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Approximation by Marcinkiewicz-Type Matrix Transform of Vilenkin–Fourier Series

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Abstract. In the present paper, we discuss the rate of the approximation by Marcinkiewicz-type matrix transform of Vilenkin–Fourier series in $L^p(G_m^2)$ spaces $(1 \leq p < \infty)$ and in $C(G_m^2)$. Moreover, we give an application for functions in Lipschitz classes $\text{Lip}(\alpha, p, G_m^2)$ $(\alpha > 0,\ 1 \leq p < \infty)$ and $\text{Lip}(\alpha, C(G_m^2))$ $(\alpha > 0)$.

Mathematics Subject Classification. 42C10.

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1. Introduction and Auxiliary Propositions

At first, we give a brief introduction to the theory of Vilenkin–Fourier analysis. We follow the notation and notion of the book [24]. Denote by $\mathbb{N}_+$ the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ be a sequence of positive integers not less than 2. Denote by $\mathbb{Z}_{m_n} := \{0, 1, \dots, m_n - 1\}$ the additive group of integers modulo m_n. Define the group G_m as the complete direct product of the groups $\mathbb{Z}_{m_n}$ with the product of the discrete topologies of $\mathbb{Z}_{m_n}$'s. It has countable base given by the family

$$I_0(x) := G_m,$$
$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in \mathbb{N}_+).$$

The direct product μ of the measures

$$\mu_n(\{j\}) := 1/m_n \quad (j \in \mathbb{Z}_{m_n})$$

is a Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence m is bounded, then G_m is called a bounded Vilenkin group; otherwise, it is called an unbounded one. In case of $m = (2, 2, \dots)$, we get G_2, the so-called Walsh group. The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_n, \ldots) \quad (x_n \in \mathbb{Z}_{m_n}).$$

Let us denote $I_n := I_n(0)$ for $n \in \mathbb{N}$. We define the so-called generalized number system based on m in the following way:

$$M_0 := 1,\ M_{n+1} := m_n M_n \quad (n \in \mathbb{N}).$$

Then, every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in \mathbb{Z}_{m_k}$ $(k \in \mathbb{N})$ and only a finite number of n_k's differ from zero. For a given $n \in \mathbb{N}$, the order of n is defined by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. Therefore, it is a natural number, such that $M_{|n|} \leq n < M_{|n|+1}$.

Next, we introduce on G_m an orthonormal system which is called Vilenkin system. At first, we define the complex-valued functions $r_k : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi \imath x_k / m_k) \quad \left(\imath^2 = -1, x \in G_m,\ k \in \mathbb{N}\right).$$

Let us define the Vilenkin system $\varphi := (\varphi_n : n \in \mathbb{N})$ on G_m as the product system of generalized Rademacher functions

$$\varphi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh–Paley system when $m = (2, 2, \ldots)$.

The usual Lebesgue spaces on G_m are denoted by $L^p(G_m)$ with the corresponding norm $\|.\|_p$. The space of continuous functions on G_m is denoted by $C(G_m)$ with the norm $\|f\|_\infty := \sup\{|f(x)| : x \in G_m\}$.

The modulus of continuity in L^p $(1 \leq p < \infty)$ of a function $f \in L^p$ is defined by

$$\omega_p(f, \delta) := \sup_{|t| < \delta} \|f(. + t) - f(.)\|_p, \quad \delta > 0,$$

with the notation

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{M_{i+1}} \quad \text{for all } x \in G.$$

Analogically, the modulus of continuity in C is denoted by $\omega_\infty(f, \delta)$. Since, the modulus of continuity is constant at the intervals $\left(\frac{1}{M_{n+1}}, \frac{1}{M_n}\right]$ $(n \in \mathbb{N})$, it is possible to choice it as a continuous parameter $\delta > 0$. We note that the original definition of Vilenkin was a sequence-type definition which reflects the group structure [28].

The Lipschitz classes in $L^p(G_m)$ for each $\alpha > 0$ are defined by

$$\mathrm{Lip}(\alpha, p, G_m) := \{f \in L^p(G_m) : \omega_p(f, \delta) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

Moreover

$$\mathrm{Lip}(\alpha, C(G_m)) := \{f \in C(G_m) :\ |f(x + y) - f(x)| \leq C|y|^\alpha,\ x, y \in G_m\}.$$

Furthermore, for the simplicity, we write $\mathrm{Lip}(\alpha, \infty, G_m) := \mathrm{Lip}(\alpha, C(G_m))$.

In dimension two, for $x = (x^1, x^2) \in G_m^2$, we define $|x|$ by $|x|^2 := (x^1)^2 + (x^2)^2$. Thus, the modulus of continuity $\omega_p(\delta, f)$ is well defined for $\delta > 0$ $(1 \leq p \leq \infty)$. The partial modulus of continuity is defined by

$$\omega_p^1(f,\delta) := \sup_{|t|<\delta} \|f(x^1+t, x^2) - f(x^1, x^2)\|_p,$$
$$\omega_p^2(f,\delta) := \sup_{|t|<\delta} \|f(x^1, x^2+t) - f(x^1, x^2)\|_p$$

$(\delta > 0)$ for $f \in L^p(G_m^2)$. In the case $f \in C(G_m^2)$, we change p by ∞. The mixed modulus of continuity is defined as follows:

$$\omega_p^{1,2}(\delta_1, \delta_2, f) := \sup\{\|f(.+x^1, .+x^2) - f(.+x^1, .) - f(., .+x^2) + f(., .)\|_p : |x^1| < \delta_1, |x^2| < \delta_2\},$$

where $\delta_1, \delta_2 > 0$.

