Mediterr. J. Math. (2022) 19:228
https://doi.org/10.1007/s00009-022-02144-w
1660-5446/22/050001-16
published online September 9, 2022

Mediterranean Journal of Mathematics

Variational Approaches for Contact Models with Multi-Contact Zones

Mariana Chivu Cojocaru and Andaluzia Matei

Abstract. We consider a contact model with two contact zones, for linearly elastic materials, under the small deformation hypothesis. We pay attention to four possible variational formulations: one of them is a variational inequality of the second kind and the other three are mixed variational problems governed by variational inequalities on convex sets of Lagrange multipliers. We study the existence and the uniqueness of the solution for each of the four variational formulations. Some connections between these four weak formulations are also discussed. Our approach requires a background knowledge in the variational inequalities theory as well as in the saddle point theory.

Mathematics Subject Classification. 35J65, 49J40, 35J50, 74M10, 74M15.

Keywords. Contact problems, Multi-contact zones, Weak solution, Variational inequalities, Mixed variational formulations, Saddle point problems.

1. Introduction

In the present paper, we draw attention to the possibility to deliver alternative variational formulations for contact models with multi-contact zones. Thus, depending on the motivation we have, one of several possible variational formulations can be picked. To exemplify, we consider a contact model with two contact zones, as follows.

Problem 1. *Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$, *such that*

$$\begin{aligned}
\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 &= \boldsymbol{0} && \text{in } \Omega,\\
\boldsymbol{\sigma} &= \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_1,\\
\boldsymbol{\sigma}\boldsymbol{\nu} &= \boldsymbol{f}_2 && \text{on } \Gamma_2,\\
u_\nu = 0,\ \|\boldsymbol{\sigma}_\tau\|_{\mathbb{S}^3} \le g,\ \boldsymbol{\sigma}_\tau &= -g\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|_{\mathbb{R}^3}} \ \text{if } \boldsymbol{u}_\tau \neq \boldsymbol{0} && \text{on } \Gamma_3,\\
\boldsymbol{\sigma}_\tau = \boldsymbol{0},\ \sigma_\nu \le 0,\ u_\nu &\le 0,\ \sigma_\nu u_\nu = 0 && \text{on } \Gamma_4,
\end{aligned}$$

where $\boldsymbol{u} = (u_i)$ is the displacement field and $\boldsymbol{\sigma} = (\sigma_{ij})$ is the Cauchy stress tensor. Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary Γ partitioned in four measurable parts, Γ_1, Γ_2, Γ_3, Γ_4, such that all the parts have positive measure; $\bar{\Omega} = \Omega \cup \Gamma$. We denote by $\boldsymbol{f}_0 : \Omega \to \mathbb{R}^3$ the density of the volume forces, by $\boldsymbol{f}_2 : \Gamma_2 \to \mathbb{R}^3$ the density of the surface traction, by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ the infinitesimal strain tensor, $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ for all $i, j \in \{1, 2, 3\}$ and by $\mathcal{E}$ the elastic tensor. As usual, $\boldsymbol{\nu}$ is the outward unit normal vector to the boundary Γ. Also, by $\cdot$, we denote the inner product on $\mathbb{R}^3$, and by $:$, we denote the inner product on $\mathbb{S}^3$, where $\mathbb{S}^3$ is the space of second-order symmetric tensors of $\mathbb{R}^3$ and by $\|\cdot\|_{\mathbb{R}^3}$ and $\|\cdot\|_{\mathbb{S}^3}$ we denote the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$, respectively. The operator Div is the divergence of a tensor, $Div\, \boldsymbol{\sigma} = (\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j})$, $i \in \{1, 2, 3\}$.

Problem 1 models the deformation of a body in contact with two rigid foundations. On Γ_3, the contact is frictional bilateral with a positive friction bound g, and on Γ_4, we have a frictionless unilateral contact condition. It is worth to underline that on Γ_3, we do not know a priori in which points the friction force vanishes, while Γ_4 is a potential contact zone, because we do not know a priori in which points we have contact. Recall that $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u_\tau} = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, $\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$, $\boldsymbol{\sigma_\tau} = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$, $\boldsymbol{\sigma\nu} \cdot \boldsymbol{u} = \sigma_\nu u_\nu + \boldsymbol{\sigma_\tau} \cdot \boldsymbol{u_\tau}$. For a background on the mathematical theory of contact mechanics models, see, e.g., [7,13,18].

If our interest consists only in computing the displacement field $\boldsymbol{u}$, then we can choose the primal variational formulation consisting in a variational inequality of the second kind, or, for a more efficient approximation of the weak solution, we can choose a mixed variational formulation consisting of a variational equation and a variational inequality.

If, in addition to $\boldsymbol{u}$, we are interested to compute the friction force $\boldsymbol{\sigma}_\tau$, then a mixed variational formulation governed by a Lagrange multiplier related to $\boldsymbol{\sigma}_\tau$ can be helpful. Or, if we are interested to compute the reaction of the foundation $-\sigma_\nu$, then a mixed variational formulation governed by a Lagrange multiplier related to σ_ν becomes convenient. In the last two cases, the weak formulations consist of systems of two variational inequalities.

The solvability of the weak formulations we deliver is based on the theory of variational inequalities and requires some elements of the saddle point theory. For a background on the variational inequalities of the second kind, the reader can consult, e.g., [18,19], and for useful elements in the theory of the saddle point theory related to the solvability of mixed variational problems, we refer to, e.g., [1,2,6,8]. Relevant to the matter are also the papers [4,5,11,12].

The present paper is structured as follows. In Sect. 2, we review four abstract results related to a variational inequality of the second kind and three saddle point problems. In Sect. 3, we introduce the functional setting and the working hypotheses. In Sect. 4, we deliver four weak formulations. Section 5 is devoted to the weak solvability of these four variational formulations paying attention to the connection between them.

2. Preliminaries

Everywhere in this section $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ are Hilbert spaces. We make the following assumptions:

A 1. $a : X \times X \to \mathbb{R}$ is a symmetric, bilinear form, such that

(i) there exists $M_a > 0 : |a(u,v)| \leq M_a \|u\|_X \|v\|_X \quad$ for all $u, v \in X$;

(ii) there exists $m_a > 0 : a(v,v) \geq m_a \|v\|_X^2 \quad$ for all $v \in X$.

A 2. $b : X \times Y \to \mathbb{R}$ is a bilinear form, such that

(a) there exists $M_b > 0 : |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y$ for all $v \in X$, $\mu \in Y$;

(b) there exists $\alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha$.

A 3. $\phi : X \to \mathbb{R}_+$ is a convex functional. Moreover, ϕ is a Lipschitz continuous functional, i.e., there exists $L_\phi > 0$:

$$|\phi(v) - \phi(w)| \leq L_\phi \|v - w\|_X \quad \text{for all } \; v, w \in X.$$

A 4. Λ is a closed, convex subset of Y that contains 0_Y.

A 5. K is a closed, convex subset of X that contains 0_X.

First, we focus on the following variational inequality of the second kind.

