



On the Number of Limit Cycles Bifurcating from a Quartic Reversible Center

Bo Huang, Linping Peng and Yong Cui 

Abstract. This paper deals with the bifurcation of limit cycles from a quartic reversible and non-Hamiltonian system. By using the averaging theory and some mathematical technique on estimating the zeros of the function, we show that under small polynomial perturbation of degree $3n + 1$, at most $3n - 3$ limit cycles bifurcate from the period annulus of the unperturbed system for $n > 3$, while at most $2n$ limit cycles appear from the period annulus of the unperturbed system for $n = 1, 2, 3$. And the upper bound for the latter case is sharp.

Mathematics Subject Classification. 37G15, 34C07, 34C05.

Keywords. Quartic reversible center, period annulus, bifurcation of limit cycles, polynomial perturbation, averaging method.

1. Introduction

The second part of the famous Hilbert's 16th problem asks for the maximum number of limit cycles of planar real polynomial differential equations of degree n [19]. To attack this problem, many interesting and profound results have been established under various conditions. For example, the bifurcation of limit cycles from the periodic orbits around a center has been extensively studied in the literatures [9, 14, 20–22, 24–26, 31, 33, 37] and the references therein. Simultaneously, quite a few innovative methods have been proposed based on the Poincaré map [5, 10, 23], the Poincaré-Pontryagin-Melnikov integrals or the Abelian integrals [2, 3, 11, 36], the inverse integrating factor [15–17, 35], and the averaging function [4, 6, 12, 18, 22, 25, 26, 32] which is actually equivalent to the Abelian integrals in the plane.

As for the averaging theory, it gives a quantitative relation between the solutions of a non-autonomous periodic differential system and its averaged differential equation which is autonomous. For some differential equations, the problem about the number of limit cycles bifurcating from the

This research is supported by the Natural Science Foundation of Beijing and China (4192033, 1202018, 12101032).

unperturbed systems can be reduced to the exploration of hyperbolic equilibrium points of the corresponding averaged equations by using the averaging method. Hence, the averaging method has played a crucial role in the study of limit cycles of the differential systems. Now some elegant results on the number of limit cycles of the differential systems have been obtained, such as [1, 7, 13, 18, 20, 22, 24, 31–33, 37] and so on.

Generally, it is challenging to estimate the number of limit cycles in perturbations of a polynomial differential system of high degree. In the present paper, we choose a quartic differential system as follow

$$\begin{aligned} \dot{x} &= -y + x^3y + xy^3, \\ \dot{y} &= x + x^2y^2 + y^4, \end{aligned} \tag{1.1}$$

and study the bifurcation of limit cycles from it under any small polynomial perturbation of degree $3n + 1$ by the averaging method and some mathematical technique on estimating the zeros of the function.

Clearly, system (1.1) has

$$H(x, y) = \frac{1}{3(x^2 + y^2)^{\frac{3}{2}}} - \frac{x}{\sqrt{x^2 + y^2}} = c$$

as its first integral with the integrating factor $\frac{1}{(x^2+y^2)^{5/2}}$, and the unique finite singularity $(0, 0)$ as its isochronous center. The period annulus, denoted by $\{(x, y) | H(x, y) = c, c \in (1, +\infty)\}$, starts at the center $(0, 0)$ and terminates at the separatrix passing the infinite degenerate singularity on the equator.

We summarize our main results as follows.

Theorem 1.1. *Consider the following system*

$$\begin{aligned} \dot{x} &= -y + x^3y + xy^3 + \varepsilon f(x, y), \\ \dot{y} &= x + x^2y^2 + y^4 + \varepsilon g(x, y), \end{aligned} \tag{1.2}$$

with any sufficiently small parameter $|\varepsilon| \neq 0$, and the real polynomials $f(x, y)$ and $g(x, y)$ of degree $3n + 1$ in x and y , given by

$$f(x, y) = \sum_{k=1}^n \sum_{l+m=1+3k} a_{lm} x^l y^m, \quad g(x, y) = \sum_{k=1}^n \sum_{l+m=1+3k} b_{lm} x^l y^m.$$

Then using the first order averaging method, we have

1. For $n = 1, 2, 3$, at most $2n$ limit cycles arise from the period annulus around the center of the unperturbed system $(1.2)|_{\varepsilon=0}$, respectively. Moreover, in every case, this upper bound is sharp.
2. For any $n > 3$, at most $3n - 3$ limit cycles arise from the period annulus around the center of the unperturbed system $(1.2)|_{\varepsilon=0}$.

Remark 1.2. The result for $n = 1$ has been proved in [29]. We list it here for the completeness.

Remark 1.3. We remark that the perturbations in (1.2) exclude all terms of degrees $3n$ and $3n + 2$, because the computations of the averaged function will become too much complicated with them. Ideally, these terms should

be included in the analysis, and more limit cycles may appear with general perturbations of degree n .

The rest of this paper is organized as follows. In Sect. 2, we give an introduction on the averaging theory and present some important results about the integrals. Section 3 is devoted to the proof of Theorem 1.1 by computing the first order averaged function and exploring the number of its simple zeros for two cases. Some discussions are stated in Sect. 4.

2. Preliminary Results

In this section, we briefly introduce the averaging theory and prove some results about the integrals which will be used in the proof of the main theorem. More details for the averaging method, including applications, can be found in [27, 34].

2.1. Averaging Theory

Consider the system

$$x'(t) = F_0(t, x), \tag{2.1}$$

with $F_0 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$ a \mathcal{C}^2 function, T -periodic in the first variable, and Ω is an open subset of \mathbb{R}^n . Assume that there exists an open and bounded V with its closure $Cl(V) \subset \Omega$, and system (2.1) has $Cl(V)$ as its submanifold of periodic solutions such that for each $z \in Cl(V)$, $x(t, z)$ is T -periodic, where $x(t, z)$ denotes the solution of system (2.1) with $x(0, z) = z$. The set $Cl(V)$ is *isochronous* for system (2.1), i.e. it is a set formed only by periodic orbits having the same period.

Then the linearization of system (2.1) along the periodic solution $x(t, z)$ takes the form

$$y' = D_x F_0(t, x(t, z))y, \tag{2.2}$$

and denote by $M_z(t, z)$ a fundamental matrix of this linear system satisfying that $M_z(0, z)$ is the identity matrix.

Let ε be sufficiently small and we consider a perturbation of system (2.1) of the form

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon), \tag{2.3}$$

with $F_1 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$ are \mathcal{C}^2 functions, T -periodic in the first variable. Then, an answer to the problem of the bifurcation of T -periodic solutions from the $x(t, z)$ contained in $Cl(V)$ is given in the following result.

Lemma 2.1. (Perturbations of an isochronous set) *Assume that there exists an open and bounded set V with $Cl(V) \subset \Omega$ such that for each $z \in Cl(V)$, the solution $x(t, z)$ is T -periodic, then we consider the function $\mathcal{F} : Cl(V) \rightarrow \mathbb{R}^n$*

$$\mathcal{F}(z) = \int_0^T M_z^{-1}(t, z) F_1(t, x(t, z)) dt. \tag{2.4}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det((d\mathcal{F}/dz)(a)) \neq 0$, then there exists a T -periodic solution $\varphi(t, \varepsilon)$ of system (2.3) such that $\varphi(0, \varepsilon) \rightarrow a$ as $\varepsilon \rightarrow 0$.

Lemma 2.1 goes back to [28, 30], see Buică et al. [8] for a shorter proof.

2.2. Some Results About Integrals

For $\rho \in (1, +\infty)$ and $j \in \mathbb{Z}$, we define

$$I_j(\rho) = \int_0^{2\pi} \frac{1}{(\cos \theta + \rho)^j} d\theta. \tag{2.5}$$

By a straightforward computation, we obtain the following result.

Lemma 2.2. For $\rho \in (1, +\infty)$, $m, j \in \mathbb{N}$ and $m, j \geq 1$, we have

$$\int_0^{2\pi} \frac{\cos^m \theta}{(\cos \theta + \rho)^j} d\theta = \sum_{k=0}^m \binom{m}{k} (-\rho)^k I_{j+k-m}(\rho).$$

Moreover, we have

Lemma 2.3. For $\rho \in (1, +\infty)$, $j \in \mathbb{Z} \setminus \{1\}$, the integrals I_j, I_{j-1} and I_{j-2} defined above satisfy

$$\rho^2 I_j(\rho) = I_j(\rho) + \frac{2j-3}{j-1} \rho I_{j-1}(\rho) + \frac{2-j}{j-1} I_{j-2}(\rho).$$

Proof. Note that

$$\begin{aligned} I_{j-1}(\rho) &= \int_0^{2\pi} \frac{\cos \theta + \rho}{(\cos \theta + \rho)^j} d\theta = \rho I_j(\rho) + \int_0^{2\pi} \frac{d(\sin \theta)}{(\cos \theta + \rho)^j} \\ &= \rho I_j(\rho) - j \int_0^{2\pi} \frac{\sin^2 \theta}{(\cos \theta + \rho)^{j+1}} d\theta \\ &= \rho I_j(\rho) - j \int_0^{2\pi} \frac{1 - \cos^2 \theta}{(\cos \theta + \rho)^{j+1}} d\theta \\ &= \rho I_j(\rho) - j I_{j+1}(\rho) + j \rho^2 I_{j+1}(\rho) + j I_{j-1}(\rho) - 2j \rho I_j(\rho). \end{aligned}$$

Then

$$j \rho^2 I_{j+1}(\rho) = j I_{j+1}(\rho) + (2j-1) \rho I_j(\rho) + (1-j) I_{j-1}(\rho).$$

Replacing j by $j-1$, we can obtain Lemma 2.3. □

Lemma 2.4. For $\rho \in (1, +\infty)$ and $j \in \mathbb{Z}$, we have

$$I_j(\rho) = \frac{1}{(\rho^2 - 1)^{j-\frac{1}{2}}} I_{1-j}(\rho). \tag{2.6}$$

Moreover,

$$I_j(\rho) = Q_{-j}(\rho) \quad j \leq 0; \quad I_j(\rho) = \frac{Q_{j-1}(\rho)}{(\rho^2 - 1)^{j-\frac{1}{2}}}, \quad j \geq 1, \tag{2.7}$$

where Q_j stands for a polynomial of degree j .

Proof. Obviously, equality (2.6) is true for $j = 0, 1$. Now we prove the result for case $j \geq 2$ by induction.