The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (see [28]). The elements of the Vilenkin system are precisely the characters of G_m, i.e., nonzero continuous functions $f\colon G_m \to \mathbb{C}$, such that

$$f(x+y) = f(x)f(y)$$

for all $x, y \in G_m$. It holds if and only if $f(x) = \varphi_n(x)$ for some $n \in \mathbb{N}$ (see [24]).

The nth Dirichlet kernel is defined by

$$D_n := \sum_{k=0}^{n-1} \varphi_k,$$

where $n \in \mathbb{N}_+$, $D_0 := 0$. The M_nth Dirichlet kernel has a closed form

$$D_{M_n}(x) = \begin{cases} 0, & \text{if } x \notin I_n(0), \\ M_n, & \text{if } x \in I_n(0). \end{cases} \tag{1}$$

Let $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers. The nth Nörlund mean of the Vilenkin–Fourier series is defined by

$$\mathbf{t}_n(f;x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f;x), \tag{2}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$ $(n \geq 1)$ and $S_k(f;x)$ denotes the kth partial sum of the Vilenkin–Fourier series of f. It is always assumed that $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty. \tag{3}$$

In this case, the summability method generated by $\{q_k\}$ is regular (see [16, 32]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0. \tag{4}$$

Móricz and Siddiqi [20] studied the rate of the approximation by Nörlund means $\mathbf{t}_n(f)$ of Walsh–Fourier series of a function f in $L^p(G_2)$ and in $C(G_2)$ (in particular, in $\mathrm{Lip}(\alpha, p, G_2)$, where $\alpha > 0$ and $1 \leq p \leq \infty$). As special cases, Móricz and Siddiqi obtained the earlier results given by Yano [31], Jastrebova [17], and Skvortsov [26] on the rate of the approximation by Cesàro means. The approximation properties of the Walsh–Cesàro means of negative order were studied by Goginava [13], and Vilenkin case was investigated by Shavardenidze [25] and Tepnadze [27]. In 2008, Fridli, Manchanda, and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [10]. Recently, the first author, Baramidze, Memić, Persson, Tephnadze and Wall presented some results with respect to this topic [2,7,18]. See [9,29], as well. Avdispahić and Pepić proved some results also for Vilenkin system in the paper [1]. For the two-dimensional results, see [6,21–23,30].

Let $\{p_k : k \geq 1\}$ be a sequence of non-negative numbers. The nth weighted mean T_n of Vilenkin–Fourier series is defined by

$$T_n(f;x) := \frac{1}{P_n}\sum_{k=1}^{n} p_k S_k(f;x), \tag{5}$$

where $P_n := \sum_{k=1}^{n} p_k$ $(n \geq 1)$. In particular case T_n are the Vilenkin–Fejér means (for all k set $p_k = 1$). It is always assumed that $p_1 > 0$ and

$$\lim_{n\to\infty} P_n = \infty, \tag{6}$$

which is the condition for regularity [16,32].

Móricz and Rhoades [19] discussed the rate of the approximation by weighted means of Walsh–Fourier series of a function in $L^p(G_2)$ and in $C(G_2)$ [in particular, in $\mathrm{Lip}(\alpha, p, G_2)$, where $\alpha > 0$ and $1 \leq p \leq \infty$]. As special cases Móricz and Rhoades obtained the earlier results given by Yano [31], Jastrebova [17] on the rate of the approximation by Walsh–Cesàro means. A common generalization of this two results of Móricz and Siddiqi [20] and Móricz and Rhoades [19] was given by the authors in the paper [4]. Recently, the generalization for linear transform of Vilenkin–Fourier series was proved by the authors [5].

Let $T := (t_{i,j})_{i,j=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is triangular. Let us define the nth linear mean (or matrix transform mean) determined by the matrix T

$$\sigma_n(f;x) := \sum_{k=1}^{n} t_{k,n} S_k(f;x),$$

where $S_k(f;x)$ denotes the kth partial sums of the Vilenkin–Fourier series of f. For matrix transform method, the conditions of regularity can be found in Zygmund's book [32, page 74] and in [16].

Since, the nth row of the matrix T determines the linear mean σ_n and its definition contains only finite number of entries; for the simplicity, we say $\{t_{k,n} : 1 \leq k \leq n,\ k \in \mathbb{N}_+\}$ is a finite sequence of numbers for each $n \in \mathbb{N}_+$.

In the further part of this paper, let $\{t_{k,n} : 1 \leq k \leq n,\ k \in \mathbb{N}_+\}$ be a finite sequence of non-negative numbers for each $n \in \mathbb{N}_+$. The nth matrix transform kernel is defined by

$$K_n^T(x) := \sum_{k=1}^{n} t_{k,n} D_k(x).$$

It is easily seen that

$$\sigma_n(f;x) = \int_{G_m} f(u) K_n^T(u+x) \mathrm{d}\mu(u).$$

It follows by simple consideration that the Nörlund means and weighted means are matrix transforms.

Our paper is motivated by the work of Móricz, Siddiqi [20] on Walsh–Nörlund mean method and the result of Móricz, Rhoades [19] on Walsh weighted mean method. It is important to note that in the paper of Chripkó [8], a generalization for Jacobi–Fourier series was discussed, and the authors found some ideas in this paper. Recently, the rate of the approximation by linear transform means $\sigma_n(f)$ of Vilenkin–Fourier series is examined in spaces $L^p(G_m)$ $(1 \le p < \infty)$ and $C(G_m)$ [5]. The authors generalized the means and the system of the Fourier series, as well. Other aspects of these methods with respect to Walsh–Fourier series are treated in the papers [9,29].