Problem 2. *Given $f \in X$, find $u \in K$, such that*

$$a(u, v-u) + \phi(v) - \phi(u) \geq (f, v-u)_X \quad \textit{for all } \; v \in K \subseteq X.$$

Theorem 1. *The assumptions A 1, A 3 and A 5 hold true. Then, Problem 2 has a unique solution, $u \in K$.*

For a proof of Theorem 1, the reader can consult, e.g., Theorem 3.1 in [19].

Notice that the unique solution of Problem 2 is the unique minimum of the functional

$$J : K \to \mathbb{R}, \quad J(v) = \frac{1}{2} a(v,v) + \phi(v) - (f,v)_X.$$

To proceed, we review three useful saddle point problems, Problems 3,4, 5, below.

Problem 3. *Given $f \in X$, find $u \in X$ and $\lambda \in \Lambda \subseteq Y$, such that*

$$\begin{aligned} a(u,v) + b(v,\lambda) &= (f,v)_X \qquad && \textit{for all } v \in X \\ b(u, \mu - \lambda) &\leq 0 && \textit{for all } \mu \in \Lambda. \end{aligned}$$

Theorem 2. *The assumptions A 1, A 2 and A 4 hold true. Then, Problem 3 has a unique solution, $(u,\lambda) \in X \times \Lambda$.*

For a proof, we send the reader to, e.g., Corollary 2 in [4].

Notice that a pair $(u, \lambda) \in X \times \Lambda$ is a solution of Problem 3 if and only if it is a saddle point of the following functional:

$$\mathcal{L}_1 : X \times \Lambda \to \mathbb{R}, \quad \mathcal{L}_1(v, \mu) = \frac{1}{2}a(v, v) - (f, v)_X + b(v, \mu);$$

this motivates us to consider Problem 3 a *saddle point problem.*

Problem 4. *Given $f \in X$, find $u \in K \subseteq X$ and $\lambda \in \Lambda \subseteq Y$, such that*

$$\begin{aligned} a(u, v-u) + b(v-u, \lambda) &\geq (f, v-u)_X \qquad \text{for all } v \in K \\ b(u, \mu - \lambda) &\leq 0 \qquad \text{for all } \mu \in \Lambda. \end{aligned}$$

Theorem 3. *The assumptions A 1, A 2 (a), A 4 and A 5 hold true. If, in addition, $\Lambda \subseteq Y$ is a bounded subset, then Problem 4 has a solution $(u, \lambda) \in K \times \Lambda$, unique in its first component.*

For details, see, e.g., Remark 3 in [4].

Notice that a pair $(u, \lambda) \in K \times \Lambda$ is a solution of Problem 4 if and only if it is a saddle point of the following functional:

$$\mathcal{L}_2 : K \times \Lambda \to \mathbb{R}, \quad \mathcal{L}_2(v, \mu) = \frac{1}{2}a(v, v) - (f, v)_X + b(v, \mu).$$

Thus, Problem 4 can be considered a *saddle point problem.*

Problem 5. *Given $f \in X$, find $u \in X$ and $\lambda \in \Lambda \subseteq Y$, such that*

$$\begin{aligned} a(u, v-u) + b(v-u, \lambda) + \phi(v) - \phi(u) &\geq (f, v-u)_X \qquad \text{for all } v \in X \\ b(u, \mu - \lambda) &\leq 0 \qquad \text{for all } \mu \in \Lambda. \end{aligned}$$

Theorem 4. *The Assumptions A 1–A 4 hold true. Then, Problem 5 has a solution $(u, \lambda) \in X \times \Lambda$, unique in its first component.*

For a proof, we send the reader to, e.g., Theorem 3 in [4]. Similar techniques can be found in, e.g., [5,11].

Notice that a pair $(u, \lambda) \in X \times \Lambda$ is a solution of Problem 5 if and only if it is a saddle point of the following functional:

$$\mathcal{L}_3 : X \times \Lambda \to \mathbb{R}, \quad \mathcal{L}_3(v, \mu) = \frac{1}{2}a(v, v) + \phi(v) - (f, v)_X + b(v, \mu).$$

Hence, Problem 5 is *a saddle point problem*, too.

3. Functional Setting and Working Hypotheses

To start, we remind some useful Hilbert Lebesgue spaces.

- $L^2(\Omega)^3 = \{\boldsymbol{v} = (v_i) \mid v_i \in L^2(\Omega),\ 1 \leq i \leq 3\}$ endowed with the inner product $(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^3} = \sum_{i=1}^3 \int_\Omega u_i\, v_i \,\mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}x$ and the associated norm $\|\boldsymbol{v}\|_{L^2(\Omega)^3} = \left(\sum_{i=1}^3 \int_\Omega v_i\, v_i \,\mathrm{d}x\right)^{1/2}$.
- $L^2(\Omega)^{3\times 3} = \{\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) \mid \boldsymbol{\tau}_{ij} \in L^2(\Omega),\ 1 \leq i, j \leq 3\}$ endowed with $(\boldsymbol{\sigma}, \boldsymbol{\tau})_{L^2(\Omega)^{3\times 3}} = \sum_{i,j=1}^3 \int_\Omega \sigma_{ij}\, \tau_{ij} \,\mathrm{d}x = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \,\mathrm{d}x$ and the corresponding norm $\|\boldsymbol{\tau}\|_{L^2(\Omega)^{3\times 3}} = \left(\sum_{i,j=1}^3 \int_\Omega \tau_{ij}\, \tau_{ij} \,\mathrm{d}x\right)^{1/2}$.

- $L^2_s(\Omega)^{3\times 3} = \{\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) \mid \boldsymbol{\tau}_{ij} = \boldsymbol{\tau}_{ji} \in L^2(\Omega),\ 1 \le i,j \le 3\}$, endowed with $(\boldsymbol{\sigma},\boldsymbol{\tau})_{L^2_s(\Omega)^{3\times 3}} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{L^2(\Omega)^{3\times 3}}$, and the norm $\|\boldsymbol{\sigma}\|_{L^2_s(\Omega)^{3\times 3}} = \|\boldsymbol{\sigma}\|_{L^2(\Omega)^{3\times 3}}$.

Subsequently, we introduce some useful Hilbert Sobolev spaces.

- $H^1(\Omega)^3 = \{\boldsymbol{v} = (v_i) \mid v_i \in H^1(\Omega),\ 1 \le i \le 3\}$ endowed with the canonical inner product

$$(\boldsymbol{u},\boldsymbol{v})_{H^1(\Omega)^3} = \sum_{i=1}^{3}(u_i,v_i)_{L^2(\Omega)} + \sum_{i=1}^{3}(\nabla u_i,\nabla v_i)_{L^2(\Omega)^3},$$

and the associated norm

$$\|\boldsymbol{v}\|_{H^1(\Omega)^3} = \sqrt{\sum_{i=1}^{3}\|v_i\|^2_{L^2(\Omega)} + \sum_{i=1}^{3}\|\nabla v_i\|^2_{L^2(\Omega)^3}}.$$

Furthermore, $H^1(\Omega)^3$ can be endowed with the following particular inner product:

$$((\boldsymbol{u},\boldsymbol{v}))_{H^1(\Omega)^3} = (\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)^3} + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}} \quad \text{for all } \boldsymbol{u},\boldsymbol{v} \in H^1(\Omega)^3$$

and the associated norm

$$|||\boldsymbol{v}|||_{H^1(\Omega)^3} = (\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{L^2(\Omega)^{3\times 3}})^{1/2} \quad \text{for all } \boldsymbol{v} \in H^1(\Omega)^3,$$

where $\boldsymbol{\varepsilon} : H^1(\Omega)^3 \to L^2_s(\Omega)^{3\times 3}$, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is a linear and continuous tensor; see, e.g., [18].

