(\frac{\log\alpha}{\log P_{n+k-1}}\right) - x\right| &< \frac{6}{(\log P_{n+k-1})(2.3^{1/50})^m} \\ &< \frac{6}{(\log P_3)(2.3^{1/50})^m} \\ &< \frac{3.8}{(2.3^{1/50})^m}. \end{aligned} \tag{5.1}$$

So, we are in situation to apply Lemma 2.3. We choose

$$u := m, \quad v := x \quad \gamma := \frac{\log\alpha}{\log P_{n+k-1}}, \ (4 \leq n+k \leq 50) \quad \text{and}$$
$$M := 1.11 \times 10^{10}.$$

Using Maple package, we obtain that $a_M \le 177$, for $4 \le n+k \le 50$. Then, Lemma 2.3 implies that

$$\left| m\left(\frac{\log\alpha}{\log P_{n+k-1}}\right) - x \right| > \frac{1}{179m}. \tag{5.2}$$

If we compare (5.1) and (5.2) and use the fact that $m < 1.11 \times 10^{10}$, we conclude that

$$m < \frac{\log(3.8 \times 179 \times 1.1 \times 10^{10})}{\log(2.3^{1/50})} < 1780.$$

Next, since $(n+k-3)x \le m$, we have

$$x \le m/(n+k-3) \le 1780/(n+k-3).$$

A computer program with Maple revealed that there are no solutions to Eq. (1.8) in the range $n+k \in \{4,5,\ldots,50\}$, $m \in [6,1780]$ and $x \in [3, 1780/(n+k-3)]$.

6. The Bound on x

From now on, we suppose that $n+k \ge 51$. We will prove the following lemma.

Lemma 6.1. *If (k,n,m,x) is any nontrivial solution in positive integers of Eq. (1.8) with $k \ge 3$, $x \ge 3$, then $x \le 5$.*

Proof. We suppose that $x \ge 6$ in order to get a contradiction. By Lemma 4.1, we have

$$\frac{x}{\alpha^{2(n+k-1)}} < \frac{2.9 \times 10^5 (n+k)\log^2(n+k)}{\alpha^{2(n+k-1)}} < \frac{1}{\alpha^{n+k}},$$

where we used the fact that the inequality $2.9 \times 10^5(n+k)\log^2(n+k) < \alpha^{n+k-2}$ holds for $n+k \ge 23$, which is the case for us. We now write

$$P_{n+k-1}^x = \frac{\alpha^{(n+k-1)x}}{8^{x/2}}\left(1 - \frac{(-1)^{n+k-1}}{\alpha^{2(n+k-1)}}\right)^x.$$

If $n+k-1$ is odd, then

$$\begin{aligned} 1 < \left(1 - \frac{(-1)^{n+k-1}}{\alpha^{2(n+k-1)}}\right)^x &= \left(1 + \frac{1}{\alpha^{2(n+k-1)}}\right)^x \\ &= \exp\left(x \log\left(1 + \frac{1}{\alpha^{2(n+k-1)}}\right)\right) \\ &< \exp\left(\frac{x}{\alpha^{2(n+k-1)}}\right) < \exp\left(\frac{1}{\alpha^{n+k}}\right) \\ &< 1 + \frac{2}{\alpha^{n+k}}, \end{aligned}$$

because $\dfrac{1}{\alpha^{n+k}} \le \alpha^{-51}$ is very small. If $n+k-1$ is even, then

$$1 > \left(1 - \frac{(-1)^{n+k-1}}{\alpha^{2(n+k-1)}}\right) > \exp\left(\frac{-x}{\alpha^{2(n+k-1)}}\right) > \exp\left(\frac{-1}{\alpha^{n+k}}\right) > 1 - \frac{2}{\alpha^{n+k}},$$

again because $\dfrac{1}{\alpha^{n+k}} \le \alpha^{-51}$ is very small. Hence, we obtain

$$P_{n+k-1}^x = \frac{\alpha^{(n+k-1)x}}{8^{x/2}}\left(1 - \frac{(-1)^{n+k-1}}{\alpha^{2(n+k-1)}}\right)^x = \frac{\alpha^{(n+k-1)x}}{8^{x/2}}(1+\zeta), \quad |\zeta| < \frac{1}{\alpha^{n+k}}.$$