Fejér kernels are defined as the arithmetical means of Dirichlet kernels, that is

$$K_n(x) := \frac{1}{n} \sum_{k=1}^{n} D_k(x).$$

In dimension 2, the Marcinkiewicz kernels are defined as follows:

$$\mathcal{K}_n(x,y) := \frac{1}{n} \sum_{k=1}^{n} D_k(x) D_k(y).$$

Let us define the Marcinkiewicz-type linear transform means and kernels as follows:

$$\sigma_n^T(f;x,y) := \sum_{k=1}^{n} t_{k,n} S_{k,k}(f;x,y), \quad K_n^T(x,y) := \sum_{k=1}^{n} t_{k,n} D_k(x) D_k(y).$$

Our main aim is to investigate the rate of the approximation by two-dimensional Marcinkiewicz-type matrix transform in terms of modulus of continuity. Moreover, our main theorem (Theorem 1) gives a kind of common two-dimensional generalization of the two results of Móricz, Siddiqi on Nörlund means [20] and Móricz, Rhoades on weighted means [19]. Moreover, we generalized the system, as well (see [21,23]). In this section, the two-dimensional kernels $K_n^T(x,y)$ are decomposed and two useful Lemmas are proved. The main theorem follows in Sect. 2, and the results are reached for two class of means. The results are stated for non-decreasing and non-increasing generating sequences $\{t_{k,n} : 1 \le k \le n\}$ $(n \in \mathbb{P})$. At the end, we present an application for Lipschitz functions.

For more about the original Marcinkiewicz–Fejér means, see e.g. [3,11,14,15].

For two-dimensional variable $(x,y) \in G_m \times G_m$, we use the notations

$$r_n^1(x,y) = r_n(x),\ D_n^1(x,y) = D_n(x),\ K_n^1(x,y) = K_n(x),$$
$$r_n^2(x,y) = r_n(y),\ D_n^2(x,y) = D_n(y),\ K_n^2(x,y) = K_n(y),$$

for any $n \in \mathbb{N}$. More generally

$$P_n^1(x,y) = P_n(x), \quad P_n^2(x,y) = P_n(y)$$

for any Vilenkin polynomial $P_n = \sum_{k=0}^{n-1} c_k \varphi_k$. Let us denote the set of Vilenkin polynomials with order less than M_n by $\mathcal{P}_{M_n}$. The two-dimensional Vilenkin polynomials are defined analogically. That is

$$P_{n,m}(x,y) = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} c_{k,l} \varphi_k(x) \varphi_l(y).$$

Let us denote the set of two-dimensional Vilenkin polynomials with order less than (M_n, M_n) by $\mathcal{P}_{M_n,M_n}$.

We introduce the notation $\Delta t_{k,n} := t_{k,n} - t_{k+1,n}$, where $k \in \{1,\dots,n\}$ and $t_{n+1,n} := 0$. In the next Lemma, we give a decomposition of the kernels $K_n^T(x,y)$.

Lemma 1. *Let $n > 2$ be a positive integer, then we have*

$$\begin{aligned}
K_n^T = & \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s^1=0}^{l-1} \sum_{s^2=0}^{l-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} (r_j^1)^{s^1} (r_j^2)^{s^2} D_{M_j}^1 D_{M_j}^2 \\
& + \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s^1=0}^{l-1} (r_j^1)^{s^1} (r_j^2)^l D_{M_j}^1 W_{l,j,n}^2 \\
& + \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s^2=0}^{l-1} (r_j^1)^l (r_j^2)^{s^2} W_{l,j,n}^1 D_{M_j}^2 + \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} (r_j^1)^l (r_j^2)^l \mathcal{W}_{l,j,n} \\
& + \sum_{l=1}^{n_{|n|}-1} \sum_{s^1=1}^{l-1} \sum_{s^2=1}^{l-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} (r_{|n|}^1)^{s^1} (r_{|n|}^2)^{s^2} D_{M_{|n|}}^1 D_{M_{|n|}}^2 \\
& + \sum_{l=1}^{n_{|n|}-1} \sum_{s^1=1}^{l-1} (r_{|n|}^1)^{s^1} (r_{|n|}^2)^l D_{M_{|n|}}^1 Q_{l,n}^2 \\
& + \sum_{l=1}^{n_{|n|}-1} \sum_{s^2=1}^{l-1} (r_{|n|}^1)^l (r_{|n|}^2)^{s^2} Q_{l,n}^1 D_{M_{|n|}}^2 + \sum_{l=1}^{n_{|n|}-1} (r_{|n|}^1)^l (r_{|n|}^2)^l \mathcal{Q}_{l,n} \\
& + \sum_{k=0}^{n-n_{|n|}M_{|n|}} \sum_{s^1=0}^{n_{|n|}-1} \sum_{s^2=0}^{n_{|n|}-1} t_{n_{|n|}M_{|n|}+k,n} (r_{|n|}^1)^{s^1} (r_{|n|}^2)^{s^2} D_{M_{|n|}}^1 D_{M_{|n|}}^2 \\
& + \sum_{s^1=0}^{n_{|n|}-1} (r_{|n|}^1)^{s^1} (r_{|n|}^2)^{n_{|n|}} D_{M_{|n|}}^1 R_n^2 + \sum_{s^2=0}^{n_{|n|}-1} (r_{|n|}^1)^{n_{|n|}} (r_{|n|}^2)^{s^2} R_n^1 D_{M_{|n|}}^2 \\
& + (r_{|n|}^1)^{n_{|n|}} (r_{|n|}^2)^{n_{|n|}} \mathcal{R}_n =: \sum_{i=1}^{12} K_{i,n}.
\end{aligned}$$