Recall that there exists a constant $c > 0$, such that, for all $\boldsymbol{v} \in H^1(\Omega)^3$, we have

$$\sum_{i=1}^{3}\int_\Omega v_i\, v_i\, \mathrm{d}x + \sum_{i=1}^{3}\int_\Omega \nabla v_i \cdot \nabla v_i\, \mathrm{d}x \le c\left(\sum_{i=1}^{3}\int_\Omega v_i\, v_i\, \mathrm{d}x + \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x\right); \quad (1)$$

see for instance [9]. Thus, we deduce that $\|\cdot\|_{H^1(\Omega)^3}$ is equivalent with $|||\cdot|||_{H^1(\Omega)^3}$. Therefore, the space $(H^1(\Omega)^3, ((\cdot,\cdot))_{H^1(\Omega)^3}, |||\cdot|||_{H^1(\Omega)^3})$ is a Hilbert space.

-

$$\begin{aligned} H^{1/2}(\Gamma)^3 &= \boldsymbol{\gamma}(H^1(\Omega)^3) \\ &= \{\boldsymbol{w} = (w_1,w_2,w_3)^T \mid w_i \in H^{1/2}(\Gamma),\ 1 \le i \le 3\}. \end{aligned}$$

This space is endowed with the following inner product:

$$\begin{aligned} (\boldsymbol{\chi},\boldsymbol{w})_{H^{1/2}(\Gamma)^3} &= \sum_{i=1}^{3}(\chi_i,w_i)_{H^{1/2}(\Gamma)} \\ &= \sum_{i=1}^{3}(\chi_i,w_i)_{L^2(\Gamma)} + \sum_{i=1}^{3}\int_\Gamma\int_\Gamma \frac{(\chi_i(\boldsymbol{x})-\chi_i(\boldsymbol{y}))(w_i(\boldsymbol{x})-w_i(\boldsymbol{y}))}{\|\boldsymbol{x}-\boldsymbol{y}\|^3} ds(\boldsymbol{x})ds(\boldsymbol{y}) \end{aligned}$$

and the corresponding norm

$$\|\boldsymbol{w}\|_{H^{1/2}(\Gamma)^3} = \left(\|\boldsymbol{w}\|^2_{L^2(\Gamma)^3} + \int_\Gamma\int_\Gamma \frac{(\boldsymbol{w}(\boldsymbol{x})-\boldsymbol{w}(\boldsymbol{y}))^2}{\|\boldsymbol{x}-\boldsymbol{y}\|^3} ds(\boldsymbol{x})ds(\boldsymbol{y})\right)^{1/2}.$$

For more details about the Hilbert spaces $H^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)^3$, the reader can consult, e.g., [7,14]. Relevant to the matter are also, e.g., [10,15].

Recall that $\boldsymbol{\gamma} : H^1(\Omega)^3 \to L^2(\Gamma)^3$ is a linear, continuous and compact operator and $\boldsymbol{\gamma} : H^1(\Omega)^3 \to H^{1/2}(\Gamma)^3$ is a linear, continuous and surjective operator. Furthermore, there exists a linear, continuous operator $\boldsymbol{l} : H^{1/2}(\Gamma)^3 \to H^1(\Omega)^3$, such that $\boldsymbol{\gamma}(\boldsymbol{l}(\boldsymbol{\xi})) = \boldsymbol{\xi}$ for all $\boldsymbol{\xi} \in H^{1/2}(\Gamma)^3$. The operator $\boldsymbol{l}$ is called the *right inverse of the trace operator* $\boldsymbol{\gamma}$.

To proceed, we introduce useful closed subspaces of the space $H^1(\Omega)^3$.

- $X_0 = \{\boldsymbol{v} \in H^1(\Omega)^3 \mid \boldsymbol{\gamma}\boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\}$, $meas(\Gamma_1) > 0$; see, e.g., [18]. Recall that there exists $c_K = c_K(\Omega, \Gamma_1) > 0$, such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)^{3\times 3}} \geq c_K |||\boldsymbol{v}|||_{H^1(\Omega)^3} \quad \text{for all } \boldsymbol{v} \in X_0. \tag{2}$$

This is the Korn's inequality and for a proof of it the reader can consult, e.g., [16]. Let us introduce the following inner product:

$$(\boldsymbol{u}, \boldsymbol{v})_{X_0} = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}}$$

with the corresponding norm

$$\|\boldsymbol{v}\|_{X_0} = \left(\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x\right)^{1/2} = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)^{3\times 3}}.$$

Using Korn's inequality, we get the equivalence between $|||\cdot|||_{H^1(\Omega)^3}$ and $\|\cdot\|_{X_0}$. Therefore, $(X_0, (\cdot,\cdot)_{X_0}, \|\cdot\|_{X_0})$ is a Hilbert space. Furthermore, there exists a positive constant c_0, such that

$$\|\boldsymbol{\gamma}\boldsymbol{v}\|_{L^2(\Gamma_3)^3} \leq c_0 \|\boldsymbol{v}\|_{X_0} \quad \text{for all } \boldsymbol{v} \in X_0. \tag{3}$$

- Let us introduce a subspace of the space X_0 as follows:

$$X = \{\, \boldsymbol{v} \in X_0 \mid v_\nu = 0 \text{ a.e. on } \Gamma_3\}, \tag{4}$$

where $v_\nu = \boldsymbol{\gamma}\boldsymbol{v}\cdot\boldsymbol{\nu}$. As it is known, the space $(X, (\cdot,\cdot)_{X_0}, \|\cdot\|_{X_0})$ is a Hilbert space; see, e.g., [18].
- $S = \boldsymbol{\gamma}(X) = \{\boldsymbol{w} = \boldsymbol{\gamma}\boldsymbol{v} \text{ a.e. on } \Gamma,\ \boldsymbol{v} \in X\}$. We sent the reader to, e.g., Proposition 2.1 in [12] for a proof of the fact that $\boldsymbol{\gamma}(X_0)$ and $\boldsymbol{\gamma}(X)$ are closed subspaces of the Hilbert space $H^{1/2}(\Gamma)^3$. Therefore, $(S, (\cdot,\cdot)_{H^{1/2}(\Gamma)^3}, \|\cdot\|_{H^{1/2}(\Gamma)^3})$ is a Hilbert space.
- $Y = S'$, Y being the dual of S.
- We also need a convex subset of X as follows:

$$K = \{\boldsymbol{v} \in X \mid v_\nu \leq 0 \text{ a.e. on } \Gamma_4\}. \tag{5}$$

The set K is a nonempty, unbounded, closed, convex subset of X containing 0_X.