Suppose that (2.6) holds for j and $j + 1$, that is

$$(\rho^2 - 1)^{j-\frac{1}{2}}I_j(\rho) = I_{1-j}(\rho), \quad (\rho^2 - 1)^{j+\frac{1}{2}}I_{j+1}(\rho) = I_{-j}(\rho). \tag{2.8}$$

We need to show that

$$(\rho^2 - 1)^{j+\frac{3}{2}}I_{j+2}(\rho) = I_{-(j+1)}(\rho). \tag{2.9}$$

Using Lemma 2.3, we have

$$(\rho^2 - 1)I_{j+2}(\rho) = \frac{2j + 1}{j + 1}\rho I_{j+1}(\rho) - \frac{j}{j + 1}I_j(\rho).$$

Multiplying the above equality by $(\rho^2 - 1)^{j+\frac{1}{2}}$, we get

$$(\rho^2 - 1)^{j+\frac{3}{2}}I_{j+2}(\rho) = \frac{2j + 1}{j + 1}\rho(\rho^2 - 1)^{j+\frac{1}{2}}I_{j+1}(\rho) - \frac{j}{j + 1}(\rho^2 - 1)^{j+\frac{1}{2}}I_j(\rho).$$

Then it follows from (2.8) that

$$(\rho^2 - 1)^{j+\frac{3}{2}}I_{j+2}(\rho) = \frac{2j + 1}{j + 1}\rho I_{-j}(\rho) - \frac{j}{j + 1}(\rho^2 - 1)I_{1-j}(\rho). \tag{2.10}$$

Using Lemma 2.3 again, we have

$$(\rho^2 - 1)I_{1-j}(\rho) = \frac{2j + 1}{j}\rho I_{-j}(\rho) - \frac{j + 1}{j}I_{-(j+1)}(\rho). \tag{2.11}$$

Substituting (2.11) into (2.10), we obtain (2.9), then (2.6) holds for $j \in \mathbb{Z}^+ \cup \{0\}$.

When $j > 1$, we have

$$I_{1-j}(\rho) = (\rho^2 - 1)^{j-\frac{1}{2}}I_j(\rho).$$

Let $m = 1 - j < 0$, then we have

$$I_m(\rho) = (\rho^2 - 1)^{\frac{1}{2}-m}I_{1-m}(\rho) = \frac{1}{(\rho^2 - 1)^{m-\frac{1}{2}}}I_{1-m}(\rho),$$

which implies that (2.6) holds for $j \in \mathbb{Z}^-$. Hence (2.6) is true for all $j \in \mathbb{Z}$.

By the definition and (2.6), we obtain (2.7). This completes the proof of Lemma 2.4. □

From the above results, we get Corollary 2.5.

Corollary 2.5. 1. For $k \in \mathbb{N}$, we have

$$I_{-(2k+1)}(\rho) = \rho Q_k(\rho^2), \quad I_{-2k}(\rho) = Q_k(\rho^2), \quad I_{2k+1}(\rho) = \frac{Q_k(\rho^2)}{(\rho^2 - 1)^{2k+\frac{1}{2}}}.$$

2. For $k \in \mathbb{N} \setminus \{0\}$, we have

$$I_{2k}(\rho) = \frac{\rho Q_{k-1}(\rho^2)}{(\rho^2 - 1)^{2k-\frac{1}{2}}},$$

where Q_j stands only for a polynomial of degree j . The difference in several equalities is ignored.

A straightforward computation leads to Lemma 2.6 immediately.

Lemma 2.6. *The following explicit expressions hold.*

$$\begin{aligned}
 I_0(\rho) &= 2\pi, \quad I_1(\rho) = \frac{2\pi}{(\rho^2 - 1)^{\frac{1}{2}}}, \quad I_2(\rho) = \frac{2\pi\rho}{(\rho^2 - 1)^{\frac{3}{2}}}, \\
 I_{-1}(\rho) &= 2\pi\rho, \quad I_{-2}(\rho) = \pi(2\rho^2 + 1), \quad I_{-3}(\rho) = \pi(2\rho^3 + 3\rho).
 \end{aligned}$$

3. Proof of Theorem 1.1

We split the proof into three steps: first derive the explicit expression of the averaged function, then prove Theorem 1.1 for the odd and even cases, respectively.

In the polar coordinates, system (1.2) becomes the form

$$\begin{aligned}
 \dot{r} &= r^4 \sin \theta + \varepsilon F(r, \theta), \\
 \dot{\theta} &= 1 + \varepsilon G(r, \theta),
 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
 F(r, \theta) &= \sum_{k=1}^n \sum_{l+m=1+3k} r^{l+m} (a_{lm} \cos^{l+1} \theta \sin^m \theta + b_{lm} \cos^l \theta \sin^{m+1} \theta), \\
 G(r, \theta) &= \sum_{k=1}^n \sum_{l+m=1+3k} r^{l+m-1} (b_{lm} \cos^{l+1} \theta \sin^m \theta - a_{lm} \cos^l \theta \sin^{m+1} \theta).
 \end{aligned}$$

Obviously, system (3.1) is equivalent to

$$\frac{dr}{d\theta} = r^4 \sin \theta + \varepsilon F_1(r, \theta) + O(\varepsilon^2) \tag{3.2}$$

where

$$\begin{aligned}
 F_1(r, \theta) &= \sum_{k=1}^n \sum_{l+m=1+3k} r^{l+m+3} (a_{lm} \cos^l \theta \sin^{m+2} \theta - b_{lm} \cos^{l+1} \theta \sin^{m+1} \theta) \\
 &\quad + \sum_{k=1}^n \sum_{l+m=1+3k} r^{l+m} (a_{lm} \cos^{l+1} \theta \sin^m \theta + b_{lm} \cos^l \theta \sin^{m+1} \theta).
 \end{aligned}$$

3.1. First Order Averaged Function

In this subsection, we first derive the formula for the first averaged function $\mathcal{F}(z)$ of system (3.2), then obtain the associated function whose zeros coincide with those of $\mathcal{F}(z)$.

It is not difficult to know that the closed orbits of system (3.2)| $_{\varepsilon=0}$ takes the form

$$r_0(\theta, z) = \frac{1}{(3 \cos \theta - 3 + \frac{1}{z^3})^{\frac{1}{3}}}, \quad z \in (0, \sqrt[3]{1/6})$$

with period 2π and the initial condition $r_0(0, z) = z$. The linearization of system (3.2)| $_{\varepsilon=0}$ along $r_0(\theta, z)$ takes the form

$$\dot{y} = \frac{4 \sin \theta}{3(\cos \theta - 1) + \frac{1}{z^3}} y.$$

Take the fundamental matrix of the above system

$$M_z(\theta, z) = \frac{1}{z^4 \left(3 \cos \theta - 3 + \frac{1}{z^3}\right)^{4/3}},$$

and its inverse

$$M_z^{-1}(\theta, z) = z^4 \left(3 \cos \theta - 3 + \frac{1}{z^3}\right)^{4/3},$$

then from Lemma 2.1, the averaged function of system (3.2) can be expressed as

$$\begin{aligned} \mathcal{F}(z) &= \int_0^{2\pi} M_z^{-1}(\theta, z) \cdot F_1(r, \theta)|_{r=r_0(\theta, z)} d\theta \\ &= \int_0^{2\pi} z^4 (3 \cos \theta + 3\rho)^{4/3} \left\{ \sum_{k=1}^n \sum_{l+m=1+3k} r^{l+m+3} (a_{lm} \cos^l \theta \sin^{m+2} \theta \right. \\ &\quad \left. - b_{lm} \cos^{l+1} \theta \sin^{m+1} \theta) \right. \\ &\quad \left. + \sum_{k=1}^n \sum_{l+m=1+3k} r^{l+m} (a_{lm} \cos^{l+1} \theta \sin^m \theta + b_{lm} \cos^l \theta \sin^{m+1} \theta) \right\} \\ &\quad \Bigg|_{r=\frac{1}{(3 \cos \theta + 3\rho)^{1/3}}} d\theta \tag{3.3} \\ &= z^4 \left\{ \sum_{k=1}^n \int_0^{2\pi} \sum_{l+m=1+3k} \frac{a_{lm} \cos^l \theta \sin^{m+2} \theta - b_{lm} \cos^{l+1} \theta \sin^{m+1} \theta}{3^k (\cos \theta + \rho)^k} d\theta \right. \\ &\quad \left. + \sum_{k=1}^n \int_0^{2\pi} \sum_{l+m=1+3k} \frac{a_{lm} \cos^{l+1} \theta \sin^m \theta + b_{lm} \cos^l \theta \sin^{m+1} \theta}{3^{k-1} (\cos \theta + \rho)^{k-1}} d\theta \right\} \\ &= z^4 \left\{ \sum_{k=1}^n \int_0^{2\pi} \frac{R_{3k+3}(\cos \theta, \sin \theta)}{3^k (\cos \theta + \rho)^k} d\theta + \sum_{k=1}^n \int_0^{2\pi} \frac{R_{3k+2}(\cos \theta, \sin \theta)}{3^{k-1} (\cos \theta + \rho)^{k-1}} d\theta \right\}, \end{aligned}$$

where $\rho = -1 + 1/(3z^3) \in (1, +\infty)$, $R_l(x, y)$ stands for a homogeneous polynomial of degree l in x and y .

Noting that

$$\begin{aligned} \int_0^{2\pi} \frac{\cos^p \theta \sin^{2q+1} \theta}{(\cos \theta + \rho)^k} d\theta &= \int_0^{2\pi} \frac{\cos^p \theta \sin^{2q+1} \theta}{(\cos \theta + \rho)^{k-1}} d\theta = 0, \\ \int_0^{2\pi} \frac{\cos^p \theta \sin^{2q} \theta}{(\cos \theta + \rho)^k} d\theta &= \int_0^{2\pi} \frac{\cos^p \theta (1 - \cos^2 \theta)^q}{(\cos \theta + \rho)^k} d\theta, \\ \int_0^{2\pi} \frac{\cos^p \theta \sin^{2q} \theta}{(\cos \theta + \rho)^{k-1}} d\theta &= \int_0^{2\pi} \frac{\cos^p \theta (1 - \cos^2 \theta)^q}{(\cos \theta + \rho)^{k-1}} d\theta \end{aligned}$$

for any nonnegative integer numbers p and q , we get

$$\mathcal{F}(z) = z^4 \left\{ \sum_{k=1}^n \int_0^{2\pi} \frac{S_{3k+3}(\cos \theta)}{(\cos \theta + \rho)^k} d\theta + \sum_{k=2}^n \int_0^{2\pi} \frac{T_{3k+2}(\cos \theta)}{(\cos \theta + \rho)^{k-1}} d\theta \right\},$$

where $S_l(x, y)$ and $T_l(x, y)$ denote the polynomials of degree l in x and y , respectively. Then define the function

$$G(\rho) = \sum_{k=1}^n \int_0^{2\pi} \frac{S_{3k+3}(\cos \theta)}{(\cos \theta + \rho)^k} d\theta + \sum_{k=2}^n \int_0^{2\pi} \frac{T_{3k+2}(\cos \theta)}{(\cos \theta + \rho)^{k-1}} d\theta,$$

we obtain Remark 3.1.

Remark 3.1. The non-zero zeros of $\mathcal{F}(z)$ coincide with those of the function $G(\rho)$. In the following, we study the zeros of the function $G(\rho)$ for $\rho \in (1, +\infty)$ instead of $\mathcal{F}(z)$ for $z \in (0, \sqrt[3]{1/6})$.