In particular, $|\zeta| < 1/2$, so that

$$\alpha^{(n+k-1)x}/8^{x/2} \in ((2/3)P_{n+k-1}^x, 2P_{n+k-1}^x).$$

Thus, we obtain

$$\alpha^m - \frac{\alpha^{(n+k-1)x}}{8^{x/2}}(1+\zeta) = \beta^m + \left(\sum_{j=n}^{n+k-2} P_j^x\right).$$

Dividing both sides by $\alpha^{(n+k-1)x}/8^{x/2}$, we get

$$\begin{aligned}&\left|\alpha^{m-(n+k-1)x}8^{x/2} - 1\right| \\ &\quad \le |\zeta| + \frac{1}{\alpha^m}\left(\frac{8^{x/2}}{\alpha^{(n+k-1)x}}\right) + \left(\frac{8^{x/2}}{\alpha^{(n+k-1)x}}\right)\left(\sum_{j=n}^{n+k-2} P_j^x\right).\end{aligned}$$

The fact that $\alpha^{(n+k-1)x}/8^{x/2} \in ((2/3)P_{n+k-1}^x, 2P_{n+k-1}^x)$ gives

$$\frac{1}{\alpha^m}\left(\frac{8^{x/2}}{\alpha^{(n+k-1)x}}\right) < \frac{3}{2\cdot\alpha^m P_{n+k-1}^x} < \frac{1}{\alpha^{n+k}}.$$

Since $P_\ell/P_{\ell+1} \le 3/7$, for $\ell \ge 2$, it results that

$$\begin{aligned}&\left(\frac{8^{x/2}}{\alpha^{(n+k-1)x}}\right)\left(\sum_{j=n}^{n+k-2} P_j^x\right) \\ &\quad < \frac{3}{2}\left(\left(\frac{P_{n+k-2}}{P_{n+k-1}}\right)^x + \left(\frac{P_{n+k-3}}{P_{n+k-1}}\right)^x + \cdots + \left(\frac{P_n}{P_{n+k-1}}\right)^x\right) \\ &\quad = \frac{3}{2}\left(\frac{P_{n+k-2}}{P_{n+k-1}}\right)^x\left(1 + \left(\frac{P_{n+k-3}}{P_{n+k-2}}\right)^x + \cdots + \left(\frac{P_n}{P_{n+k-2}}\right)^x\right) \\ &\quad < \frac{1}{2.3^x}\left(2 + \left(\frac{3}{7}\right)^2 + \left(\frac{3}{7}\right)^4 + \cdots\right) \\ &\quad < \frac{2.23}{2.3^x}.\end{aligned}$$

Thus, we deduce that

$$\left|\alpha^{m-(n+k-1)x}8^{x/2} - 1\right| < \frac{2}{\alpha^{n+k}} + \frac{2.23}{2.3^x} < \frac{5}{2.3^{\min\{x,n+k\}}}.$$

As $x \ge 6$, the above upper bound is smaller than $1/2$, so

$$|(m-(n+k-1)x)\log\alpha - x\log(2\sqrt{2})| < \frac{10}{2.3^{\min\{x,n+k\}}}. \tag{6.1}$$

The expression on the right is smaller than $1/2$, so $|m-(n+k-1)x| < 2x$. Next, we apply Theorem 2.1 by taking

$$(\alpha_1, b_1) := (\alpha, m-(n+k-1)x), \quad \text{and} \quad (\alpha_2, b_2) := (2\sqrt{2}, x).$$