In the study of Problem 1, we admit the following hypotheses:

H 1. $\mathcal{E} = (\mathcal{E}_{ijkl}) : \mathbb{S}^3 \to \mathbb{S}^3$ *is a fourth-order tensor, such that*

(a) $\mathcal{E}\boldsymbol{\sigma} : \boldsymbol{\tau} = \boldsymbol{\sigma} : \mathcal{E}\boldsymbol{\tau}$ *for all* $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^3$;

(b) *there exists* $m_\mathcal{E} > 0$ *such that* $\mathcal{E}\boldsymbol{\tau} : \boldsymbol{\tau} \geq m_\mathcal{E}\|\boldsymbol{\tau}\|^2_{\mathbb{S}^3}$ *for all* $\boldsymbol{\tau} \in \mathbb{S}^3$.

H 2. $\boldsymbol{f}_0 \in L^2(\Omega)^3$, $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$, $g > 0$.

4. Weak Formulations

Let $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ be smooth enough functions which verify Problem 1. For all $\boldsymbol{v} \in X$, we have

$$\int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, \mathrm{d}x + \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, \mathrm{d}x = 0.$$

Using a Green formula for tensors, see, e.g., page 89 in [18] and taking into account the boundary conditions, we get

$$\int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x + \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, \mathrm{d}x = 0 \quad \text{for all } \boldsymbol{v} \in X.$$

Notice that $\boldsymbol{\sigma\nu} \cdot \boldsymbol{\gamma v} = \sigma_\nu\, v_\nu + \boldsymbol{\sigma_\tau} \cdot \boldsymbol{v_\tau} = \boldsymbol{\sigma_\tau} \cdot \boldsymbol{v_\tau}$ on Γ_3. Furthermore, since $\boldsymbol{v_\tau} = \boldsymbol{\gamma v} - v_\nu \boldsymbol{\nu} = \boldsymbol{\gamma v}$ on Γ_3, we obtain that $\boldsymbol{\sigma\nu} \cdot \boldsymbol{\gamma v} = \boldsymbol{\sigma_\tau} \cdot \boldsymbol{\gamma v}$ on Γ_3. Consequently

$$\int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma = \int_{\Gamma_3} \boldsymbol{\sigma_\tau} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in X.$$

On the other hand, $\boldsymbol{\sigma\nu} \cdot \boldsymbol{\gamma v} = \sigma_\nu\, v_\nu$ on Γ_4. Therefore

$$\int_{\Gamma_4} \boldsymbol{\sigma\nu} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma = \int_{\Gamma_4} \sigma_\nu \boldsymbol{\nu} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in X.$$

Then, for all $\boldsymbol{v} \in X$, we obtain

$$\begin{aligned} &\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\mathrm{d}x - \int_{\Gamma_3} \boldsymbol{\sigma_\tau} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma - \int_{\Gamma_4} \sigma_\nu \boldsymbol{\nu} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma \\ &\quad = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma. \end{aligned} \tag{6}$$

By (6), for all $\boldsymbol{v} \in X$, we can write

$$\begin{aligned} &\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\mathrm{d}x - \int_{\Gamma_3} \boldsymbol{\sigma_\tau} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})\mathrm{d}\Gamma \\ &\quad - \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\mathrm{d}x - \int_{\Gamma_4} \sigma_\nu \boldsymbol{\nu} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})\mathrm{d}\Gamma = \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})\mathrm{d}\Gamma. \end{aligned} \tag{7}$$

Let us define a bilinear form as follows:

$$a : X \times X \to \mathbb{R}, \quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x \quad \text{for all} \quad \boldsymbol{u}, \boldsymbol{v} \in X. \tag{8}$$

The form a is well defined. Indeed

$$\int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\, \mathrm{d}x = (\mathcal{N}_\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}},$$

where $\mathcal{N}_\mathcal{E} : L^2(\Omega)^{3\times 3} \to L^2(\Omega)^{3\times 3}$, $\mathcal{N}_\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}))(\boldsymbol{x}) = \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Omega$ is the Nemytskii operator; see, e.g., page 370 in [17].

In addition, by the Riesz's representation theorem, we define $\boldsymbol{f} \in X$ as follows:

$$(\boldsymbol{f}, \boldsymbol{v})_X = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma \ \text{ for all } \boldsymbol{v} \in X. \tag{9}$$

4.1. The First Weak Formulation

Due to the boundary conditions on Γ_4, by taking $\boldsymbol{v} \in K$, we can write

$$-\int_{\Gamma_4} \sigma_\nu \boldsymbol{\nu} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})\, \mathrm{d}\Gamma = -\int_{\Gamma_4} \sigma_\nu \boldsymbol{\nu} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma \leq 0.$$

Based on relations (7), (8) and (9), for all $\boldsymbol{v} \in K$, we get

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \int_{\Gamma_3} \boldsymbol{\sigma_\tau} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})\, \mathrm{d}\Gamma \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_X. \tag{10}$$

Let us define a functional as follows:

$$\phi : X \to \mathbb{R}_+, \qquad \phi(\boldsymbol{v}) = \int_{\Gamma_3} g \, \|\boldsymbol{v}_\tau\|_{\mathbb{R}^3}\, \mathrm{d}\Gamma. \tag{11}$$

We observe that for all $\boldsymbol{v} \in K$

$$-\int_{\Gamma_3} \boldsymbol{\sigma_\tau} \cdot \boldsymbol{\gamma v} \mathrm{d}\Gamma \leq \int_{\Gamma_3} \|\boldsymbol{\sigma_\tau}\|_{\mathbb{S}^3} \|\boldsymbol{\gamma v}\|_{\mathbb{R}^3} \mathrm{d}\Gamma \leq \int_{\Gamma_3} g \|\boldsymbol{v_\tau}\|_{\mathbb{R}^3} \mathrm{d}\Gamma = \phi(\boldsymbol{v}),$$

$$-\int_{\Gamma_3} \boldsymbol{\sigma_\tau} \cdot \boldsymbol{\gamma u} \mathrm{d}\Gamma = \int_{\Gamma_3} g \|\boldsymbol{u_\tau}\|_{\mathbb{R}^3} \mathrm{d}\Gamma = \phi(\boldsymbol{u}).$$

As a result

$$\phi(\boldsymbol{v}) - \phi(\boldsymbol{u}) \geq -\int_{\Gamma_3} \boldsymbol{\sigma_\tau} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u}) \mathrm{d}\Gamma.$$

Therefore, we arrive at the following weak formulation.

Problem 6. *Given $\boldsymbol{f} \in X$, find $\boldsymbol{u} \in K \subset X$, such that*

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \phi(\boldsymbol{v}) - \phi(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_X \quad \textit{for all } \boldsymbol{v} \in K \subset X.$$

4.2. The Second Weak Formulation

Let us introduce the Lagrange multiplier $\boldsymbol{\lambda}_2 \in X'$ as follows:

$$(\boldsymbol{\lambda}_2, \boldsymbol{v})_{X',X} = -\int_{\Gamma_3} \boldsymbol{\sigma_\tau} \cdot \boldsymbol{\gamma v}\, \mathrm{d}\Gamma - \int_{\Gamma_4} \sigma_\nu v_\nu \, \mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in X,$$

where $(\cdot, \cdot)_{X',X}$ denotes the duality pairing between X' and X; herein and everywhere below, X' stands for the dual of X.