with the notation $W_{l,j,n} := \sum_{k=0}^{M_j-1} t_{lM_j+k,n} D_k$, $\mathcal{W}_{l,j,n} := \sum_{k=0}^{M_j-1} t_{lM_j+k,n} D_k^1 D_k^2$,

$$Q_{l,n} := \sum_{k=1}^{M_{|n|}-1} t_{lM_{|n|}+k,n} D_k,\ \mathcal{Q}_{l,n} := \sum_{k=1}^{M_{|n|}-1} t_{lM_{|n|}+k,n} D_k^1 D_k^2,$$

$$R_n := \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D_k \text{ and } \mathcal{R}_n := \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D_k^1 D_k^2.$$

Proof. Let us set $0 \leq k < M_j$ and $0 < l < m_j$, then

$$D_{lM_j+k} = \sum_{s=0}^{l-1} \sum_{i=0}^{M_j-1} \varphi_{sM_j+i} + \sum_{i=0}^{k-1} \varphi_{lM_j+i} = \sum_{s=0}^{l-1} r_j^s D_{M_j} + r_j^l D_k. \tag{7}$$

We write

$$K_n^T = \sum_{j=0}^{|n|-1} \sum_{l=M_j}^{M_{j+1}-1} t_{l,n} D_l^1 D_l^2 + \sum_{l=M_{|n|}}^{n} t_{l,n} D_l^1 D_l^2 =: K_n^1 + K_n^2.$$

For the expression K_n^1, the equality (7) yields

$$\begin{aligned} K_n^1 &= \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} D_{lM_j+k}^1 D_{lM_j+k}^2 \\ &= \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} \sum_{s^1=0}^{l-1} (r_j^1)^{s^1} D_{M_j}^1 \sum_{s^2=0}^{l-1} (r_j^2)^{s^2} D_{M_j}^2 \\ &\quad + \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} \sum_{s^1=0}^{l-1} (r_j^1)^{s^1} D_{M_j}^1 (r_j^2)^l D_k^2 \\ &\quad + \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} (r_j^1)^l D_k^1 \sum_{s^2=0}^{l-1} (r_j^2)^{s^2} D_{M_j}^2 \\ &\quad + \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n} (r_j^1)^l (r_j^2)^l D_k^1 D_k^2 \\ &=: K_n^{1,1} + K_n^{1,2} + K_n^{1,3} + K_n^{1,4}. \end{aligned}$$

For the expression $K^{1,2}$, we write

$$K_n^{1,2} = \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s^1=0}^{l-1} (r_j^1)^{s^1} (r_j^2)^l D_{M_j}^1 W_{l,j,n}^2.$$

We apply the notation of our Lemma for the expressions $K^{1,3}$ and $K^{1,4}$ analogically. Applying Abel's transformation, we have

$$W_{l,j,n} = \sum_{k=1}^{M_j-2} \Delta t_{lM_j+k,n} k K_k + t_{(l+1)M_j-1,n} (M_j - 1) K_{M_j-1} \tag{8}$$

and

$$\mathcal{W}_{l,j,n} = \sum_{k=1}^{M_j-2} \Delta t_{lM_j+k,n} k\mathcal{K}_k + t_{(l+1)M_j-1,n}(M_j-1)\mathcal{K}_{M_j-1}. \tag{9}$$

For the expression K_n^2, we write

$$\begin{aligned} K_n^2 &= \sum_{k=0}^{n-M_{|n|}} t_{M_{|n|}+k,n} D^1_{M_{|n|}+k} D^2_{M_{|n|}+k} \\ &= \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} D^1_{lM_{|n|}+k} D^2_{lM_{|n|}+k} \\ &\quad + \sum_{k=0}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D^1_{n_{|n|}M_{|n|}+k} D^2_{n_{|n|}M_{|n|}+k} \\ &=: K_n^{2,1} + K_n^{2,2}. \end{aligned}$$

Moreover, equality (7) yields

$$\begin{aligned} K_n^{2,1} &= \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} \sum_{s^1=0}^{l-1} (r^1_{|n|})^{s^1} D^1_{M_{|n|}} \sum_{s^2=0}^{l-1} (r^2_{|n|})^{s^2} D^2_{M_{|n|}} \\ &\quad + \sum_{l=1}^{n_{|n|}-1} \sum_{s^1=0}^{l-1} (r^1_{|n|})^{s^1} (r^2_{|n|})^l D^1_{M_{|n|}} \sum_{k=1}^{M_{|n|}-1} t_{lM_{|n|}+k,n} D^2_k \\ &\quad + \sum_{l=1}^{n_{|n|}-1} (r^1_{|n|})^l \sum_{s^2=0}^{l-1} (r^2_{|n|})^{s^2} D^2_{M_{|n|}} \sum_{k=1}^{M_{|n|}-1} t_{lM_{|n|}+k,n} D^1_k \\ &\quad + \sum_{l=1}^{n_{|n|}-1} (r^1_{|n|})^l (r^2_{|n|})^l \sum_{k=1}^{M_{|n|}-1} t_{lM_{|n|}+k,n} D^1_k D^2_k \\ &=: K_n^{2,1,1} + K_n^{2,1,2} + K_n^{2,1,3} + K_n^{2,1,4} \end{aligned}$$