Again $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ has $D = 2$. We take $\log B_1 := 1/2$, $\log B_2 := (\log 8)/2$. Thus,

$$b' := \frac{|m - (n + k - 1)x|}{\log 8} + x < \frac{2x}{\log 8} + x < 2x.$$

So, Theorem 2.1 tells us that the left-hand side of (6.1) is bounded by

$$-203(\max\{\log(2.4x), 10.5\})^2.$$

This and (6.1) imply that

$$\min\{x, n + k\}\log(2.3) - \log 10 < 203(\max\{\log(2.4x), 10.5\})^2.$$

If the maximum in the right above is 10.5, then $\log(2.4x) \leq 10.5$ and so

$$x < 15{,}132. \tag{6.2}$$

Otherwise, we get

$$\min\{x, n + k\}\log(2.3) - \log 10 < 203(\log(2.4x))^2,$$

which leads to

$$\min\{x, n + k\} < 550\log^2 x,$$

where we used the fact that $\log(2.4x) < 1.5\log x$, for $x \geq 6$. If

$$\min\{x, n + k\} = x,$$

we get $x < 550\log^2 x$. This implies

$$x < 70{,}000. \tag{6.3}$$

Finally, it remains to consider the possibility

$$\min\{x, n + k\} = n + k.$$

In this case, we get $n + k < 550\log^2 x$. So, by Lemma 4.1, we get

$$\begin{aligned} n + k &< 550\left(\log(2.9 \times 10^5(n + k)\log^2(n + k))\right)^2 \\ &< 550\left(12.58 + 2\log(n + k)\right)^2 \\ &< 14{,}872\log^2(n + k), \end{aligned}$$

where we used the fact that $2\log\log(n+k) < \log(n+k)$ and $12.58+2\log(n+k) < 5.2\log(n + k)$, for $n + k \geq 51$. Thus,

$$n + k < 3.4 \times 10^6.$$

So, again by Lemma 4.1, we get

$$x < 2.9 \times 10^5 \times 3.4 \times 10^6 \log^2(3.4 \times 10^6) < 2.3 \times 10^{14}. \tag{6.4}$$

In conclusion, from (6.2), (6.3) and (6.4), we have inequality (6.4). We now go back to inequality (6.1) and divide it across by $x\log\alpha$ to obtain

$$\left|\frac{\log(2\sqrt{2})}{\log\alpha} - \frac{(n + k - 1)x - m}{x}\right| < \frac{10}{x(\log\alpha)2.3^{\min\{x,n+k\}}}. \tag{6.5}$$

Since $x \geq 6$, we have $2.3^x > (20/\log\alpha)x$. Furthermore, since $n+k \geq 51$, we have

$$\frac{2.3^{n+k}}{(20/\log\alpha)} \geq \frac{2.3^{51}}{(20/\log\alpha)} > 1.2 \times 10^{17} > x.$$

To summarize, the assumption $x \geq 6$ implies

$$\frac{2.3^{\min\{x,n+k\}}}{(10/\log\alpha)} > 2x,$$

and, therefore, inequality (6.5) gives that

$$\left|\frac{\log(2\sqrt{2})}{\log\alpha} - \frac{(n+k-1)x-m}{x}\right| < \frac{1}{2x^2}.$$

Thus, Lemma 2.2 implies that that $((n+k-1)x-m)/x = p_t/q_t$ for some convergent p_t/q_t of $\tau := \log(2\sqrt{2})/\log\alpha$. The continued fraction of τ starts as

$$[1, 5, 1, 1, 3, 3, 1, 1, 7, 3, 1, 1, \ldots]$$

with the 32st convergent p_{32}/q_{32} satisfying $q_{32} > 4.16 \times 10^{14} > x$. Thus, by Lemma 2.2, we have

$$\begin{aligned}|(m-(n+k-1)x)\log\alpha - x\log(2\sqrt{2})| &\geq (\log\alpha)|m-(n+k-1)x-x\tau| \\ &> (\log\alpha)|p_{31}-q_{31}\tau| > 1.83 \times 10^{-15},\end{aligned}$$