Let us introduce the following set of the Lagrange multipliers:

$$\Lambda_2 = \{\boldsymbol{\mu} \in X' \mid (\boldsymbol{\mu}, \boldsymbol{v})_{X',X} \leq \int_{\Gamma_3} g \|\boldsymbol{v}_\tau\|_{\mathbb{R}^3} \, \mathrm{d}\Gamma \text{ for all } \boldsymbol{v} \in K\}.$$

Keeping in mind the boundary conditions on Γ_3 and Γ_4, we immediately deduce that $\boldsymbol{\lambda}_2 \in \Lambda_2$. Afterwards, we define a bilinear form

$$\widetilde{b} : X \times X' \to \mathbb{R}, \qquad \widetilde{b}(\boldsymbol{v}, \boldsymbol{\mu}) = (\boldsymbol{\mu}, \boldsymbol{v})_{X',X} \quad \text{for all } \boldsymbol{v} \in X,\, \boldsymbol{\mu} \in X'. \tag{12}$$

According to (6), we can write

$$a(\boldsymbol{u}, \boldsymbol{v}) + \widetilde{b}(\boldsymbol{v}, \boldsymbol{\lambda}_2) = (\boldsymbol{f}, \boldsymbol{v})_X \quad \text{for all } \boldsymbol{v} \in X,$$

where the forms a and $\widetilde{b}$ were defined in (8) and (12), respectively, and $\boldsymbol{f}$ was introduced in (9). On the other hand

$$\widetilde{b}(\boldsymbol{u}, \boldsymbol{\mu}) \leq \int_{\Gamma_3} g \|\boldsymbol{u_\tau}\|_{\mathbb{R}^3} \, \mathrm{d}\Gamma \quad \text{for all } \boldsymbol{\mu} \in \Lambda_2,$$

and

$$\begin{aligned}\widetilde{b}(\boldsymbol{u},\boldsymbol{\lambda}_2) &= (\boldsymbol{\lambda}_2,\boldsymbol{u})_{X',X}\\ &= -\int_{\Gamma_3} \boldsymbol{\sigma_\tau}\cdot\boldsymbol{\gamma u}\,\mathrm{d}\Gamma - \int_{\Gamma_4} \sigma_\nu u_\nu\,\mathrm{d}\Gamma\\ &= \int_{\Gamma_3} g\|\boldsymbol{u_\tau}\|_{\mathbb{R}^3}\mathrm{d}\Gamma.\end{aligned}$$

As a consequence

$$\widetilde{b}(\boldsymbol{u},\boldsymbol{\mu}-\boldsymbol{\lambda}_2) \le 0 \quad \text{for all } \boldsymbol{\mu}\in\Lambda_2. \tag{13}$$

Therefore, we can state the following variational formulation.

Problem 7. *Given $\boldsymbol{f}\in X$, find $\boldsymbol{u}\in X$ and $\boldsymbol{\lambda}_2\in\Lambda_2\subset X'$, such that*

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + \widetilde{b}(\boldsymbol{v},\boldsymbol{\lambda}_2) &= (\boldsymbol{f},\boldsymbol{v})_X && \textit{for all } \boldsymbol{v}\in X\\ \widetilde{b}(\boldsymbol{u},\boldsymbol{\mu}-\boldsymbol{\lambda}_2) &\le \boldsymbol{0} && \textit{for all } \boldsymbol{\mu}\in\Lambda_2.\end{aligned}$$

4.3. The Third Weak Formulation

Let us define the Lagrange multiplier $\boldsymbol{\lambda}_3\in Y$

$$\langle\boldsymbol{\lambda}_3,\boldsymbol{w}\rangle = -\int_{\Gamma_3}\boldsymbol{\sigma}_\tau\cdot\boldsymbol{w}\,\mathrm{d}\Gamma \text{ for all } \boldsymbol{w}\in S;$$

herein and everywhere below, $\langle\cdot,\cdot\rangle$ denotes the duality product between Y and S.

We also define a set of Lagrange multipliers as follows:

$$\Lambda_3 = \{\boldsymbol{\mu}\in Y \mid \langle\boldsymbol{\mu},\boldsymbol{w}\rangle \le \int_{\Gamma_3} g\,\|\boldsymbol{w}\|_{\mathbb{R}^3}\,\mathrm{d}\Gamma \text{ for all } \boldsymbol{w}\in S\}. \tag{14}$$

Due to the boundary conditions on Γ_3, we deduce that $\boldsymbol{\lambda}_3\in\Lambda_3$.

We define now a bilinear form as follows:

$$b: X\times Y\to\mathbb{R},\qquad b(\boldsymbol{v},\boldsymbol{\mu}) = \langle\boldsymbol{\mu},\boldsymbol{\gamma v}\rangle \quad \text{for all } \boldsymbol{v}\in X,\boldsymbol{\mu}\in Y. \tag{15}$$

According to (10), we can write

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + b(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\lambda}_3) \ge (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_X \quad \text{for all } \boldsymbol{v}\in K, \tag{16}$$

where a and b are the forms introduced in (8) and (15), $\boldsymbol{f}\in X$ is the element defined in (9), and K is the set defined in (5).

Furthermore

$$b(\boldsymbol{u},\boldsymbol{\mu}) \le \int_{\Gamma_3} g\|\boldsymbol{u_\tau}\|_{\mathbb{R}^3}\,\mathrm{d}\Gamma \quad \text{for all } \boldsymbol{\mu}\in\Lambda_3.$$

On the other hand

$$b(\boldsymbol{u},\boldsymbol{\lambda}_3) = \langle\boldsymbol{\lambda}_3,\boldsymbol{\gamma u}\rangle = -\int_{\Gamma_3}\boldsymbol{\sigma_\tau}\cdot\boldsymbol{\gamma u}\,\mathrm{d}\Gamma = \int_{\Gamma_3} g\|\boldsymbol{u_\tau}\|_{\mathbb{R}^3}\,\mathrm{d}\Gamma.$$

As a consequence

$$b(\boldsymbol{u},\boldsymbol{\mu}-\boldsymbol{\lambda}_3) \le 0 \quad \text{for all } \boldsymbol{\mu}\in\Lambda_3. \tag{17}$$

By (16) and (17), we arrive at the following weak formulation.

Problem 8. *Given* $\boldsymbol{f} \in X$*, find* $\boldsymbol{u} \in K \subset X$ *and* $\boldsymbol{\lambda}_3 \in \Lambda_3 \subset Y$*, such that*

$$a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + b(\boldsymbol{v}-\boldsymbol{u}, \boldsymbol{\lambda}_3) \geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_X \qquad \text{for all } \boldsymbol{v} \in K$$
$$b(\boldsymbol{u}, \boldsymbol{\mu}-\boldsymbol{\lambda}_3) \leq \mathbf{0} \qquad \text{for all } \boldsymbol{\mu} \in \Lambda_3.$$

4.4. The Fourth Weak Formulation

Let us consider a Lagrange multiplier $\boldsymbol{\lambda}_4 \in Y$, such that

$$\langle \boldsymbol{\lambda}_4, \boldsymbol{w} \rangle = -\int_{\Gamma_4} \sigma_\nu \, \boldsymbol{w} \cdot \boldsymbol{\nu} \, \mathrm{d}\Gamma \quad \text{for all } \boldsymbol{w} \in S$$

and the set of the Lagrange multipliers as follows:

$$\Lambda_4 = \{\boldsymbol{\mu} \in Y \mid \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle \leq 0 \quad \text{for all } \boldsymbol{w} \in S \text{ such that } \boldsymbol{w} \cdot \boldsymbol{\nu} \leq 0 \text{ a.e. on } \Gamma_4\}.$$

Since $\sigma_\nu \leq 0$ a.e. on Γ_4, it results that $\boldsymbol{\lambda}_4 \in \Lambda_4$.