From the definitions, we know that the degree of S_{3k+3} is odd when k is even, and even when k is odd, while it is opposite for T_{3k+2} . This fact yields

$$\begin{aligned} & \sum_{k=1}^n \frac{S_{3k+3}(\cos \theta)}{(\cos \theta + \rho)^k} \\ &= \frac{S_{6,0} + S_{6,2} \cos^2 \theta + S_{6,4} \cos^4 \theta + S_{6,6} \cos^6 \theta}{\cos \theta + \rho} \\ &+ \frac{S_{9,1} \cos \theta + S_{9,3} \cos^3 \theta + S_{9,5} \cos^5 \theta + S_{9,7} \cos^7 \theta + S_{9,9} \cos^9 \theta}{(\cos \theta + \rho)^2} \\ &+ \dots + \frac{S_{3n+3}(\cos \theta)}{(\cos \theta + \rho)^n}, \tag{3.4} \\ & \sum_{k=2}^n \frac{T_{3k+2}(\cos \theta)}{(\cos \theta + \rho)^{k-1}} \\ &= \frac{T_{8,0} + T_{8,2} \cos^2 \theta + T_{8,4} \cos^4 \theta + T_{8,6} \cos^6 \theta + T_{8,8} \cos^8 \theta}{\cos \theta + \rho} \\ &+ \dots + \frac{T_{3n+2}(\cos \theta)}{(\cos \theta + \rho)^{n-1}}, \end{aligned}$$

where $S_{i,j}$ and $T_{i,j}$ are dependent on the perturbation coefficients a_{lm} and b_{lm} . After some simplification, we have

$$\begin{aligned} G(\rho) &= \sum_{k=1}^n \int_0^{2\pi} \frac{S_{3k+3}(\cos \theta)}{(\cos \theta + \rho)^k} d\theta + \sum_{k=2}^n \int_0^{2\pi} \frac{T_{3k+2}(\cos \theta)}{(\cos \theta + \rho)^{k-1}} d\theta \\ &= \int_0^{2\pi} \left[B_{0,0} \cdot (\cos \theta + \rho)^{2n+3} + B_{1,1} \rho \cdot (\cos \theta + \rho)^{2n+2} \right. \\ &+ (B_{2,0} + B_{2,2} \rho^2) \cdot (\cos \theta + \rho)^{2n+1} \\ &+ \dots + (B_{2n+2,0} + B_{2n+2,2} \rho^2 + \dots + B_{2n+2,2n+2} \rho^{2n+2}) \cdot (\cos \theta + \rho) \\ &+ (B_{2n+3,1} \rho + B_{2n+3,3} \rho^3 + \dots + B_{2n+3,2n+3} \rho^{2n+3}) \\ &+ \frac{B_{2n+4,0} + B_{2n+4,2} \rho^2 + \dots + B_{2n+4,2n+4} \rho^{2n+4}}{\cos \theta + \rho} \\ &+ \left. \frac{B_{2n+5,1} \rho + B_{2n+5,3} \rho^3 + \dots + B_{2n+5,2n+5} \rho^{2n+5}}{(\cos \theta + \rho)^2} \right] d\theta \end{aligned}$$

$$\begin{aligned}
 & + \cdots + \frac{B_{3n+3}\rho^{m(i)}}{(\cos \theta + \rho)^n} \Big] d\theta \\
 = & B_{0,0} \cdot I_{-(2n+3)}(\rho) + B_{1,1}\rho \cdot I_{-(2n+2)}(\rho) + (B_{2,0} + B_{2,2}\rho^2) \cdot I_{-(2n+1)}(\rho) \\
 & + \cdots + (B_{2n+2,0} + B_{2n+2,2}\rho^2 + \cdots + B_{2n+2,2n+2}\rho^{2n+2}) \cdot I_{-1}(\rho) \\
 & + (B_{2n+3,1}\rho + B_{2n+3,3}\rho^3 + \cdots + B_{2n+3,2n+3}\rho^{2n+3}) \cdot I_0(\rho) \\
 & + (B_{2n+4,0} + B_{2n+4,2}\rho^2 + \cdots + B_{2n+4,2n+4}\rho^{2n+4})I_1(\rho) \\
 & + (B_{2n+5,1}\rho + B_{2n+5,3}\rho^3 + \cdots + B_{2n+5,2n+5}\rho^{2n+5})I_2(\rho) \\
 & + \cdots + B_{3n+3}\rho^{m(n)} I_n(\rho), \tag{3.5}
 \end{aligned}$$

where $I_n(\rho)$ is defined as before, the coefficient $B_{i,j}$ depends on the coefficients of $S_{i,j}$ and $T_{i,j}$, that is, they are the functions of any real perturbation coefficients a_{lm} and b_{lm} . Moreover, we obtain

$$B_{3n+3}\rho^{m(n)} = \sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i}\rho^{2i} \tag{3.6}$$

for the odd number n , and

$$B_{3n+3}\rho^{m(n)} = \sum_{i=0}^{\frac{3n+2}{2}} B_{3n+3,2i+1}\rho^{2i+1}$$

for the even number n .

Lemma 3.2. *For the coefficients $B_{3n+3,2i}$ and $B_{3n+3,2i+1}$ defined above, the following statements are true.*

1. When n is odd, we have

$$\sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i} = 0;$$

2. When n is even, we have

$$\sum_{i=0}^{\frac{3n+2}{2}} B_{3n+3,2i+1} = 0.$$

Proof. For the odd number n , it follows from (3.3) and (3.4) that

$$\begin{aligned}
 \frac{S_{3n+3}(\cos \theta)}{(\cos \theta + \rho)^n} &= \frac{S_{3n+3,0} + S_{3n+3,2} \cos^2 \theta + \cdots + S_{3n+3,3n+3} \cos^{3n+3} \theta}{(\cos \theta + \rho)^n} \\
 &= \sum_{2p+2q=1+3n} \frac{a_{2p,2q} \cos^{2p} \theta \sin^{2q+2} \theta}{3^n (\cos \theta + \rho)^n} \\
 &\quad - \sum_{2p+2q=3n-1} \frac{b_{2p+1,2q+1} \cos^{2p+2} \theta \sin^{2q+2} \theta}{3^n (\cos \theta + \rho)^n} .s \tag{3.7}
 \end{aligned}$$

Comparing (3.6) with (3.7), we have

$$B_{3n+3,2i} = S_{3n+3,2i} \binom{2i}{0} = S_{3n+3,2i}.$$

Substituting the above equality and $\cos \theta = 1$ into (3.7), we can obtain the first result.

Similarly, the second result follows from

$$\begin{aligned} \frac{S_{3n+3}(\cos \theta)}{(\cos \theta + \rho)^n} &= \frac{S_{3n+3,1} \cos \theta + S_{3n+3,3} \cos^3 \theta + \cdots + S_{3n+3,3n+3} \cos^{3n+3} \theta}{(\cos \theta + \rho)^n} \\ &= \sum_{2p+2q=3n} \frac{a_{2p+1,2q} \cos^{2p+1} \theta \sin^{2q+2} \theta - b_{2p,2q+1} \cos^{2p+1} \theta \sin^{2q+2} \theta}{3^n (\cos \theta + \rho)^n}. \end{aligned}$$

This completes the proof of Lemma 3.2. □

3.2. Proof of Theorem 1.1 for the Odd Case

This subsection aims at proving Theorem 1.1 for the odd number n by studying the zeros of the function $G(\rho)$. In addition, an example is given to illustrate that the upper bound of the number of limit cycles for $n = 3$ can be reached.

When n is odd, we have

$$\begin{aligned} G(\rho) &= B_{0,0} \cdot I_{-(2n+3)}(\rho) + B_{1,1} \rho \cdot I_{-(2n+2)}(\rho) + (B_{2,0} + B_{2,2} \rho^2) \cdot I_{-(2n+1)}(\rho) \\ &\quad + \cdots + (B_{2n+2,0} + B_{2n+2,2} \rho^2 + \cdots + B_{2n+2,2n+2} \rho^{2n+2}) \cdot I_{-1}(\rho) \\ &\quad + (B_{2n+3,1} \rho + B_{2n+3,3} \rho^3 + \cdots + B_{2n+3,2n+3} \rho^{2n+3}) \cdot I_0(\rho) \\ &\quad + (B_{2n+4,0} + B_{2n+4,2} \rho^2 + \cdots + B_{2n+4,2n+4} \rho^{2n+4}) I_1(\rho) \\ &\quad + (B_{2n+5,1} \rho + B_{2n+5,3} \rho^3 + \cdots + B_{2n+5,2n+5} \rho^{2n+5}) I_2(\rho) \\ &\quad + \cdots + \sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i} \rho^{2i} I_n(\rho) \\ &= \sum_{t=0}^{n+1} I_{-(2t+1)}(\rho) \sum_{i=0}^{n-t+1} B_{2n-2t+2,2i} \rho^{2i} \\ &\quad + \sum_{t=0}^{n+1} I_{-(2t)}(\rho) \sum_{i=0}^{n-t+1} B_{2n-2t+3,2i+1} \rho^{2i+1} \\ &\quad + \sum_{t=0}^{\frac{n-1}{2}} I_{2t+1}(\rho) \sum_{i=0}^{n+t+2} B_{2n+2t+4,2i} \rho^{2i} \\ &\quad + \sum_{t=1}^{\frac{n-1}{2}} I_{2t}(\rho) \sum_{i=0}^{n+t+1} B_{2n+2t+3,2i+1} \rho^{2i+1}. \end{aligned} \tag{3.8}$$

To simplify (3.8), we list Lemmas 3.3-3.4 which can be derived from Corollary 2.5.

Lemma 3.3. *For any integer $k \geq 0$, we get*

$$\begin{aligned} I_{-(2k+1)}(\rho) &= \sum_{i=0}^{n-k+1} B_{2n-2k+2,2i} \rho^{2i} \\ &= M_{2n+3,1}^{(k)} \rho + M_{2n+3,3}^{(k)} \rho^3 + \dots + M_{2n+3,2n+3}^{(k)} \rho^{2n+3}, \\ I_{-(2k)}(\rho) &= \sum_{i=0}^{n-k+1} B_{2n-2k+3,2i+1} \rho^{2i+1} \\ &= N_{2n+3,1}^{(k)} \rho + N_{2n+3,3}^{(k)} \rho^3 + \dots + N_{2n+3,2n+3}^{(k)} \rho^{2n+3}, \end{aligned}$$

where all coefficients $M_{2n+3,2i+1}^{(k)}$ and $N_{2n+3,2i+1}^{(k)}$ are the linear combination of $B_{2n-2k+2,2i}$ and $B_{2n-2k+3,2i+1}$ given above.