and now inequality (6.1) shows that

$$2.3^{\min\{x,n+k\}} < \frac{10 \times 10^{15}}{1.83} < 5.5 \times 10^{14}.$$

This gives $\min\{x, n+k\} \leq 40$, so $x \leq 40$, since $n+k \geq 51$. The sequence of convergents of τ is

$$1, \quad \frac{6}{5}, \quad \frac{7}{6}, \quad \frac{13}{11}, \quad \frac{46}{39}, \quad \frac{151}{128}, \quad \ldots.$$

The only convergents of the form p_t/q_t with q_t a divisor of x and $x \in [6, 40]$ are the first 5 numbers above. Thus, $t \in \{0, 1, 2, 3, 4\}$. For each one of them, we get that $q_t \mid x$ so $x \geq q_t$. Thus, $x \geq \max\{6, q_t\}$. Now, inequality (6.5) implies that

$$\left|\frac{\log(2\sqrt{2})}{\log\alpha} - \frac{p_t}{q_t}\right| < \frac{10}{(\log\alpha)\max\{6,q_t\}2.3^{\max\{6,q_t\}}}.$$

We checked that this last inequality fails for $t \in \{0, 1, 2, 3, 4\}$. Thus, the assumption $x \geq 6$ is false, therefore, $x \leq 5$ which is what we wanted. □

7. Final Computation

From now, we assume that $3 \le x \le 5$ and $n+k \ge 51$. We take l to be some number in $\{n, n+1, \ldots, n+k-1\}$ such that $l \ge 25$. For example, we can take $l = n + \lfloor k/2 \rfloor$ and then certainly $l \ge (n+k)/2 \ge 25$ since $n+k \ge 51$. Furthermore, if $k \le 26$, then we can take $l = n = (n+k) - k \ge 51 - 26 = 25$. We make these choices more precise later. Let $j \in \{l+1, \ldots, n+k-1\}$. We have

$$\frac{x}{\alpha^{2j}} \le \frac{5}{\alpha^{2l+2}} < \frac{\alpha^2}{\alpha^{2l+2}} = \frac{1}{\alpha^{2l}}.$$

We now write

$$P_j^x = \frac{\alpha^{jx}}{8^{x/2}} \left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^x.$$

If j is odd, then

$$\begin{aligned} 1 < \left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^x = \left(1 + \frac{1}{\alpha^{2j}}\right)^x &= \exp\left(x \log\left(1 + \frac{1}{\alpha^{2j}}\right)\right) \\ &< \exp\left(\frac{x}{\alpha^{2j}}\right) < \exp\left(\frac{1}{\alpha^{2l}}\right) \\ &< 1 + \frac{2}{\alpha^{2l}}, \end{aligned}$$

because $\frac{1}{\alpha^{2l}} \le \alpha^{-50}$. If j is even, then $1 > \left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^x > 1 - \frac{2}{\alpha^{2l}}$. So, we have

$$\left|P_j^x - \frac{\alpha^{jx}}{8^{x/2}}\right| = \frac{\alpha^{jx}}{8^{x/2}} \left|\left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^x - 1\right| < \frac{\alpha^{jx}}{8^{x/2}} \left(\frac{2}{\alpha^{2l}}\right).$$

We now return to our Eq. (1.8) and rewrite it as

$$\alpha^m + \beta^m = P_n^x + P_{n+1}^x + \cdots + P_l^x + \sum_{j=l+1}^{n+k-1} \frac{\alpha^{jx}}{8^{x/2}} + \sum_{j=l+1}^{n+k-1} \left(P_j^x - \frac{\alpha^{jx}}{8^{x/2}}\right).$$