Keeping in mind (7), (8), (9) (11) and (15), we get

$$a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + b(\boldsymbol{v}-\boldsymbol{u}, \boldsymbol{\lambda}_4) + \phi(\boldsymbol{v}) - \phi(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_X \quad \text{for all } \boldsymbol{v} \in X. \tag{18}$$

Furthermore

$$b(\boldsymbol{u}, \boldsymbol{\mu}) \leq 0 \quad \text{for all } \boldsymbol{\mu} \in \Lambda_4$$

and

$$b(\boldsymbol{u}, \boldsymbol{\lambda}_4) = \langle \boldsymbol{\lambda}_4, \gamma \boldsymbol{u} \rangle = -\int_{\Gamma_4} \sigma_\nu \gamma \boldsymbol{u} \cdot \boldsymbol{\nu} \, \mathrm{d}\Gamma = 0.$$

As a result

$$b(\boldsymbol{u}, \boldsymbol{\mu}-\boldsymbol{\lambda}_4) \leq 0 \quad \text{for all } \boldsymbol{\mu} \in \Lambda_4. \tag{19}$$

By (18) and (19), we arrive at the following variational problem.

Problem 9. *Given* $\boldsymbol{f} \in X$*, find* $\boldsymbol{u} \in X$ *and* $\boldsymbol{\lambda}_4 \in \Lambda_4 \subset Y$*, such that*

$$a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + b(\boldsymbol{v}-\boldsymbol{u}, \boldsymbol{\lambda}_4) + \phi(\boldsymbol{v}) - \phi(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_X \quad \text{for all } \boldsymbol{v} \in X$$
$$b(\boldsymbol{u}, \boldsymbol{\mu}-\boldsymbol{\lambda}_4) \leq 0 \qquad \text{for all } \boldsymbol{\mu} \in \Lambda_4,$$

where a, ϕ, $\boldsymbol{f}$, and b are those introduced in (8), (11), (9), and (15).

5. Main Results

In this section, we focus on the weak solvability of Problem 1 using successively the four variational formulations delivered in Sect. 4. Our first result is the following existence and uniqueness result regarding Problem 6.

Theorem 5. *We admit hypotheses H 1 and H 2. Then, Problem 6 has a unique solution,* $\boldsymbol{u}_1 \in K$*.*

Proof. We note that X defined in (4) is a Hilbert space. Obviously, K defined in (5) is a closed, convex subset of X, such that it contains 0_X. Therefore, the assumption A 5 holds true.

The form a in (8) is a symmetric and bilinear form. Moreover, (i) in the assumption A 1 holds true. Indeed

$$\begin{aligned}
|a(\boldsymbol{u},\boldsymbol{v})| &= \left|\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}x\right| \\
&\le \int_\Omega \|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})\|_{\mathbb{S}^3}\|\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\|_{\mathbb{S}^3}\,\mathrm{d}x \\
&\le \max_{i,j,k,l}|\mathcal{E}_{ijkl}| \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})\|_{\mathbb{S}^3}\|\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\|_{\mathbb{S}^3}\,\mathrm{d}x \\
&\le \max_{i,j,k,l}|\mathcal{E}_{ijkl}| \left(\int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})\|^2_{\mathbb{S}^3}\right)^{1/2} \left(\int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\|^2_{\mathbb{S}^3}\,\mathrm{d}x\right)^{1/2} \\
&= \max_{i,j,k,l}|\mathcal{E}_{ijkl}|\,\|\boldsymbol{u}\|_{X_0}\,\|\boldsymbol{v}\|_{X_0}.
\end{aligned}$$

Thus, we can choose $M_a = \max_{i,j,k,l}|\mathcal{E}_{ijkl}|$.

To prove (ii) in the assumption A 1, we evaluate

$$\begin{aligned}
a(\boldsymbol{v},\boldsymbol{v}) &= \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}x \ge \int_\Omega m_{\mathcal{E}}\|\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\|^2_{\mathbb{S}^3}\,\mathrm{d}x \\
&= m_{\mathcal{E}}\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{L^2(\Omega)^{3\times 3}} = m_{\mathcal{E}}\|\boldsymbol{v}\|^2_{X_0}.
\end{aligned}$$

We can take $m_a = m_{\mathcal{E}}$. Therefore, the assumption A 1 holds true.

Furthermore, the assumption A 3 is fulfilled. It is obviously that the functional ϕ in (11) is convex. Let us show that ϕ is a Lipschitz continuous functional. We write

$$\begin{aligned}
|\phi(\boldsymbol{v}) - \phi(\boldsymbol{u})| &= \left|\int_{\Gamma_3} (g\|\boldsymbol{v}_{\boldsymbol{\tau}}(\boldsymbol{x})\|_{\mathbb{R}^3} - g\|\boldsymbol{u}_{\boldsymbol{\tau}}(\boldsymbol{x})\|_{\mathbb{R}^3})\,\mathrm{d}\Gamma\right| \\
&\le g\int_{\Gamma_3} \|\boldsymbol{v}_{\boldsymbol{\tau}}(\boldsymbol{x}) - \boldsymbol{u}_{\boldsymbol{\tau}}(\boldsymbol{x})\|_{\mathbb{R}^3}\,\mathrm{d}\Gamma \\
&\le g\sqrt{meas(\Gamma_3)}\|\boldsymbol{\gamma}\boldsymbol{v} - \boldsymbol{\gamma}\boldsymbol{u}\|_{L^2(\Gamma_3)^3} \\
&\le c_0\, g\sqrt{meas(\Gamma_3)}\|\boldsymbol{v} - \boldsymbol{u}\|_{X_0},
\end{aligned}$$

where $c_0 > 0$ is the constant in (3). Therefore, we can take $L_\phi = c_0\, g\sqrt{meas(\Gamma_3)}$.

Therefore, we can apply Theorem 1. □

To proceed, we focus on the solvability of Problem 7.

Theorem 6. *We admit the hypotheses H 1 and H 2. Then, Problem 7 has a unique solution,* $(\boldsymbol{u}_2, \boldsymbol{\lambda}_2) \in X \times \Lambda_2$.

Proof. The assumptions A 1, A 2 and A4 are fulfilled. Hence, we can apply Theorem 2. □

Subsequently, we deliver a characterization of the solution $(\boldsymbol{u}_2, \boldsymbol{\lambda}_2)$ in terms of the unique solution of Problem 6, $\boldsymbol{u}_1$.