Lemma 3.4. *The following statements are true.*

1. *For any integer $k \geq 1$, we have*

$$I_{2k}(\rho) \sum_{i=0}^{n+k+1} B_{2n+2k+3,2i+1} \rho^{2i+1} = \frac{\rho}{(\rho^2 - 1)^{2k-\frac{1}{2}}} \cdot \sum_{i=0}^{n+2k} B_{2n+4k+1,2i+1}^{(k)} \rho^{2i+1}.$$

2. *For any integer $k \geq 0$, we have*

$$I_{2k+1}(\rho) \sum_{i=0}^{n+k+2} B_{2n+2k+4,2i} \rho^{2i} = \frac{1}{(\rho^2 - 1)^{2k+\frac{1}{2}}} \cdot \sum_{i=0}^{n+2k+2} B_{2n+4k+4,2i}^{(k)} \rho^{2i}.$$

where the coefficients $B_{2n+4k+1,2i+1}^{(k)}$ and $B_{2n+4k+4,2i}^{(k)}$ are the linear combinations of $B_{2n+2k+3,2i+1}$ and $B_{2n+4k+4,2i}$, respectively.

Moreover, we have

Lemma 3.5. 1. *For the odd number n , define*

$$h_1(\omega^2) := \sum_{i=0}^{2n+1} B_{4n+2,2i}^{(\frac{n-1}{2})} (1 + \omega^2)^{2i} (1 - \omega^2)^{4n-2i+2},$$

then we have $h_1(\omega^2) = \omega^2 \cdot R_{4n}^*(\omega^2)$, where $R_{4n}^*(x)$ denotes a symmetrical polynomials of degree $4n$ in x .

2. *For the even number n , define*

$$h_2(\omega^2) := \sum_{i=0}^{2n} B_{4n+1,2i+1}^{(\frac{n}{2})} (1 + \omega^2)^{2i+2} (1 - \omega^2)^{4n-2i},$$

then we have $h_2(\omega^2) = \omega^2 \cdot R_{4n}^{**}(\omega^2)$, where $R_{4n}^{**}(x)$ denotes a symmetrical polynomials of degree $4n$ in x .

Proof. Here we only prove the first result, the second one is similar.

In fact, using Corollary 2.5, we have

$$\begin{aligned}
 I_n(\rho) \sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i} \rho^{2i} &= \frac{Q_{\frac{n-1}{2}}(\rho^2)}{(\rho^2 - 1)^{n-\frac{1}{2}}} \sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i} \rho^{2i} \\
 &= \frac{1}{(\rho^2 - 1)^{n-\frac{1}{2}}} \left[\sum_{j=0}^{\frac{n-1}{2}} q_{\frac{n-1}{2},2j} \rho^{2j} \right] \sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i} \rho^{2i}.
 \end{aligned}$$

On the other hand, when $k = \frac{n-1}{2}$ in the second equality of Lemma 3.4, we also get

$$I_n(\rho) \sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i} \rho^{2i} = \frac{1}{(\rho^2 - 1)^{n-\frac{1}{2}}} \sum_{i=0}^{2n+1} B_{4n+2,2i}^{(\frac{n-1}{2})} \rho^{2i}.$$

Hence we have

$$\left[\sum_{j=0}^{\frac{n-1}{2}} q_{\frac{n-1}{2},2j} \rho^{2j} \right] \sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i} \rho^{2i} = \sum_{i=0}^{2n+1} B_{4n+2,2i}^{(\frac{n-1}{2})} \rho^{2i}. \tag{3.9}$$

Let $\rho = \frac{1+\omega^2}{1-\omega^2}$ in (3.9) and define

$$\begin{aligned}
 h_1(\omega^2) &:= \sum_{i=0}^{2n+1} B_{4n+2,2i}^{(\frac{n-1}{2})} (1 + \omega^2)^{2i} (1 - \omega^2)^{4n-2i+2} \\
 &= \left[\sum_{j=0}^{\frac{n-1}{2}} q_{\frac{n-1}{2},2j} (1 + \omega^2)^{2j} (1 - \omega^2)^{n-2j-1} \right] \\
 &\quad \left[\sum_{i=0}^{\frac{3n+3}{2}} B_{3n+3,2i} (1 + \omega^2)^{2i} (1 - \omega^2)^{3n-2i+3} \right].
 \end{aligned}$$

From Lemma 3.2, it is not difficult to know that $h_1(x)$ is a symmetrical polynomial of degree $4n + 1$ with zero constant term. Then we obtain

$$h_1(\omega^2) = \omega^2 \cdot R_{4n}^*(\omega^2),$$

where $R^*(x)$ is a symmetrical polynomial of degree $4n$ in x . This is just the first result of Lemma 3.5. □

Using Lemmas 3.3–3.5, (3.8) can be simplified as

$$\begin{aligned}
 G(\rho) &= \sum_{t=0}^{n+1} I_{-(2t+1)}(\rho) \sum_{i=0}^{n-t+1} B_{2n-2t+2,2i} \rho^{2i} \\
 &\quad + \sum_{t=0}^{n+1} I_{-2t}(\rho) \sum_{i=0}^{n-t+1} B_{2n-2t+3,2i+1} \rho^{2i+1} \\
 &\quad + \sum_{t=0}^{\frac{n-1}{2}} I_{2t+1}(\rho) \sum_{i=0}^{n+t+2} B_{2n+2t+4,2i} \rho^{2i}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{t=1}^{\frac{n-1}{2}} I_{2t}(\rho) \sum_{i=0}^{n+t+1} B_{2n+2t+3,2i+1} \rho^{2i+1} \\
 & = \sum_{t=0}^{\frac{n-1}{2}} \frac{1}{(\rho^2 - 1)^{2t+\frac{1}{2}}} \cdot \sum_{i=0}^{n+2t+2} B_{2n+4t+4,2i}^{(t)} \rho^{2i} \\
 & + \sum_{t=1}^{\frac{n-1}{2}} \frac{\rho}{(\rho^2 - 1)^{2t-\frac{1}{2}}} \cdot \sum_{i=0}^{n+2t} B_{2n+4t+1,2i+1}^{(t)} \rho^{2i+1} \\
 & + (D_{2n+3,1}\rho + D_{2n+3,3}\rho^3 + \dots + D_{2n+3,2n+3}\rho^{2n+3}), \quad (3.10)
 \end{aligned}$$

where

$$D_{2n+3,2i+1} = \sum_{k=0}^{n+1} \left(M_{2n+3,2i+1}^{(k)} + N_{2n+3,2i+1}^{(k)} \right)$$

for $i = 1, 2, \dots, n$. Making the transformation $\rho = (1 + \omega^2)/(1 - \omega^2)$ for $\omega \in (0, 1)$ in (3.10), we have

$$\begin{aligned}
 \tilde{G}(\omega) & := G(\rho) \Big|_{\rho=\frac{1+\omega^2}{1-\omega^2}} \\
 & = \left[D_{2n+3,1} \left(\frac{1 + \omega^2}{1 - \omega^2} \right) + D_{2n+3,3} \left(\frac{1 + \omega^2}{1 - \omega^2} \right)^3 \right. \\
 & \quad \left. + \dots + D_{2n+3,2n+3} \left(\frac{1 + \omega^2}{1 - \omega^2} \right)^{2n+3} \right] \\
 & + \sum_{t=0}^{\frac{n-1}{2}} \left[\frac{(1 - \omega^2)^{4t+1}}{(2\omega)^{4t+1}} \cdot \sum_{i=0}^{n+2t+2} B_{2n+4t+4,2i}^{(t)} \left(\frac{1 + \omega^2}{1 - \omega^2} \right)^{2i} \right] \\
 & + \sum_{t=1}^{\frac{n-1}{2}} \left[\frac{(1 + \omega^2) \cdot (1 - \omega^2)^{4t-2}}{(2\omega)^{4t-1}} \cdot \sum_{i=0}^{n+2t} B_{2n+4t+1,2i+1}^{(t)} \left(\frac{1 + \omega^2}{1 - \omega^2} \right)^{2i+1} \right] \\
 & = \frac{1}{(2\omega)^{2n-1} \cdot (1 - \omega^2)^{2n+3}} \left\{ \sum_{t=0}^{\frac{n-3}{2}} (2\omega)^{2n-4t-2} \cdot \right. \\
 & \quad \left[\sum_{m=0}^{n+2t+2} B_{2n+4t+4,2m}^{(t)} (1 - \omega^2)^{2n+4t-2m+4} \cdot (1 + \omega^2)^{2m} \right] \\
 & + h_1(\omega^2) + \sum_{t=1}^{\frac{n-1}{2}} (2\omega)^{2n-4t} \cdot \\
 & \quad \left[\sum_{m=0}^{n+2t} B_{2n+4t+1,2m+1}^{(t)} (1 - \omega^2)^{2n+4t-2m} \cdot (1 + \omega^2)^{2m+2} \right] \\
 & \left. + (2\omega)^{2n-1} \cdot \left[\sum_{m=0}^{n+1} D_{2n+3,2m+1} (1 - \omega^2)^{2n-2m+2} \cdot (1 + \omega^2)^{2m+1} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(2\omega)^{2n-3} \cdot (1 - \omega^2)^{2n+3}} \left\{ \sum_{t=0}^{\frac{n-3}{2}} (2\omega)^{2n-4t-4} \cdot \right. \\
 &\quad \left[\sum_{m=0}^{n+2t+2} B_{2n+4t+4,2m}^{(t)} (1 - \omega^2)^{2n+4t-2m+4} \cdot (1 + \omega^2)^{2m} \right] \\
 &\quad + \frac{1}{4} R_{4n}^*(\omega^2) + \sum_{t=1}^{\frac{n-1}{2}} (2\omega)^{2n-4t-2} \cdot \\
 &\quad \left[\sum_{m=0}^{n+2t} B_{2n+4t+1,2m+1}^{(t)} (1 - \omega^2)^{2n+4t-2m} \cdot (1 + \omega^2)^{2m+2} \right] \\
 &\quad \left. + (2\omega)^{2n-3} \cdot \left[\sum_{m=0}^{n+1} D_{2n+3,2m+1} (1 - \omega^2)^{2n-2m+2} \cdot (1 + \omega^2)^{2m+1} \right] \right\} \\
 &= \frac{1}{(2\omega)^{2n-3} \cdot (1 - \omega^2)^{2n+3}} \left[W_{4n}(\omega^2) + (2\omega)^{2n-3} \cdot W_{2n+3}(\omega^2) \right],
 \end{aligned}$$

where

$$\begin{aligned}
 W_{4n}(\omega^2) &= \sum_{t=0}^{\frac{n-3}{2}} (2\omega)^{2n-4t-4} \\
 &\quad \cdot \left[\sum_{m=0}^{n+2t+2} B_{2n+4t+4,2m}^{(t)} (1 - \omega^2)^{2n+4t-2m+4} \cdot (1 + \omega^2)^{2m} \right] \\
 &\quad + \frac{1}{4} R_{4n}^*(\omega^2) + \sum_{t=1}^{\frac{n-1}{2}} (2\omega)^{2n-4t-2} \\
 &\quad \cdot \left[\sum_{m=0}^{n+2t} B_{2n+4t+1,2m+1}^{(t)} (1 - \omega^2)^{2n+4t-2m} \cdot (1 + \omega^2)^{2m+2} \right], \\
 W_{2n+3}(\omega^2) &= \sum_{m=0}^{n+1} D_{2n+3,2m+1} (1 - \omega^2)^{2n-2m+2} \cdot (1 + \omega^2)^{2m+1}.
 \end{aligned}$$

A straightforward calculation yields that the coefficients in the function $W_{4n}(\omega^2) + (2\omega)^{2n-3} \cdot W_{2n+3}(\omega^2)$ are symmetrical with respect to ω , then when $\omega_0 \neq 0$ is one root of $\tilde{G}(\omega) = 0$, so is $1/\omega_0$.