Thus, we obtain

$$\begin{aligned} &\left|\alpha^m - \frac{\alpha^{(l+1)x}}{8^{x/2}} \sum_{i=0}^{n+k-l-2} \alpha^{ix}\right| \\ &\quad = \left|-\beta^m + \sum_{j=l+1}^{n+k-1} \left(P_j^x - \frac{\alpha^{jx}}{8^{x/2}}\right) + P_n^x + P_{n+1}^x + \cdots + P_l^x\right| \\ &\quad \le \frac{1}{\alpha^m} + \sum_{j=l+1}^{n+k-1} \left|P_j^x - \frac{\alpha^{jx}}{8^{x/2}}\right| + P_n^x + P_{n+1}^x + \cdots + P_l^x \end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{\alpha^m} + \sum_{j=l+1}^{n+k-1} \left| P_j^x - \frac{\alpha^{jx}}{8^{x/2}} \right| + P_l^x \left(1 + \left(\frac{P_{l-1}}{P_l} \right)^x + \left(\frac{P_{l-2}}{P_l} \right)^x + \cdots \right) \\
&< \frac{1}{\alpha^m} + \sum_{j=l+1}^{n+k-1} \frac{\alpha^{jx}}{8^{x/2}} \left(\frac{2}{\alpha^{2l}} \right) + P_l^x \left(2 + \frac{1}{3} + \frac{1}{3^2} + \cdots \right) \\
&= \frac{1}{\alpha^m} + \frac{\alpha^{(l+1)x}}{8^{x/2}} \left(\frac{2}{\alpha^{2l}} \right) \sum_{i=0}^{n+k-l-2} \alpha^{ix} + 2.5\alpha^{(l-1)x}. \qquad (7.1)
\end{aligned}
$$

In the above chain of inequalities, we used the facts that $P_i/P_{i+1} \leq 3/7 < 1/2.3$ for $i \geq 2$, the fact that $2.3^x > 3$ since $x \geq 2$ and $P_l^x < \alpha^{(l-1)x}$. Multiplying both sides of the above inequality (7.1) by $\alpha^{-(n+k-1)x}8^{x/2}$, we obtain

$$
\begin{aligned}
&\left| \alpha^{m-(n+k-1)x}8^{x/2} - \sum_{i=0}^{n+k-l-2} \alpha^{-ix} \right| \\
&\quad \leq \frac{8^{x/2}}{\alpha^{m+(n+k-1)x}} + \frac{2}{\alpha^{2l}} \sum_{i=0}^{n+k-l-2} \alpha^{-ix} + \frac{2.5 \times 8^{x/2}}{\alpha^{(n+k-l)x}} \\
&\quad < \frac{1}{\alpha^{(n+k-1)x}} + \frac{2}{\alpha^{2l}} \sum_{i=0}^{n+k-l-2} \alpha^{-ix} + \frac{2.5}{\alpha^{(n+k-l-1.2)x}} \\
&\quad < \frac{3.5}{\alpha^{(n+k-l-1.2)x}} + \frac{2}{\alpha^{2l}} \sum_{i\geq 0} \frac{1}{3^i} < \frac{3.5}{\alpha^{(n+k-l-1.2)x}} + \frac{3}{\alpha^{2l}} \\
&\quad < \frac{6.5}{\alpha^{\min\{(n+k-l-1.2)x, 2l\}}}.
\end{aligned}
$$

In the above, we used the fact that $m > (n+k-3)x \geq 48x$ (see Lemma 3.1), so $\alpha^m > \alpha^{48x} > 8^{x/2}$, the fact that $\alpha^x \geq \alpha^3 > 3$, the fact that

$$
\sum_{i\geq 0} 3^{-i} = \frac{3}{2},
$$

as well as the fact that $\sqrt{8} < \alpha^{1.2}$. On the other hand, one has

$$
\begin{aligned}
\sum_{i=0}^{n+k-l-2} \alpha^{-ix} &= \sum_{i\geq 0} \frac{1}{\alpha^{ix}} - \sum_{i\geq n+k-l-1} \frac{1}{\alpha^{ix}} \\
&= \frac{1}{1-1/\alpha^x} - \frac{1}{\alpha^{(n+k-l-1)x}} \left(1 + \frac{1}{\alpha^x} + \frac{1}{\alpha^{2x}} + \cdots \right) \\
&= \frac{\alpha^x}{\alpha^x - 1} + \eta,
\end{aligned}
$$

where

$$
|\eta| < \frac{1}{\alpha^{(n+k-l-1)x}} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \right) < \frac{1.5}{\alpha^{(n+k-l-1)x}}.
$$