and

$$\begin{aligned} K_n^{2,2} &= \sum_{k=0}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} \sum_{s^1=0}^{n_{|n|}-1} \sum_{s^2=0}^{n_{|n|}-1} (r^1_{|n|})^{s^1} (r^2_{|n|})^{s^2} D^1_{M_{|n|}} D^2_{M_{|n|}} \\ &\quad + \sum_{s^1=0}^{n_{|n|}-1} (r^1_{|n|})^{s^1} (r^2_{|n|})^{n_{|n|}} D^1_{M_{|n|}} \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D^2_k \\ &\quad + (r^1_{|n|})^{n_{|n|}} \sum_{s^2=0}^{n_{|n|}-1} (r^2_{|n|})^{s^2} D^2_{M_{|n|}} \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D^1_k \\ &\quad + (r^1_{|n|})^{n_{|n|}} (r^2_{|n|})^{n_{|n|}} \sum_{k=1}^{n-n_{|n|}M_{|n|}} t_{n_{|n|}M_{|n|}+k,n} D^1_k D^2_k. \end{aligned}$$

Now, we use Abel's transform for the expressions $Q_{l,n}$ and $\mathcal{Q}_{l,n}$ in formula $K_n^{2,1}$. We have

$$Q_{l,n} = \sum_{k=1}^{M_{|n|}-2} \Delta t_{lM_{|n|}+k,n} k K_k + t_{(l+1)M_{|n|}-1,n}(M_{|n|}-1)K_{M_{|n|}-1}, \quad (10)$$

$$\mathcal{Q}_{l,n} = \sum_{k=1}^{M_{|n|}-2} \Delta t_{lM_{|n|}+k,n} k \mathcal{K}_k + t_{(l+1)M_{|n|}-1,n}(M_{|n|}-1)\mathcal{K}_{M_{|n|}-1}. \quad (11)$$

Later, in the proof of the main Theorem, we will substitute the result to the expressions $K_n^{2,1,2}$, $K_n^{2,1,3}$ and $K_n^{2,1,4}$. Moreover, we apply Abel's transform for the formulas R_n and $\mathcal{R}_n$

$$R_n = \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} \Delta t_{n_{|n|}M_{|n|}+k,n} k K_k + t_{n,n}(n - n_{|n|}M_{|n|})K_{n-n_{|n|}M_{|n|}}, \quad (12)$$

$$\mathcal{R}_n = \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} \Delta t_{n_{|n|}M_{|n|}+k,n} k \mathcal{K}_k + t_{n,n}(n - n_{|n|}M_{|n|})\mathcal{K}_{n-n_{|n|}M_{|n|}}. \quad (13)$$

It completes the proof of Lemma 1. □

Lemma 2. *Let* $P \in \mathcal{P}_{M_A},\ f \in L^p(G_m^2)(A \in \mathbb{P},\ 1 \le p < \infty)$ *or* $f \in C(G_m^2)$. *Then*

$$\left\| \int_{G_m^2} (f(.+u) - f(.)) r_A^q(u^1) r_A^s(u^2) P(u^1) D_{M_A}(u^2) \mathrm{d}\mu(u) \right\|_p \le m_A \|P\|_1 \omega_p^1 (f, 1/M_A)$$

for any $s, q \in \mathbb{N}$, *where* $q \ne km_A,\ k \in \mathbb{N}$ *(for* $f \in C(G_m^2)$*, we change* p *by* ∞*).*

Proof. We carry out the proof in spaces $L^p(G_m^2)$ $(1 \le p < \infty)$, in space $C(G_m^2)$ the proof is similar, even simpler

$$\begin{aligned}
&\left\| \int_{G_m^2} r_A^q(u^1) r_A^s(u^2) P(u^1) D_{M_A}(u^2)(f(.+u) - f(.)) \mathrm{d}\mu(u) \right\|_p \\
&= \Bigg(\int_{G_m^2} \Bigg| \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} P(y^1) \\
&\quad \times \int_{I_A(y^1)\times I_A} r_A^q(u^1) r_A^s(u^2) D_{M_A}(u^2)(f(x+u) - f(x)) \mathrm{d}\mu(u) \Bigg|^p \mathrm{d}\mu(x) \Bigg)^{\frac{1}{p}} \\
&\le \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} |P(y^1)| \\
&\quad \times \left(\int_{G_m^2} \left| \int_{I_A(y^1)\times I_A} r_A^q(u^1) r_A^s(u^2) M_A (f(x+u) - f(x)) \mathrm{d}\mu(u) \right|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}}
\end{aligned}$$

$$= \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} |P(y^1)|$$

$$\times \left(\int_{G_m^2} \left| \int_{I_A(y^1)\times I_A} r_A^q(u^1) r_A^s(u^2) M_A f(x+u) \mathrm{d}\mu(u) \right|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}} = (*).$$

For any fixed y^1, let us investigate the expression

$$I(y^1) := \left| \int_{I_A(y^1)\times I_A} r_A^q(u^1) r_A^s(u^2) f(x+u) \mathrm{d}\mu(u) \right|.$$

We write

$$I(y^1) = \left| \sum_{y_A^1=0}^{m_A-1} \int_{I_{A+1}(y^1)\times I_A} \exp\left(\frac{2\pi\imath y_A^1 q}{m_A} \right) r_A^s(u^2) f(x+u) \mathrm{d}\mu(u) \right|.$$