Moreover, we have Lemma 3.6.

Lemma 3.6. *For the odd number $n \geq 3$, the function $\tilde{G}(\omega)$ can be expressed as*

$$\tilde{G}(\omega) = \frac{1 - \omega}{(2\omega)^{2n-3} \cdot (1 + \omega)^{2n+3}} \cdot g(\omega), \tag{3.11}$$

where $g(\omega)$ is a symmetrical polynomial of degree $6n - 4$, and the ordered list of coefficients of $g(\omega)$ changes its sign at most $6n - 6$ times. Consequently, the function $\tilde{G}(\omega)$ has at most $3n - 3$ simple zeros in $\omega \in (0, 1)$.

Proof. Note the fact

$$\lim_{z \rightarrow 0} \int_0^{2\pi} \left[\sum_{k=1}^n \frac{S_{3k+3}(\cos \theta)}{(\cos \theta - 1 + \frac{1}{3z^3})^k} + \sum_{k=1}^n \frac{T_{3k+2}(\cos \theta)}{(\cos \theta - 1 + \frac{1}{3z^3})^{k-1}} \right] d\theta = 0,$$

which implies $\tilde{G}(1) = 0$. Hence we have

$$\begin{aligned} \tilde{G}(\omega) &= \frac{1}{(2\omega)^{2n-3} \cdot (1 - \omega^2)^{2n+3}} \left[W_{4n}(\omega^2) + (2\omega)^{2n-3} \cdot W_{2n+3}(\omega^2) \right] \\ &= \frac{1 - \omega}{(2\omega)^{2n-3} \cdot (1 + \omega)^{2n+3}} \cdot g(\omega), \end{aligned}$$

where $g(\omega)$ is such a polynomial of degree $6n - 4$ that satisfies the property

$$W_{4n}(\omega^2) + (2\omega)^{2n-3} \cdot W_{2n+3}(\omega^2) = (1 - \omega)^{2n+4} \cdot g(\omega). \tag{3.12}$$

Without loss of generality, suppose $g(\omega) = \sum_{i=0}^{6n-4} a_i \omega^i$. Since the coefficients of ω and ω^{8n-1} in (3.12) are identically equal to zero, we get

$$a_1 - (2n + 4)a_0 = 0, \quad a_{6n-5} - (2n + 4)a_{6n-4} = 0, \tag{3.13}$$

that is $a_1 = (2n + 4)a_0$, $a_{6n-5} = (2n + 4)a_{6n-4}$.

Thus the ordered list of coefficients of $g(\omega)$ changes its sign at most $6n - 6$ times. Recall that $g(\omega)$ are symmetrical with respect to ω , then its roots appear in such the pairs as $w_0 \neq 0$ and $1/w_0$. Hence $g(\omega)$ has at most $3n - 3$ zeros in $\omega \in (0, 1)$. The proof of Lemma 3.6 is completed. \square

Based on Lemma 3.6, we have the following result.

Corollary 3.7. *System (3.2) for the odd number n has at most $3n - 3$ periodic solutions arising from the periodic annulus around the center $(0, 0)$ of system (3.2) $_{|\varepsilon=0}$, and for the case $n = 3$, this upper bound can be reached.*

Proof. The first result follows directly from Lemma 3.6. Now we prove the second one.

Consider the following perturbed system

$$\begin{aligned} \dot{x} &= -y + x^3y + xy^3 + \varepsilon \sum_{k=1}^3 \sum_{l+m=1+3k} c_{lm} x^l y^m, \\ \dot{y} &= x + x^2y^2 + y^4 + \varepsilon \sum_{k=1}^3 \sum_{l+m=1+3k} d_{lm} x^l y^m, \end{aligned} \tag{3.14}$$

From (3.3), (3.11) and (3.14), we get the averaged function

$$\begin{aligned} \mathcal{F}(z) = z^4 \cdot \tilde{G}(\omega) &= \frac{(1 - \omega^2)^{4/3}}{6^{4/3}} \cdot \frac{1 - \omega}{\omega^3 \cdot (1 + \omega)^9} \cdot g(\omega) \\ &= \frac{(1 - \omega)^{7/3}}{6^{4/3} \cdot \omega^3 \cdot (1 + \omega)^{23/3}} \cdot g(\omega) \end{aligned} \tag{3.15}$$

where

$$g(\omega) = -\frac{\pi}{1728} \cdot \left(\frac{f_1}{10} \omega^{14} + f_1 \omega^{13} + f_2 \omega^{12} + f_3 \omega^{11} + f_4 \omega^{10} + f_5 \omega^9 + f_6 \omega^8 \right. \\ \left. + f_7 \omega^7 + f_6 \omega^6 + f_5 \omega^5 + f_4 \omega^4 + f_3 \omega^3 + f_2 \omega^2 + f_1 \omega + \frac{f_1}{10} \right)$$

with

$$f_1 = 80 d_{91} + 400 c_{100},$$

$$f_2 = 96 d_{73} + 1240 c_{100} + 96 c_{82} + 56 d_{91} - 192 d_{61} - 384 c_{70},$$

$$f_3 = 41 d_{37} + 175 d_{55} - 55 d_{73} - 120 d_{25} + 7 c_{46} + 65 c_{64} + 295 c_{82} + 47 d_{19} \\ - 24 c_{16} - 120 c_{52} - 408 d_7 - 432 c_4 - 25 c_{10} + 144 d_{13} - 24 c_{34} - 312 d_{43} \\ - 144 c_{22} - 432 c_{40} + 432 d_{31} - 600 d_{61} + 1825 c_{100} \\ + 335 d_{91} - 3000 c_{70} + c_{28},$$

$$f_4 = 266 c_{64} + 70 c_{46} - 170 d_{55} - 1584 d_{61} - 9648 c_{70} + 410 d_{37} + 314 d_{73} \\ + 10 c_{28} - 250 c_{10} + 1810 c_{100} + 470 d_{19} - 4320 c_4 - 240 c_{16} - 1440 c_{22} \\ - 240 c_{34} - 2016 c_{40} - 1200 c_{52} - 26 c_{82} - 4080 d_7 - 816 d_{43} + 2016 d_{31} \\ - 1200 d_{25} + 1440 d_{13} - 130 d_{91},$$

$$f_5 = -400 d_{73} + 320 d_{91} - 2880 d_{61} - 800 c_{100} + 1424 d_{19} - 4608 c_{22} \\ + 400 d_{55} - 400 d_{37} - 17088 d_7 + 4608 d_{13} - 4608 c_{40} + 400 c_{82} \\ - 1728 c_{34} - 2880 c_{52} - 880 c_{10} - 18432 c_4 - 1344 c_{16} + 112 c_{28} \\ + 400 c_{46} - 400 c_{64} - 20160 c_{70} - 2880 d_{25} + 4608 d_{31} - 1728 d_{43},$$

$$f_6 = -5520 c_{16} + 534 c_{64} - 870 c_{46} - 870 c_{82} - 7776 c_{22} - 30288 c_{70} + 790 c_{28} \\ - 41760 c_4 - 4560 d_{25} - 2290 c_{100} - 3216 c_{34} - 36240 d_7 - 4560 c_{52} \\ - 7200 c_{40} - 1270 d_{19} + 7776 d_{13} - 630 d_{55} + 390 d_{37} + 7200 d_{31} \\ - 2640 d_{43} + 390 d_{73} - 734 d_{91} - 3984 d_{61} - 550 c_{10},$$

$$f_7 = -8928 c_{22} - 8784 c_{16} - 5520 d_{25} - 882 d_{37} - 54432 c_4 - 690 d_{73} \\ - 4450 c_{100} + 8352 d_{31} - 3408 c_{34} - 45648 d_7 - 1826 c_{28} + 8928 d_{13} \\ + 3410 c_{10} + 450 d_{55} - 4560 d_{61} - 1342 d_{19} - 34320 c_{70} - 2832 d_{43} \\ + 210 c_{82} + 130 d_{91} + 786 c_{46} - 930 c_{64} - 5520 c_{52} - 8352 c_{40}.$$

Evidently, the function $g(\omega)$ is the symmetrical polynomial of degree 14, and the ordered list of coefficients of $g(\omega)$ changes its sign at most 12 times, so $g(\omega)$ has at most 6 simple zeros in $\omega \in (0, 1)$, which means that the averaged function $\mathcal{F}(z)$ in (3.15) also has at most six zeros locating at $(0, \sqrt[3]{1/6})$. In the following, we construct a family of systems whose first averaged functions have exactly six simple zeros in this interval.

For example, consider a family of systems

$$\dot{x} = -y + x^3 y + x y^3 + \varepsilon \left[c_{40} x^4 + c_{31} x^3 y + c_{22} x^2 y^2 + c_{13} x y^3 \right. \\ \left. + \left(-\frac{2}{5} c_{100} - \frac{1}{5} c_{82} - \frac{1982312879}{70778880} \right) x^7 + c_{61} x^6 y + c_{52} x^5 y^2 + c_{43} x^4 y^3 \right]$$

$$\begin{aligned}
 & + \frac{14377520035}{382205952} x^3 y^4 + c_{25} x^2 y^5 + c_{07} y^7 + c_{100} x^{10} + c_{91} x^9 y + c_{82} x^8 y^2 \\
 & + c_{73} x^7 y^3 + c_{55} x^5 y^5 + c_{37} x^3 y^7 - \frac{311748125}{21233664} x^2 y^8 + c_{19} x y^9 \Big], \\
 \dot{y} = & x + x^2 y^2 + y^4 + \varepsilon \left[d_{40} x^4 + c_{40} x^3 y + d_{22} x^2 y^2 \right. \\
 & + \left(\frac{14}{5} c_{100} + \frac{7}{5} c_{82} - \frac{5736812501}{212336640} \right) x^6 y \\
 & + c_{22} x y^3 + d_{04} y^4 + d_{70} x^7 + d_{52} x^5 y^2 \\
 & + \frac{127718550845}{382205952} x^4 y^3 + d_{34} x^3 y^4 - c_{52} x^2 y^5 + d_{16} x y^6 \\
 & + d_{100} x^{10} + \left(-5 c_{100} + \frac{1}{8} \right) x^9 y + d_{82} x^8 y^2 \\
 & + \left(-\frac{3627079267}{21233664} - 6 c_{100} + c_{82} \right) x^7 y^3 + d_{64} x^6 y^4 \\
 & \left. + d_{46} x^4 y^6 + d_{28} x^2 y^8 + d_{010} y^{10} \right], \tag{3.16}
 \end{aligned}$$

where the coefficients c_{ij} and d_{ij} are any real constants.