Let $\boldsymbol{u}_1 \in K$ be the unique solution of Problem 6. Next, we define $\boldsymbol{\lambda}_1 \in X'$ as follows:

$$(\boldsymbol{\lambda}_1, \boldsymbol{v})_{X',X} = (\boldsymbol{f}, \boldsymbol{v})_X - a(\boldsymbol{u}_1, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in X. \tag{20}$$

Proposition 1. *If* $\boldsymbol{u}_1 \in K$ *is the unique solution of Problem 6 and* $\boldsymbol{\lambda}_1$ *is the element of* X' *defined in (20), then* $(\boldsymbol{u}_1, \boldsymbol{\lambda}_1)$ *is the unique solution of Problem 7.*

Proof. Setting $\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}_1$, respectively, in Problem 6, we deduce that

$$a(\boldsymbol{u}_1, \boldsymbol{u}_1) + \phi(\boldsymbol{u}_1) = (\boldsymbol{f}, \boldsymbol{u}_1)_X. \tag{21}$$

And from this, keeping in mind Problem 6

$$a(\boldsymbol{u}_1, \boldsymbol{v}) + \phi(\boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v})_X \text{ for all } \boldsymbol{v} \in K.$$

As a result

$$\phi(\boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v})_X - a(\boldsymbol{u}_1, \boldsymbol{v}) \text{ for all } \boldsymbol{v} \in K.$$

Hence, keeping in mind (20), we immediately observe that $\boldsymbol{\lambda}_1 \in \Lambda_2$.

On the other hand, by (20), it is straightforward to see that

$$a(\boldsymbol{u}_1, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}_1) = (\boldsymbol{f}, \boldsymbol{v})_X \text{ for all } \boldsymbol{v} \in X. \tag{22}$$

Moreover, for all $\boldsymbol{\mu} \in \Lambda_2$, keeping in mind the definition of Λ_2, we have

$$b(\boldsymbol{u}_1, \boldsymbol{\mu}) \leq \phi(\boldsymbol{u}_1). \tag{23}$$

Due to (21), we can write

$$\phi(\boldsymbol{u}_1) = (\boldsymbol{f}, \boldsymbol{u}_1)_X - a(\boldsymbol{u}_1, \boldsymbol{u}_1).$$

Using now (20), we get

$$b(\boldsymbol{u}_1, \boldsymbol{\lambda}_1) = \phi(\boldsymbol{u}_1). \tag{24}$$

By (23) and (24), we get

$$b(\boldsymbol{u}_1, \boldsymbol{\mu} - \boldsymbol{\lambda}_1) \leq 0 \quad \text{ for all } \boldsymbol{\mu} \in \Lambda_2. \tag{25}$$

As a consequence, due to (22) and (25), the pair $(\boldsymbol{u}_1, \boldsymbol{\lambda}_1) \in K \times \Lambda_2$ where $\boldsymbol{u}_1$ is the unique solution of Problem 6 and $\boldsymbol{\lambda}_1$ is defined in (20) is a solution of Problem 7. However, Problem 7 has a unique solution $(\boldsymbol{u}_2, \boldsymbol{\lambda}_2) \in X \times \Lambda_2$. We conclude that $\boldsymbol{u}_2 = \boldsymbol{u}_1$ and $\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1$. □

Remark 1. According to Proposition 1, the first component of the unique pair solution of Problem 7 is the unique solution of Problem 6. Also, it is worth to underline that $\boldsymbol{u}_2 \in K$.

Remark 2. Proposition 1 allows us to give a new characterization of the unique solution of Problem 6. Indeed, the unique solution $\boldsymbol{u}_1$ of Problem 6 is the first component of the unique saddle point $(\boldsymbol{u}_2, \boldsymbol{\lambda}_2)$ of the functional

$$\mathcal{L} : X \times \Lambda_2 \to \mathbb{R}, \quad \mathcal{L}(\boldsymbol{v}, \boldsymbol{\mu}) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_X + b(\boldsymbol{v}, \boldsymbol{\mu}).$$

Afterwards, we pay attention to Problem 8.

Theorem 7. *We admit the hypotheses H 1 and H 2. Then, Problem 8 has a solution* $(\boldsymbol{u}_3, \boldsymbol{\lambda}_3) \in K \times \Lambda_3$, *unique in its first component.*

Proof. The form b defined in (15) is bilinear. To verify (a) in the assumption A 2, we can write

$$\begin{aligned}|b(\boldsymbol{v},\boldsymbol{\mu})| &= |\langle \boldsymbol{\mu},\boldsymbol{\gamma v}\rangle| \le \|\boldsymbol{\mu}\|_Y \|\boldsymbol{\gamma v}\|_{H^{1/2}(\Gamma)^3} \le c_\gamma \|\boldsymbol{\mu}\|_Y \|\boldsymbol{v}\|_{H^1(\Omega)^3}\\ &\le c_\gamma\, c\, \|\boldsymbol{\mu}\|_Y |||\boldsymbol{v}|||_{H^1(\Omega)^3} \le c_K^{-1}\, c_\gamma\, c\, \|\boldsymbol{\mu}\|_Y \|\boldsymbol{v}\|_{X_0},\end{aligned}$$

where $c_K > 0$ is the constant in the Korn's inequality (2), $c > 0$ is the constant in (1) and $c_\gamma > 0$ in the constant in the trace theorem. We can choose $M_b = c_K^{-1}\, c_\gamma\, c$.

The assumptions A 1, A 2 (a), A 4 and A 5 hold true. We claim that Λ_3 in (14) is a bounded set. Indeed, let $\boldsymbol{\mu} \in \Lambda_3$. Then

$$\begin{aligned}\frac{\langle \boldsymbol{\mu},\boldsymbol{\gamma v}\rangle}{\|\boldsymbol{v}\|_{X_0}} &\le \frac{\int_{\Gamma_3} g \|\boldsymbol{v}_{\boldsymbol{\tau}}\|_{\mathbb{R}^3}\, \mathrm{d}\Gamma}{\|\boldsymbol{v}\|_{X_0}} \le \frac{g\,\sqrt{meas(\Gamma_3)}\,\|\boldsymbol{\gamma v}\|_{L^2(\Gamma_3)^3}}{\|\boldsymbol{v}\|_{X_0}}\\ &\le c_0\, g\,\sqrt{meas(\Gamma_3)},\end{aligned}$$

where $c_0 > 0$ is the constant in (3). Consequently

$$\|\boldsymbol{\mu}\|_Y \le c_0\, g \sqrt{meas(\Gamma_3)} \quad \text{for all} \ \ \boldsymbol{\mu} \in \Lambda_3.$$

As Λ_3 is a bounded subset of the Hilbert space Y, we can apply Theorem 3 to obtain the existence of a solution $(\boldsymbol{u}_3, \boldsymbol{\lambda}_3) \in K \times \Lambda_3$ which is unique in its first component. □

Remark 3. We observe that $\boldsymbol{u}_3 \in K$. However, the equality $\boldsymbol{u}_3 = \boldsymbol{u}_1$ is left open.

Finally, we address Problem 9.

Theorem 8. *We admit the hypotheses H 1 and H 2. Then, Problem 9 has a solution* $(\boldsymbol{u}_4, \boldsymbol{\lambda}_4) \in X \times \Lambda_4$*, unique in its first component.*

Proof. Clearly, the set of the Lagrange multipliers Λ_4 is a closed, convex subset of Y which contains 0_Y.