We set $y^{1'} := (y_0^1, \ldots, y_{A-1}^1, 0, y_{A+1}^1, \ldots)$, where the Ath coordinate of y^1 is changed by 0. Let us set, $e_A := (0, \ldots, 0, 1, 0, \ldots)$ (only the Ath coordinate is 1, the others are 0), $e_A^{(1)} := (e_A, 0)$ $(0 \in G_m)$ and $e := \exp\left(\frac{2\pi\imath q}{m_A}\right)$, we get

$$I(y^1) = \left| \sum_{k=0}^{m_A-1} \int_{I_{A+1}(y^{1'}+ke_A)\times I_A} r_A^s(u^2) \exp\left(\frac{2\pi\imath k q}{m_A} \right) f(x+u) \mathrm{d}\mu(u) \right|$$

$$= \left| \int_{I_{A+1}(y^{1'})\times I_A} r_A^s(u^2) \sum_{k=0}^{m_A-1} e^k f(x+u+ke_A^{(1)}) \mathrm{d}\mu(u) \right|$$

$$= \left| \int_{I_{A+1}(y^{1'})\times I_A} r_A^s(u^2) \left(\sum_{k=0}^{m_A-2} (f(x+u+ke_A^{(1)}) - f(x+u+(k+1)e_A^{(1)}) \right) \right.$$

$$\times \sum_{j=0}^{k} e^j + f(x+u+(m_A-1)e_A^{(1)}) \sum_{k=0}^{m_A-2} e^k$$

$$\left. + e^{m_A-1} f(x+u+(m_A-1)e_A^{(1)})) \mathrm{d}\mu(u) \right|.$$

Since, e is an nth root of unity, we have

$$\sum_{k=0}^{m_A-1} e^k = 0 \quad \text{and} \quad 0 < \max_{k\in\{0,\ldots,m_A-2\}} \left\{ \left| \sum_{j=0}^{k} e^j \right| \right\} \le m_A. \tag{14}$$

These yield

$$I(y^1) \le \int_{I_{A+1}(y^{1'})\times I_A} \sum_{k=0}^{m_A-2} m_A$$

$$\times \left| (f(x+u+ke_A^{(1)}) - f(x+u+(k+1)e_A^{(1)}) \right| \mathrm{d}\mu(u)$$

$$= m_A \sum_{k=0}^{m_A-2} \int_{I_{A+1}(y^{1'})\times I_A}$$
$$\times \left|(f(x+u+ke_A^{(1)}) - f(x+u+(k+1)e_A^{(1)})\right| \mathrm{d}\mu(u).$$
$$\leq m_A \int_{I_A(y^1)\times I_A} \left|(f(x+u) - f(x+u+e_A^{(1)})\right| \mathrm{d}\mu(u).$$

The generalized Minkowski inequality gives

$$(*) \leq m_A \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} |P(y^1)|$$
$$\times \left(\int_{G_m^2} \left(\int_{I_A(y^1)\times I_A} M_A \left|(f(x+u) - f(x+u+e_A^{(1)})\right| \mathrm{d}\mu(u)\right)^p d\mu(x)\right)^{\frac{1}{p}}$$
$$\leq m_A \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} |P(y^1)| \int_{I_A(y^1)\times I_A}$$
$$\times M_A \left(\int_{G_m^2} \left|(f(x+u) - f(x+u+e_A^{(1)})\right|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} d\mu(u)$$
$$\leq m_A \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} |P(y^1)| \int_{I_A(y^1)} \mathrm{d}\mu(u^1)\ \omega_p^1\left(f, \frac{1}{M_n}\right)$$
$$= m_A \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} \int_{I_A(y^1)} |P(u^1)|\mathrm{d}\mu(u^1)\ \omega_p^1\left(f, \frac{1}{M_n}\right)$$
$$= m_A \|P\|_1 \omega_p^1\left(f, \frac{1}{M_n}\right).$$

This completes the proof of Lemma 2. □

Analogically, we prove the next Lemma.

Lemma 3. *Let $P \in \mathcal{P}_{M_A}$, $f \in L^p(G_m^2)$ ($A \in \mathbb{P}$, $1 \leq p < \infty$) or $f \in C(G_m^2)$. Then*

$$\left\|\int_{G_m^2} (f(.+u) - f(.)) r_A^q(u^1) r_A^s(u^2) D_{M_A}(u^1) P(u^2) \mathrm{d}\mu(u)\right\|_p$$
$$\leq m_A \|P\|_1 \omega_p^2 (f, 1/M_A)$$

for any $s, q \in \mathbb{N}$, where $s \neq km_A$, $k \in \mathbb{N}$ (for $f \in C(G_m^2)$, we change p by ∞).

It is important to note that in the previous Lemma 3, it is possible to choose $q = km_A$ ($k \in \mathbb{N}$), specially $q = 0$ can be chosen. The situation changes in Lemma 4.