In the polar coordinates, system (3.16) takes the form

$$\begin{aligned}
 \dot{r} = & r^4 \sin \theta + \varepsilon \left\{ \left[c_{40} \cos^5 \theta + (c_{31} + d_{40}) \cos^4 \theta \sin \theta + (c_{22} + c_{40}) \cos^3 \theta \sin^2 \theta \right. \right. \\
 & \left. + (c_{13} + d_{22}) \cos^2 \theta \sin^3 \theta + c_{22} \cos \theta \sin^4 \theta \right] r^3 \\
 & + \left[\left(-\frac{2}{5} c_{100} - \frac{c_{82}}{5} - \frac{1982312879}{7077880} \right) \cos^8 \theta \right. \\
 & + (c_{61} + d_{70}) \cos^7 \theta \sin \theta + c_{52} \cos^6 \theta \sin^2 \theta + (c_{43} + d_{52}) \cos^5 \theta \sin^3 \theta \\
 & + \frac{142096070880}{382205952} \cos^4 \theta \sin^4 \theta + (c_{25} + d_{34}) \cos^3 \theta \sin^5 \theta - c_{52} \cos^2 \theta \sin^6 \theta \\
 & \left. + (c_{07} + d_{16}) \cos \theta \sin^7 \theta \right] r^7 + \left[(c_{100} + d_{100}) \cos^{11} \theta + c_{91} \cos^{10} \theta \sin \theta \right. \\
 & + \left(c_{82} - 5 c_{100} + \frac{1}{8} \right) \cos^9 \theta \sin^2 \theta + (c_{73} + d_{82}) \cos^8 \theta \sin^3 \theta \\
 & + \left(-\frac{3627079267}{21233664} - 6 c_{100} + c_{82} \right) \cos^7 \theta \sin^4 \theta + (c_{55} + d_{64}) \cos^6 \theta \sin^5 \theta \\
 & + (c_{37} + d_{46}) \cos^4 \theta \sin^7 \theta - \frac{311748125}{21233664} \cos^3 \theta \sin^8 \theta \\
 & \left. \left. + (c_{19} + d_{28}) \cos^2 \theta \sin^9 \theta + d_{010} \sin^{11} \theta \right] r^{10} \right\}, \\
 \dot{\theta} = & 1 + \varepsilon \left\{ \left[d_{40} \cos^5 \theta + (d_{22} - c_{31}) \cos^3 \theta \sin^2 \theta + (d_{04} - c_{13}) \cos \theta \sin^4 \theta \right] r^3 \right. \\
 & \left. + \left[\left(\frac{16}{5} c_{100} + \frac{8}{5} c_{82} + \frac{26265767}{26542080} \right) \cos^7 \theta \sin \theta + d_{70} \cos^8 \theta \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 &+(d_{52} - c_{61}) \cos^8 \theta \sin^2 \theta \\
 &+\left(\frac{127718550845}{382205952} - c_{52}\right) \cos^5 \theta \sin^3 \theta + (d_{34} - c_{43}) \cos^4 \theta \sin^4 \theta \\
 &-\left(c_{52} + \frac{14377520035}{382205952}\right) \cos^3 \theta \sin^5 \theta \\
 &+(d_{16} - c_{25}) \cos^2 \theta \sin^6 \theta - c_{07} \sin^8 \theta \Big] r^6 \\
 &+\left[d_{100} \cos^{11} \theta + \left(-6c_{100} + \frac{1}{8}\right) \cos^{10} \theta \sin \theta + (d_{82} - c_{91}) \cos^9 \theta \sin^2 \theta \right. \\
 &+\left(-\frac{3627079267}{21233664} - 6c_{100}\right) \cos^8 \theta \sin^3 \theta + (d_{64} - c_{73}) \cos^8 \theta \sin^4 \theta \\
 &+(d_{46} - c_{55}) \cos^5 \theta \sin^6 \theta + (d_{28} - c_{37}) \cos^3 \theta \sin^8 \theta \\
 &\left. + \frac{311748125}{21233664} \cos^2 \theta \sin^9 \theta + (d_{010} - c_{19}) \cos \theta \sin^{10} \theta \right] r^9 \Big\}. \tag{3.17}
 \end{aligned}$$

Using the first order averaging theory, we obtain the averaged function of system (3.17)

$$\begin{aligned}
 \mathcal{F}(z) &= \frac{(1 - \omega^2)^{4/3}}{6^{4/3}} \cdot \tilde{G}(\omega) \\
 &= -\frac{(1 - \omega)^{7/3} \pi}{1728 \cdot 6^{4/3} \omega^3 \cdot (1 + \omega)^{23/3}} \cdot \left(\omega^{14} + 10 \omega^{13} - \frac{64709}{144} \omega^{12} + \frac{71885}{16} \omega^{11} \right. \\
 &\quad - \frac{14576899}{648} \omega^{10} + \frac{345891785}{5184} \omega^9 - \frac{2603509747}{20736} \omega^8 + \frac{1602359665}{10368} \omega^7 \\
 &\quad - \frac{2603509747}{20736} \omega^6 + \frac{345891785}{5184} \omega^5 - \frac{14576899}{648} \omega^4 \\
 &\quad \left. + \frac{71885}{16} \omega^3 - \frac{64709}{144} \omega^2 + 10 \omega + 1 \right) \\
 &= -\frac{(1 - \omega)^{7/3} \pi}{1728 \cdot 6^{4/3} \omega^3 \cdot (1 + \omega)^{23/3}} \cdot \left(\omega + \frac{61}{4} + \frac{1}{4} \sqrt{3705} \right) \\
 &\quad \left(\omega + \frac{61}{4} - \frac{1}{4} \sqrt{3705} \right) \left(\omega - \frac{4}{3} \right) \left(\omega - \frac{3}{4} \right) \\
 &\quad \cdot \left(\omega - \frac{3}{2} \right) \left(\omega - \frac{2}{3} \right) \left(\omega - \frac{1}{2} \right) \left(\omega - \frac{1}{3} \right) \left(\omega - \frac{1}{4} \right) \\
 &\quad \left(\omega - \frac{1}{6} \right) (\omega - 6)(\omega - 4)(\omega - 3)(\omega - 2),
 \end{aligned}$$

which has just six simple zeros $\omega_1 = 3/4, \omega_2 = 2/3, \omega_3 = 1/2, \omega_4 = 1/3, \omega_5 = 1/4$ and $\omega_6 = 1/6$ in $(0, 1)$. Then we can get six corresponding closed orbits, that is

$$\begin{aligned}
 r_1\left(\theta, \frac{7^{1/3}}{96^{1/3}}\right) &= \frac{1}{(3 \cos \theta + \frac{75}{7})^{1/3}}, & r_2\left(\theta, \frac{5^{1/3}}{54^{1/3}}\right) &= \frac{1}{(3 \cos \theta + \frac{39}{5})^{1/3}}, \\
 r_3\left(\theta, \frac{1}{2}\right) &= \frac{1}{(3 \cos \theta + 5)^{1/3}}, & r_4\left(\theta, \frac{4^{1/3}}{3}\right) &= \frac{1}{(3 \cos \theta + \frac{15}{4})^{1/3}}, \\
 r_5\left(\theta, \frac{5^{1/3}}{32^{1/3}}\right) &= \frac{1}{(3 \cos \theta + \frac{17}{5})^{1/3}}, & r_6\left(\theta, \frac{35^{1/3}}{6}\right) &= \frac{1}{(3 \cos \theta + \frac{111}{35})^{1/3}}.
 \end{aligned}$$

Using Lemma 2.1, system (3.17) has exactly six periodic solutions bifurcating from the above six closed orbits, respectively. This completes the proof of Corollary 3.7. □

By virtue of Corollary 3.7 and the averaging theory, we obtain

Lemma 3.8. *For the odd number $n \geq 3$ and any sufficiently small $|\varepsilon| \neq 0$, system (1.2) has at most $3n - 3$ limit cycles bifurcating from the periodic annulus around the center $(0, 0)$ of the unperturbed system $(1.2)|_{\varepsilon=0}$, and this upper bound for the case $n = 3$ is sharp.*

3.3. Proof of Theorem 1.1 for the Even Case

We first simplify the function $G(\rho)$ defined by (3.5), then give the estimate on the number of limit cycles bifurcating from the period annulus around the center of the unperturbed system $(1.2)|_{\varepsilon=0}$ for the even number n .

Similar to (3.10), we have

$$\begin{aligned}
 G(\rho) &= B_{0,0} \cdot I_{-(2n+3)}(\rho) + B_{1,1}\rho \cdot I_{-(2n+2)}(\rho) + (B_{2,0} + B_{2,2}\rho^2) \cdot I_{-(2n+1)}(\rho) \\
 &\quad + \cdots + (B_{2n+2,0} + B_{2n+2,2}\rho^2 + \cdots + B_{2n+2,2n+2}\rho^{2n+2}) \cdot I_{-1}(\rho) \\
 &\quad + (B_{2n+3,1}\rho + B_{2n+3,3}\rho^3 + \cdots + B_{2n+3,2n+3}\rho^{2n+3}) \cdot I_0(\rho) \\
 &\quad + (B_{2n+4,0} + B_{2n+4,2}\rho^2 + \cdots + B_{2n+4,2n+4}\rho^{2n+4}) I_1(\rho) \\
 &\quad + (B_{2n+5,1}\rho + B_{2n+5,3}\rho^3 + \cdots + B_{2n+5,2n+5}\rho^{2n+5}) I_2(\rho) \\
 &\quad + \cdots + \sum_{i=0}^{\frac{3n+2}{2}} B_{3n+3,2i+1}\rho^{2i+1} I_n(\rho) \\
 &= \sum_{t=0}^{n+1} I_{-(2t+1)}(\rho) \sum_{i=0}^{n-t+1} B_{2n-2t+2,2i}\rho^{2i} \\
 &\quad + \sum_{t=0}^{n+1} I_{-(2t)}(\rho) \sum_{i=0}^{n-t+1} B_{2n-2t+3,2i+1}\rho^{2i+1} \\
 &\quad + \sum_{t=0}^{\frac{n-2}{2}} I_{2t+1}(\rho) \sum_{i=0}^{n+t+2} B_{2n+2t+4,2i}\rho^{2i} \\
 &\quad + \sum_{t=1}^{\frac{n}{2}} I_{2t}(\rho) \sum_{i=0}^{n+t+1} B_{2n+2t+3,2i+1}\rho^{2i+1} \\
 &= \sum_{t=0}^{\frac{n-2}{2}} \frac{1}{(\rho^2 - 1)^{2t+\frac{1}{2}}} \cdot \sum_{i=0}^{n+2t+2} B_{2n+4t+4,2i}^{(t)} \rho^{2i}
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{t=1}^{\frac{n}{2}} \frac{\rho}{(\rho^2 - 1)^{2t - \frac{1}{2}}} \cdot \sum_{i=0}^{n+2t} B_{2n+4t+1, 2i+1}^{(t)} \rho^{2i+1} \\
 &+ (D_{2n+3, 1} \rho + D_{2n+3, 3} \rho^3 + \cdots + D_{2n+3, 2n+3} \rho^{2n+3}),
 \end{aligned}$$