In addition, Λ_4 is an unbounded subset of Y, since there exists a sequence $(\mu_n)_n \subset \Lambda_4$, such that $\|\mu_n\|_Y \to \infty$ as $n \to \infty$. Indeed, we can construct a sequence $(\mu_n)_n$, such that for each positive integer n, $\mu_n = n\,\mu_0$, where $\mu_0 \in \Lambda_4$ is defined as follows:

$$\langle \mu_0, \boldsymbol{w}\rangle = \int_{\Gamma_4} \boldsymbol{w}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, \mathrm{d}\Gamma \quad \text{for all} \ \ \boldsymbol{w} \in S.$$

If $\boldsymbol{w} \in S$, such that $\boldsymbol{w} \cdot \boldsymbol{\nu} \le 0$ a.e. on Γ_4, then

$$\int_{\Gamma_4} \boldsymbol{w}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, \mathrm{d}\Gamma \le 0.$$

Thus, $\mu_0 \in \Lambda_4$. As a result, $\mu_n = n\,\mu_0 \in \Lambda_4$ for all $n \in \mathbb{N}$.

Furthermore, for all positive integers n

$$\|\mu_n\|_Y = n\|\mu_0\|_Y.$$

Passing to the limit as $n \to \infty$ in this last relation, we are lead to $\|\mu_n\|_Y \to \infty$.

Let us prove the inf-sup property of the form b, i.e., (b) in the assumption A 2. Using similar arguments with those used in [3,12], we can write

$$\begin{aligned}\|\boldsymbol{\mu}\|_Y &= \sup_{\boldsymbol{w}\in S,\,\boldsymbol{w}\neq 0_S} \frac{\langle \boldsymbol{\mu}, \boldsymbol{w}\rangle}{\|\boldsymbol{w}\|_{H^{1/2}(\Gamma)^3}} \leq \sup_{\boldsymbol{z}\in X,\,\gamma\boldsymbol{z}\neq 0_S} \frac{b(\boldsymbol{z},\boldsymbol{\mu})}{\|\boldsymbol{\gamma z}\|_{H^{1/2}(\Gamma)^3}} \\ &\leq \sup_{\boldsymbol{z}\in X,\,\gamma\boldsymbol{z}\neq 0_S} \frac{b(\boldsymbol{l}(\boldsymbol{\gamma z}),\boldsymbol{\mu})}{\|\boldsymbol{\gamma z}\|_{H^{1/2}(\Gamma)^3}} \leq c_l \sup_{\boldsymbol{z}\in X,\,\boldsymbol{l}(\gamma\boldsymbol{z})\neq 0_X} \frac{b(\boldsymbol{l}(\boldsymbol{\gamma z}),\boldsymbol{\mu})}{|||\boldsymbol{l}(\boldsymbol{\gamma z})|||_{H^1(\Omega)^3}} \\ &\leq \widetilde{c}_l \sup_{\boldsymbol{v}\in X,\,\boldsymbol{v}\neq 0_X} \frac{b(\boldsymbol{v},\boldsymbol{\mu})}{\|\boldsymbol{v}\|_{X_0}}.\end{aligned}$$

Thus, we can take $\alpha = \widetilde{c}_l^{\,-1}$ to conclude that the assumption A 2 (b) is fulfilled.

Since the assumptions A 1- A 4 hold true, we are going to apply Theorem 4 to get the conclusion. □

Remark 4. The equality $\boldsymbol{u}_4 = \boldsymbol{u}_1$ is an open question.

Author contributions Both authors have contributed equally in writing this article. Both authors approved the final manuscript.

Funding None.

Availability of Data and Materials Not applicable.

Declarations

Conflict of Interest The authors have no competing interests to declare.

References

[1] Boffi, D., Brezzi, F., Fortin, M.: Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics 44. Springer, Berlin (2013)

[2] Braess, D.: Finite Elements. Theory, Fast Solvers, and Applications in Solid Mechanics, 2nd edition, Cambridge University Press, Cambridge (2001)

[3] Cojocaru, M.C., Matei, A.: Well-posedness for a class of frictional contact models via mixed variational formulations. Nonlinear Anal. Real World Appl. **47**, 127–141 (2019). https://doi.org/10.1016/j.nonrwa.2018.10.009

[4] Cojocaru, M.C., Matei, A.: On a class of saddle point problems and convergence results. Math. Model. Anal. **25**(4), 608–621 (2020). https://doi.org/10.3846/mma.2020.11140

[5] Ciurcea, R., Matei, A.: Solvability of a mixed variational problem. Ann. Univ. Craiova **36**(1), 105–111 (2009)

[6] Ekeland, I., Témam, R.: Convex Analysis and Variational Problems, Classics in Applied Mathematics, 28, SIAM (1999)

[7] Han, W., Sofonea, M.: Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, American Mathematical Society/International Press. Studies in Advanced Mathematics, **30** (2002)

[8] Haslinger, J., Hlaváček, I., Nečas, J.: Numerical methods for unilateral problems in solid mechanics, in Handbook of Numerical Analysis, J.-L. L.P Ciarlet, ed., **IV**, North-Holland, Amsterdam, pp. 313–485 (1996)

[9] Kikuchi, N., Oden, J.T.: Contact Problems in Elasticity: a Study of Variational Inequalities and Finite Element Methods. SIAM, Philadelphia (1988)

[10] Kufner, A., John, O., Fučik, S.: Function Spaces, in: Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: 406 Analysis, Noordhoff International Publishing, Leyden (1977)

[11] Matei, A.: Weak solvability via Lagrange multipliers for contact problems involving multi-contact zones. Math. Mech. Solids **21**(7), 826–841 (2016)

[12] Matei, A., Ciurcea, R.: Contact problems for nonlinearly elastic materials: weak solvability involving dual Lagrange multipliers. ANZIAM J. **52**, 160–178 (2010)

[13] Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Springer, New York (2013)

[14] Monk, P.: Numerical Mathematics and Scientific Computation. Finite Element Methods for Maxwell's Equations. Oxford University Press, Oxford (2003)

[15] Nečas, J.: Direct Methods in the Theory of Elliptic Equations, Springer, New York (2012)

[16] Nečas, J., Hlaváček, I.: Mathematical Theory of Elastic and Elastico-Plastic Bodies: an Introduction. Elsevier, Amsterdam (1981)

[17] Renardy, M., Rogers, R.C.: An Introduction to Partial Differential Equations, 2nd edition, Springer, New York (2004)

[18] Sofonea, M., Matei, A.: Mathematical Models in Contact Mechanics. Cambridge University Press, Cambridge (2012)

[19] Sofonea, M., Matei, A.: Variational Inequalities with Applications. A study of Antiplane Frictional Contact Problems, Springer, New York (2009)

Mariana Chivu Cojocaru
Doctoral School of Sciences
University of Craiova
A.I. Cuza 13
200585 Craiova
Romania
e-mail: chivumarianaflorentina@yahoo.com

Andaluzia Matei
Department of mathematics
University of Craiova
A.I. Cuza 13
200585 Craiova
Romania
e-mail: andaluzia.matei@edu.ucv.ro

Received: November 9, 2021.
Revised: January 22, 2022.
Accepted: August 6, 2022.