Lemma 4. *Let* $P \in \mathcal{P}_{M_A,M_A}$, $f \in L^p(G_m^2)$ *(*$A \in \mathbb{P}$, $1 \le p < \infty$*) or* $f \in C(G_m^2)$*. Then*

$$\left\| \int_{G_m^2} (r_A(u^1))^q (r_A(u^2))^s P(u)(f(.+u) - f(.))\mathrm{d}\mu(u) \right\|_p \le m_A^2 \|P\|_1 \omega_p^{1,2}(f, 1/M_A, 1/M_A)$$

for any $q, s \in \mathbb{P}$*, where* $q, s \neq k m_A$, $k \in \mathbb{N}$ *(for* $f \in C(G_m^2)$*, we change* p *by* ∞*).*

Proof. We carry out the proof in spaces $L^p(G_m^2)$ $(1 \le p < \infty)$, in space $C(G_m^2)$ the proof is similar

$$\begin{aligned} &\left\| \int_{G_m^2} r_A^q(u^1) r_A^s(u^2) P(u)(f(.+u) - f(.))\mathrm{d}\mu(u) \right\|_p \\ &= \left(\int_{G_m^2} \left| \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} \sum_{y_0^2=0}^{m_0-1} \cdots \sum_{y_{A-1}^2=0}^{m_{A-1}-1} P(y^1, y^2) \right.\right. \\ &\quad \left.\left. \times \int_{I_A(y^1)\times I_A(y^2)} r_A^q(u^1) r_A^s(u^2)(f(x+u) - f(x))\mathrm{d}\mu(u) \right|^p d\mu(x) \right)^{\frac{1}{p}} \\ &\le \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} \sum_{y_0^2=0}^{m_0-1} \cdots \sum_{y_{A-1}^2=0}^{m_{A-1}-1} |P(y^1, y^2)| \\ &\quad \times \left(\int_{G_m^2} \left| \int_{I_A(y^1)\times I_A(y^2)} r_A^q(u^1) r_A^s(u^2) f(x+u)\mathrm{d}\mu(u) \right|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}} = (*). \end{aligned}$$

For any fixed (y^1, y^2), let us investigate the expression

$$I(y^1, y^2) := \left| \int_{I_A(y^1)\times I_A(y^2)} r_A^q(u^1) r_A^s(u^2) f(x+u) d\mu(u) \right|.$$

Following the discussion of the expression $I(y^1)$, we write:

$$\begin{aligned} I(y^1, y^2) = &\left| \int_{I_{A+1}(y^{1'})\times I_A(y^{2'})} r_A^s(u^2) \sum_{k=0}^{m_A-2} (f(x+u+ke_A^{(1)}) \right. \\ &\left. - f(x+u+(k+1)e_A^{(1)}) \sum_{j=0}^{k} e_q^j \mathrm{d}\mu(u) \right|, \end{aligned}$$

where $y^{1'}$ is defined in that way as in Lemma 2 we did. Let us set $y^{2'} := (y_0^2, \ldots, y_{A-1}^2, 0, y_{A+1}^2, \ldots)$, where the Ath coordinate of y^2 is changed by 0. ($e_A^{(1)} := (e_A, 0)$, $e_A^{(2)} := (0, e_A)$, $0 \in G_m$), $e_q := \exp\left(\frac{2\pi \imath q}{m_A}\right)$ and $e_s := \exp\left(\frac{2\pi \imath s}{m_A}\right)$. We introduce the notion $F(x+u) := \sum_{k=0}^{m_A-2}(f(x+u+ke_n^1)$

$-\ f(x+u+(k+1)e_n^1)\sum_{j=0}^{k} e_q^j$. Applying the same method for the second variable and inequalities (14), we have

$$\begin{aligned} I(y^1,y^2) &= \left|\sum_{l=0}^{m_A-1}\int_{I_{A+1}(y^{1'})\times I_{A+1}(y^{2'}+le_A)} e_s^l F(x+u)\mathrm{d}\mu(u)\right| \\ &= \left|\int_{I_{A+1}(y^{1'})\times I_{A+1}(y^{2'})}\sum_{l=0}^{m_A-1} e_s^l F(x+u+le_A^{(2)})\mathrm{d}\mu(u)\right| \\ &= \left|\int_{I_{A+1}(y^{1'})\times I_{A+1}(y^{2'})}\sum_{l=0}^{m_A-2}(F(x+u+le_A^{(2)})\right. \\ &\quad \left. -F(x+u+(l+1)e_A^{(2)}))\sum_{i=0}^{l} e_s^i \mathrm{d}\mu(u)\right| \end{aligned}$$

and

$$\begin{aligned} I(y^1,y^2) \le \int_{I_{A+1}(y^{1'})\times I_{A+1}(y^{2'})}\sum_{l=0}^{m_A-2} m_A \\ \times\left|F(x+u+le_A^{(2)})-F(x+u+(l+1)e_A^{(2)})\right|\mathrm{d}\mu(u). \end{aligned}$$

It is easily seen that

$$\begin{aligned} &\left|F(x+u+le_A^{(2)})-F(x+u+(l+1)e_A^{(2)})\right| \\ &\quad\le \sum_{k=0}^{m_A-2} m_A|f(x+u+ke_A^1+le_A^{(2)})-f(x+u+(k+1)e_A^{(1)}+le_A^{(2)}) \\ &\qquad -f(x+u+ke_A^{(1)}+(l+1)e_A^{(2)})+f(x+u+(k+1)e_A^{(1)}+(l+1)e_A^{(2)})| \end{aligned}$$

and

$$\begin{aligned} &I(y^1,y^2) \\ &\quad\le m_A^2\int_{I_A(y^1)\times I_A(y^2)}|(f(x+u)-f(x+u+e_A^{(1)})-f(x+u+e_A^{(2)}) \\ &\qquad +f(x+u+e_A^{(1)}+e_A^{(2)})|\mathrm{d}\mu(u). \end{aligned}$$