where $D_{2n+3, 2i+1}, i = 1, 2, \dots, n$ are defined as before. Making the same transformation $\rho = (1 + \omega^2)/(1 - \omega^2)$ for $\omega \in (0, 1)$, the above formula becomes

$$\begin{aligned}
 \bar{G}(\omega) &:= G(\rho) \Big|_{\rho = \frac{1+\omega^2}{1-\omega^2}} \\
 &= \left[D_{2n+3, 1} \left(\frac{1 + \omega^2}{1 - \omega^2} \right) + D_{2n+3, 3} \left(\frac{1 + \omega^2}{1 - \omega^2} \right)^3 \right. \\
 &\quad \left. + \cdots + D_{2n+3, 2n+3} \left(\frac{1 + \omega^2}{1 - \omega^2} \right)^{2n+3} \right] \\
 &+ \sum_{t=0}^{\frac{n-2}{2}} \left[\frac{(1 - \omega^2)^{4t+1}}{(2\omega)^{4t+1}} \cdot \sum_{i=0}^{n+2t+2} B_{2n+4t+4, 2i}^{(t)} \left(\frac{1 + \omega^2}{1 - \omega^2} \right)^{2i} \right] \\
 &+ \sum_{t=1}^{\frac{n}{2}} \left[\frac{(1 + \omega^2) \cdot (1 - \omega^2)^{4t-2}}{(2\omega)^{4t-1}} \cdot \sum_{i=0}^{n+2t} B_{2n+4t+1, 2i+1}^{(t)} \left(\frac{1 + \omega^2}{1 - \omega^2} \right)^{2i+1} \right] \\
 &= \frac{1}{(2\omega)^{2n-1} \cdot (1 - \omega^2)^{2n+3}} \left\{ \sum_{t=0}^{\frac{n-2}{2}} (2\omega)^{2n-4t-2} \cdot \right. \\
 &\quad \left[\sum_{m=0}^{n+2t+2} B_{2n+4t+4, 2m}^{(t)} (1 - \omega^2)^{2n+4t-2m+4} \cdot (1 + \omega^2)^{2m} \right] \\
 &\quad + \sum_{t=1}^{\frac{n-2}{2}} (2\omega)^{2n-4t} \cdot \left[\sum_{m=0}^{n+2t} B_{2n+4t+1, 2m+1}^{(t)} (1 - \omega^2)^{2n+4t-2m} \right. \\
 &\quad \left. \cdot (1 + \omega^2)^{2m+2} \right] + h_2(\omega^2) \\
 &\quad \left. + (2\omega)^{2n-1} \cdot \left[\sum_{m=0}^{n+1} D_{2n+3, 2m+1} (1 - \omega^2)^{2n-2m+2} \cdot (1 + \omega^2)^{2m+1} \right] \right\} \\
 &= \frac{1}{(2\omega)^{2n-3} \cdot (1 - \omega^2)^{2n+3}} \left\{ \sum_{t=0}^{\frac{n-2}{2}} (2\omega)^{2n-4t-4} \right. \\
 &\quad \cdot \left[\sum_{m=0}^{n+2t+2} B_{2n+4t+4, 2m}^{(t)} (1 - \omega^2)^{2n+4t-2m+4} \cdot (1 + \omega^2)^{2m} \right] \\
 &\quad + \sum_{t=1}^{\frac{n-2}{2}} (2\omega)^{2n-4t-2} \cdot \left[\sum_{m=0}^{n+2t} B_{2n+4t+1, 2m+1}^{(t)} (1 - \omega^2)^{2n+4t-2m} \right.
 \end{aligned}$$

$$\begin{aligned}
 & \cdot (1 + \omega^2)^{2m+2} \Bigg] + \frac{1}{4} R_{4n}^{**}(\omega^2) \\
 & + (2\omega)^{2n-3} \cdot \left[\sum_{m=0}^{n+1} D_{2n+3,2m+1} (1 - \omega^2)^{2n-2m+2} \cdot (1 + \omega^2)^{2m+1} \right] \Bigg\} \\
 & = \frac{1}{(2\omega)^{2n-3} \cdot (1 - \omega^2)^{2n+3}} \left[U_{4n}(\omega^2) + (2\omega)^{2n-3} \cdot U_{2n+3}(\omega^2) \right], \tag{3.18}
 \end{aligned}$$

where

$$\begin{aligned}
 & U_{4n}(\omega^2) \\
 & = \sum_{t=0}^{\frac{n-2}{2}} (2\omega)^{2n-4t-4} \cdot \left[\sum_{m=0}^{n+2t+2} B_{2n+4t+4,2m}^{(t)} (1 - \omega^2)^{2n+4t-2m+4} \cdot (1 + \omega^2)^{2m} \right] \\
 & + \sum_{t=1}^{\frac{n-2}{2}} (2\omega)^{2n-4t-2} \cdot \left[\sum_{m=0}^{n+2t} B_{2n+4t+1,2m+1}^{(t)} (1 - \omega^2)^{2n+4t-2m} \right. \\
 & \left. \cdot (1 + \omega^2)^{2m+2} \right] + \frac{1}{4} R_{4n}^{**}(\omega^2), \tag{3.19}
 \end{aligned}$$

$$\begin{aligned}
 & U_{2n+3}(\omega^2) \\
 & = \sum_{m=0}^{n+1} D_{2n+3,2m+1} (1 - \omega^2)^{2n-2m+2} \cdot (1 + \omega^2)^{2m+1}.
 \end{aligned}$$

Hence, for the even number n , we have the following results.

Lemma 3.9. 1. For $n = 2$, the function $\bar{G}(\omega)$ can be expressed as

$$\bar{G}(\omega) = \frac{1 - \omega}{\omega \cdot (1 + \omega)^7} \cdot \bar{g}(\omega),$$

where $\bar{g}(\omega)$ is a symmetrical polynomial of degree 8, having at most 4 simple zeros in $\omega \in (0, 1)$. Consequently, the function $\bar{G}(\omega)$ has at most 4 simple zeros in $\omega \in (0, 1)$. Moreover, this upper bound is sharp.

2. For the even number $n \geq 4$, the function $\bar{G}(\omega)$ can be expressed as

$$\bar{G}(\omega) = \frac{1 - \omega}{(2\omega)^{2n-3} \cdot (1 + \omega)^{2n+3}} \cdot \bar{g}(\omega),$$

where $\bar{g}(\omega)$ is a symmetrical polynomial of degree $6n - 4$, and the ordered list of coefficients of $\bar{g}(\omega)$ changes its sign at most $6n - 6$ times. Consequently, the function $\bar{G}(\omega)$ has at most $3n - 3$ simple zeros in $\omega \in (0, 1)$.

Proof. The proof of the second result is similar to Lemma 3.6. Now we begin to prove the first one.

Consider the following quartic system

$$\begin{aligned} \dot{x} &= -y + x^3y + xy^3 + \varepsilon \sum_{k=1}^2 \sum_{l+m=1+3k} c_{lm}x^l y^m, \\ \dot{y} &= x + x^2y^2 + y^4 + \varepsilon \sum_{k=1}^2 \sum_{l+m=1+3k} d_{lm}x^l y^m. \end{aligned}$$

The formula (3.18) associated to the above system takes the form

$$\bar{G}(\omega) = \frac{1 - \omega}{\omega \cdot (1 + \omega)^7} \cdot \bar{g}(\omega),$$

where

$$\bar{g}(\omega) = \frac{\pi}{72} \cdot (f_0\omega^8 + f_1\omega^7 + f_2\omega^6 + f_3\omega^5 + f_4\omega^4 + f_3\omega^3 + f_2\omega^2 + f_1\omega + f_0)$$

with

$$f_0 = 16c_{70} + 8d_{61},$$

$$f_1 = 18c_{04} + c_{16} + 6c_{22} + c_{34} + 18c_{40} + 5c_{52} + 93c_{70} + 17d_{07} - 6d_{13} + 5d_{25} - 18d_{31} + 13d_{43} + 9d_{61},$$

$$f_2 = 144c_{04} + 8c_{16} + 48c_{22} + 8c_{34} + 48c_{40} + 40c_{52} + 200c_{70} + 136d_{07} - 48d_{13} + 40d_{25} - 48d_{31} + 8d_{43} + 40d_{61},$$

$$f_3 = 462c_{04} + 39c_{16} + 90c_{22} + 55c_{34} + 78c_{40} + 35c_{52} + 347c_{70} + 423d_{07} - 90d_{13} + 35d_{25} - 78d_{31} + 43d_{43} + 31d_{61},$$

$$f_4 = 672c_{04} + 144c_{16} + 96c_{22} + 16c_{34} + 96c_{40} + 80c_{52} + 368c_{70} + 528d_{07} - 96d_{13} + 80d_{25} - 96d_{31} + 16d_{43} + 64d_{61}.$$

Since $\bar{g}(\omega)$ is a symmetrical polynomial of degree 8, it has at most 4 simple zeros in $\omega \in (0, 1)$. Hence $\bar{G}(\omega)$ also has four as the upper bound of the number of its zeros in $\omega \in (0, 1)$. To show this upper bound can be reached, we consider the following quartic system

$$\begin{aligned} \dot{x} &= -y + x^3y + xy^3 + \varepsilon \left[c_{40}x^4 + c_{31}x^3y + c_{22}x^2y^2 + c_{13}xy^3 \right. \\ &\quad + \left(\frac{13483}{7680} - c_{70} - d_{07} \right) y^4 + c_{70}x^7 \\ &\quad + c_{61}x^6y + c_{52}x^5y^2 + c_{43}x^4y^3 + \left(-\frac{114853}{7680} + 5c_{70} \right) x^3y^4 + c_{25}x^2y^5 \\ &\quad \left. + \left(-\frac{29399}{7680} + 3c_{70} + d_{07} \right) xy^6 + c_{07}y^7 \right], \\ \dot{y} &= x + x^2y^2 + y^4 + \varepsilon \left[d_{40}x^4 + c_{40}x^3y + d_{22}x^2y^2 + c_{22}xy^3 + d_{04}y^4 + d_{70}x^7 \right. \\ &\quad \left. + \left(\frac{1}{8} - 2c_{70} \right) x^6y + d_{52}x^5y^2 - \left(5c_{70} + \frac{8633}{3840} \right) x^4y^3 + d_{34}x^3y^4 \right] \end{aligned}$$

$$\left. -c_{52}x^2y^5 + d_{16}xy^6 + d_{07}y^7 \right],$$

where the coefficients c_{ij} and d_{ij} are any real constants. The corresponding function $\bar{G}(\omega)$ can be expressed as

$$\begin{aligned} \bar{G}(\omega) = & \frac{\pi(1-\omega)}{72\omega(1+\omega)^7} \cdot \left(\omega^8 - \frac{917}{60}\omega^7 + \frac{10747}{120}\omega^6 - \frac{15211}{60}\omega^5 + \frac{21677}{60}\omega^4 \right. \\ & \left. - \frac{15211}{60}\omega^3 + \frac{10747}{120}\omega^2 - \frac{917}{60}\omega + 1 \right) \\ & c \frac{\pi(1-\omega)}{72\omega(1+\omega)^7} \cdot \left(\omega - \frac{1}{2} \right) (\omega - 2) \left(\omega - \frac{1}{3} \right) (\omega - 3) \left(\omega - \frac{1}{4} \right) \\ & (\omega - 4)(\omega - 5) \left(\omega - \frac{1}{5} \right), \end{aligned}$$

which has exactly four simple zeros $\omega_1 = 1/2, \omega_2 = 1/3, \omega_3 = 1/4, \omega_4 = 1/5$. So we complete the proof of the first result. \square

Based on Lemma 3.9, we have Corollary 3.10.