Hence, we obtain

$$\begin{aligned}\left|\alpha^{m-(n+k-1)x}8^{x/2}-\frac{\alpha^x}{\alpha^x-1}\right| &< \frac{6.5}{\alpha^{\min\{(n+k-l-1.2)x,2l\}}}+|\eta| \\ &< \frac{6.5}{\alpha^{\min\{(n+k-l-1.2)x,2l\}}}+\frac{1.5}{\alpha^{(n+k-l-1)x}} \\ &< \frac{8}{\alpha^{\min\{(n+k-l-1.2)x,2l\}}}. \end{aligned} \tag{7.2}$$

We want to show that $(n+k-l-1.2)x \leq 5$. Suppose that $(n+k-l-1.2)x > 6$. Since $2l \geq 50$, inequality (7.2) certainly implies that

$$\left|\alpha^{m-(n+k-1)x}8^{x/2}-1\right| < \frac{8}{\alpha^6}+\frac{\alpha^x}{\alpha^x-1}-1=\frac{8}{\alpha^6}+\frac{1}{\alpha^x-1}<\frac{1}{4}, \tag{7.3}$$

since $x \geq 3$. So, we obtain

$$|(m-(n+k-1)x)\log\alpha-(x-1)\log(2\sqrt{2})| < \frac{1}{2}.$$

Hence, we have that $|m-(n+k-1)x| < 2x$. We now take $l := n+\lfloor k/2\rfloor$. Note that $2l \geq 50$ since $l \geq 25$. We then get

$$\left|\alpha^{m-(n+k-1)x}8^{x/2}-\frac{\alpha^x}{\alpha^x-1}\right| < \frac{8}{\alpha^{\min\{(k-\lfloor k/2\rfloor-1.2)x,50\}}}.$$

We checked that for $x \in [3,5]$, there is no integer $t := m-(n+k-1)x$, $t \in (-2x, 2x)$ such that

$$\left|\alpha^t 8^{x/2}-\frac{\alpha^x}{\alpha^x-1}\right| < \frac{8}{\alpha^5}.$$

The way we checked that was to check numerically that for every x in our range and for all $t \in [-2x+1, 2x-1]$, the minimum of $\left|\alpha^t 8^{x/2}-\frac{\alpha^x}{\alpha^x-1}\right|$ is $> 0.098 > \frac{8}{\alpha^5}$, which certainly shows that such t cannot exist. This shows that $(k-\lfloor k/2\rfloor-1.2)x \leq 5$. Since $x \geq 3$, this shows that $k-\lfloor k/2\rfloor-1.2 \leq 1.7$, so $k-\lfloor k/2\rfloor \leq 2.9$, showing that $k \leq 5$. We now take $l = n$. Then, $l = (n+k)-k \geq 51-5 > 25$, so this choice of l is also valid. In this case, we see that $n+k-l-1.2 = k-1.2 > 0$, so inequality (7.2) becomes

$$\left|\alpha^{m-(n+k-1)x}8^{x/2}-\frac{\alpha^x}{\alpha^x-1}\right| < \frac{8}{\alpha^{\min\{(k-1.2)x,50\}}}.$$

The preceding argument shows that $(k-1.2)x \leq 5$ and since $x \geq 3$, we get $k \leq 2$, which is a contradiction. Thus, there are no solutions with $n+k \geq 51$, and this completes the proof of Theorem 1.1.

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