Corollary 3.10. *For any sufficiently small $|\varepsilon| \neq 0$, the following properties hold.*

1. *For $n = 2$, system (3.2) has at most 4 periodic solutions bifurcating from the periodic annulus around the center $(0, 0)$ of system (3.2)| $_{\varepsilon=0}$, and this upper bound is sharp.*
2. *For the even number $n \geq 4$, system (3.2) has at most $3n - 3$ periodic solutions bifurcating from the periodic annulus around the center $(0, 0)$ of system (3.2)| $_{\varepsilon=0}$.*

Equivalently, we have Lemma 3.11.

Lemma 3.11. *For any sufficiently small $|\varepsilon| \neq 0$, the following statements are true.*

1. *For $n = 2$, system (1.2) has at most 4 limit cycles bifurcating from the periodic annulus around the center $(0, 0)$ of system (1.2)| $_{\varepsilon=0}$, and this upper bound is sharp.*
2. *For the even number $n \geq 4$, system (1.2) has at most $3n - 3$ limit cycles bifurcating from the periodic annulus around the center $(0, 0)$ of system (1.2)| $_{\varepsilon=0}$.*

Proof of Theorem 1.1. Theorem 1.1 follows directly from Lemmas 3.8 and 3.11. \square

4. Discussions

In this paper, we obtained a bound for the maximum number of limit cycles that bifurcate from a non-Hamiltonian quartic reversible center by adding perturbed terms which are the sum of homogeneous polynomials of degree

$3k + 1$ for $1 \leq k \leq n$. Our initial idea is to consider general degree n perturbations of the center, but the main difficulty exists in the technical and cumbersome computations of the averaged function. We leave this as a future research problem.

By observing the proofs of Lemma 3.6, Corollary 3.7, and Lemma 3.9, we have an intuition that the difference between the obtained number of limit cycles in cases $n = 2$ and $n \geq 3$ lies in the relation of coefficients of the resulting polynomial $g(\omega)$, see equation (3.13). In addition, in the proof of Corollary 3.7 for $n = 3$, we noticed a fact that the variables in the coefficients of the function $g(\omega)$ (see (3.15)) are quite enough. So we have a conjecture that for any $n \geq 4$, the bound $3n - 3$ is also sharp. However, we cannot prove this conjecture. Maybe the problem can be solved with the aid of some beautiful mathematical techniques or most advanced computing technologies. How to explain the upper bound is sharp remains a question for further investigation.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

- [1] Álvarez, M., Gasull, A., Prohens, R.: Limit cycles for two families of cubic systems. *Nonlinear Anal.* **75**, 6402–6417 (2012)
- [2] Arnold, V.I., Ilyashenko, Y.S.: *Dynamical Systems I: Ordinary Differential Equations*, Encyclopaedia Math. Sci., vol. 1. Springer, Berlin (1986)
- [3] Atabaigi, A., Nyamoradi, N., Zangeneh, H.R.Z.: The number of limit cycles of a quintic polynomial system with a center. *Nonlinear Anal.* **71**, 3008–3017 (2009)
- [4] Benterki, R., Llibre, J.: Limit cycles of polynomial differential equations with quintic homogeneous nonlinearities. *J. Math. Anal. Appl.* **407**, 16–22 (2013)
- [5] Blows, T.R., Perko, L.M.: Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems. *SIAM Rev.* **36**, 341–376 (1994)
- [6] Buică, A., Llibre, J.: Averaging methods for finding periodic orbits via Brouwer degree. *Bull. Sci. Math.* **128**, 7–22 (2004)
- [7] Buică, A., Llibre, J.: Limit cycles of a perturbed cubic polynomial differential center. *Chaos Solit. Fract.* **32**, 1059–1069 (2007)
- [8] Buică, A., Françoise, J.P., Llibre, J.: Periodic solutions of nonlinear periodic differential systems with a small parameter. *Commun. Pure Appl. Anal.* **6**, 103–111 (2007)
- [9] Chen, F.D., Li, C., Llibre, J., Zhang, Z.H.: A unified proof on the weak Hilbert 16th problem for $n = 2$. *J. Differ. Equ.* **221**, 309–342 (2006)

- [10] Chicone, C., Jacobs, M.: Bifurcation of limit cycles from quadratic isochrones. *J. Differ. Equ.* **91**, 268–326 (1991)
- [11] Coll, B., Gasull, A., Prohens, R.: Bifurcation of limit cycles from two families of centers. *Dyn. Contin. Discrete Impuls. Syst. Ser. A (Math. Anal.)* **12**, 275–287 (2005)
- [12] Coll, B., Llibre, J., Prohens, R.: Limit cycles bifurcating from a perturbed quartic center. *Chaos Solitons Fract.* **44**, 317–334 (2011)
- [13] García-Saldaña, J., Gasull, A., Giacomini, H.: A new approach for the study of limit cycles. *J. Differ. Equ.* **269**, 6269–6292 (2020)
- [14] Gautier, S., Gavrilov, L., Iliev, I.D.: Perturbations of quadratic centers of genus one. *Discret. Contin. Dyn. Syst.* **25**, 511–535 (2009)
- [15] Giacomini, H., Llibre, J., Viano, M.: On the nonexistence, existence and uniqueness of limit cycles. *Nonlinearity* **9**, 501–516 (1996)
- [16] Giacomini, H., Llibre, J., Viano, M.: On the shape of limit cycles that bifurcate from Hamiltonian centers. *Nonlinear Anal.* **41**, 523–537 (2000)
- [17] Giacomini, H., Llibre, J., Viano, M.: On the shape of limit cycles that bifurcate from non-Hamiltonian centers. *Nonlinear Anal.* **43**, 837–859 (2001)
- [18] Giné, J., Llibre, J.: Limit cycles of cubic polynomial vector fields via the averaging theory. *Nonlinear Anal.* **66**, 1707–1721 (2007)
- [19] Hilbert, D.: Mathematische probleme. *Arch. Math. Phys.* **1**, 213–237 (1901)
- [20] Huang, J., Liang, H.: Limit cycles of planar system defined by the sum of two quasi-homogeneous vector fields. *Discret. Contin. Dyn. Syst. Ser. B* **26**, 861–873 (2021)
- [21] Iliev, I.D.: Perturbations of quadratic centers. *Bull. Sci. Math.* **122**, 107–161 (1998)
- [22] Li, C., Llibre, J.: Quadratic perturbations of a quadratic reversible Lotka–Volterra system. *Qual. Theory Dyn. Syst.* **9**, 235–249 (2010)
- [23] Li, C., Llibre, J., Zhang, Z.: Weak focus, limit cycles and bifurcations for bounded quadratic systems. *J. Differ. Equ.* **115**, 193–223 (1995)
- [24] Liu, C., Xiao, D.: The smallest upper bound on the number of zeros of Abelian integrals. *J. Differ. Equ.* **269**, 3816–3852 (2020)
- [25] Llibre, J.: Averaging theory and limit cycles for quadratic systems. *Radovi Mat.* **11**, 1–14 (2002)
- [26] Llibre, J., Pérez del Río, J.S., Rodríguez, J.A.: Averaging analysis of a perturbed quadratic center. *Nonlinear Anal.* **46**, 45–51 (2001)
- [27] Llibre, J., Moeckel, R., Simó, C.: Central Configuration, Periodic Orbits, and Hamiltonian Systems. *Advanced Courses in Mathematics-CRM Barcelona*. Birkhäuser, Basel (2015)
- [28] Malkin, I.G.: *Some Problems of the Theory of Nonlinear Oscillations*. Gosudarstv. Izdat. Tehn. Teor. Lit, Moscow (1956). (Russian)
- [29] Peng, L., Feng, Z.: Bifurcation of limit cycles from quartic isochronous systems. *Electron. J. Differ. Equ.* **2014**, 1–14 (2014)
- [30] Roseau, M.: *Vibrations Non Liné aries et Théorie de la Stabilité*, Springer Tracts in Natural Philosophy, vol. 18. Springer, Berlin (1966)
- [31] Sheng, L., Wang, S., Li, X., Han, M.: Bifurcation of periodic orbits of periodic equations with multiple parameters by averaging method. *J. Math. Anal. Appl.* **490**, 124311 (2020)

- [32] Shi, J., Wang, W., Zhang, X.: Limit cycles of polynomial Liénard systems via the averaging method. *Nonlinear Anal. Real World Appl.* **45**, 650–667 (2019)
- [33] Tian, Y., Han, M., Xu, F.: Bifurcations of small limit cycles in Liénard systems with cubic restoring terms. *J. Differ. Equ.* **267**, 1561–1580 (2019)
- [34] Verhulst, F.: *Nonlinear Differential Equations and Dynamical Systems*. Springer, Berlin (1991)
- [35] Viano, M., Llibre, J., Giacomini, H.: Arbitrary order bifurcations for perturbed Hamiltonian planar systems via the reciprocal of an integrating factor. *Nonlinear Anal.* **48**, 117–136 (2002)
- [36] Xiang, G., Han, M.: Global bifurcation of limit cycles in a family of polynomial systems. *J. Math. Anal. Appl.* **295**, 633–644 (2004)
- [37] Xiong, Y., Han, M.: Limit cycles bifurcations by perturbing a class of planar quantic vector fields. *J. Differ. Equ.* **269**, 10964–10994 (2020)

Bo Huang and Linping Peng
LMIB, School of Mathematical Science
Beihang University
Beijing 100191
China

Yong Cui
School of Automation Science and Electrical Engineering
Beihang University
Beijing 100191
China
e-mail: cuiyongsas@163.com

Received: August 9, 2021.

Revised: February 4, 2022.

Accepted: August 5, 2022.