The generalized Minkowski's inequality gives

$$\begin{aligned} \sum &\le m_A^2\sum_{y_0^1=0}^{m_0-1}\cdots\sum_{y_{A-1}^1=0}^{m_{A-1}-1}\sum_{y_0^2=0}^{m_0-1}\cdots\sum_{y_{A-1}^2=0}^{m_{A-1}-1}|P(y^1,y^2)|\int_{I_A(y^1)\times I_A(y^2)}\left(\int_{G_m^2}|(f(x+u)\right. \\ &\quad \left. -f(x+u+e_A^{(1)})-f(x+u+e_A^{(2)})+f(x+u+e_A^{(1)}+e_A^{(2)})|^p d\mu(x)\right)^{\frac{1}{p}}\mathrm{d}\mu(u) \\ &\le m_A^2\sum_{y_0^1=0}^{m_0-1}\cdots\sum_{y_{A-1}^1=0}^{m_{A-1}-1}\sum_{y_0^2=0}^{m_0-1}\cdots\sum_{y_{A-1}^2=0}^{m_{A-1}-1}|P(y^1,y^2)|\int_{I_A(y^1)\times I_A(y^2)}\mathrm{d}\mu(u)\omega_p^{1,2}\left(f,\frac{1}{M_A},\frac{1}{M_A}\right) \end{aligned}$$

$$= m_A^2 \sum_{y_0^1=0}^{m_0-1} \cdots \sum_{y_{A-1}^1=0}^{m_{A-1}-1} \sum_{y_0^2=0}^{m_0-1} \cdots \sum_{y_{A-1}^2=0}^{m_{A-1}-1} \int_{I_A(y^1)\times I_A(y^2)} |P(u^1,u^2)| \mathrm{d}\mu(u)\, \omega_p^{1,2}\left(f, \frac{1}{M_A}, \frac{1}{M_A}\right)$$
$$= m_A^2 \|P\|_1 \omega_p^{1,2}\left(f, \frac{1}{M_A}, \frac{1}{M_A}\right).$$

This completes the proof of Lemma 4. □

From now, we discuss bounded Vilenkin groups, i.e., we suppose that $\sup_n m_n < \infty$.

In this case, it is well known that the $L^1(G_m)$ norm of the Fejér kernels is uniformly bounded. Namely, there exists a positive constant c, such that

$$\|K_n\|_1 \leq c. \tag{15}$$

Next lemma was proved by Glukhov [12].

Lemma 5. (Glukhov [12]) *Let $\alpha_1, \ldots, \alpha_n$ be real numbers. Then*

$$\frac{1}{n}\left\|\sum_{k=1}^{n} \alpha_k D_k(.) D_k(..)\right\|_1 \leq \frac{c}{\sqrt{n}}\left(\sum_{k=1}^{n} \alpha_k^2\right)^{1/2},$$

where c is an absolute constant.

As a corollary of Lemma 5, there exists a positive constant c, such that

$$\|\mathcal{K}_n\|_1 \leq c \quad \text{for all } n \in \mathbb{N}. \tag{16}$$

2. The Main Theorem and an Application

Theorem 1. *Let $f \in C(G_m^2)$ or $f \in L^p(G_m^2)$ $(1 \leq p < \infty)$. For every $n \in \mathbb{N}$, let $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers, such that*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

is satisfied.

(a) If the finite sequence $\{t_{k,n} : 1 \leq k \leq n\}$ is non-decreasing for a fixed n and the condition

$$t_{n,n} = O\left(\frac{1}{n}\right) \tag{17}$$

is satisfied, then

$$\|\sigma_n^T(f) - f\|_p \leq c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n}\left(\omega_p^1\left(f, \frac{1}{M_j}\right) + \omega_p^2\left(f, \frac{1}{M_j}\right)\right)$$
$$+c \sum_{k=M_{|n|}}^{n} t_{k,n}\left(\omega_p^1\left(f, \frac{1}{M_{|n|}}\right) + \omega_p^2\left(f, \frac{1}{M_{|n|}}\right)\right)$$
$$+O\left(\omega_p^1\left(f, \frac{1}{M_{|n|}}\right) + \omega_p^2\left(f, \frac{1}{M_{|n|}}\right)\right)$$

holds (for $f \in C(G_m^2)$, we change p by ∞).

(b) If the finite sequence $\{t_{k,n} : 1 \le k \le n\}$ is non-increasing for a fixed n, then

$$\|\sigma_n^T(f) - f\|_p \le c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n} \left(\omega_p^1\left(f, \frac{1}{M_j}\right) + \omega_p^2\left(f, \frac{1}{M_j}\right)\right)$$
$$+c \sum_{k=M_{|n|}}^{n} t_{k,n} \left(\omega_p^1\left(f, \frac{1}{M_{|n|}}\right) + \omega_p^2\left(f, \frac{1}{M_{|n|}}\right)\right)$$

holds (for $f \in C(G_m^2)$, we change p by ∞).

We note that in our Theorem, the expression $c\sum_{k=M_{|n|}}^{n} t_{k,n}\omega_p \left(f, 1/M_{|n|}\right)$ can be replaced by $O\left(\omega_p\left(f, 1/M_{|n|}\right)\right)$. Moreover, we remark that condition $\sum_{k=1}^{n} t_{k,n} = 1$ is natural, many well-known means satisfy this condition.

Proof of Theorem 1. We prove the theorem for $L^p(G_m^2)$ spaces $1 \le p < \infty$. For $C(G_m^2)$, the proof is similar. Let us set $f \in L^p(G_m^2)$. From the condition $\sum_{k=1}^{n} t_{k,n} = 1$, it follows:

$$\begin{aligned}
\|\sigma_n^T(f) - f\|_p &= \left(\int_{G_m^2} |\sigma_n^T(f, x) - f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \\
&= \left(\int_{G_m^2} \left|\int_{G_m^2} K_n^T(u)(f(x+u) - f(x))\mathrm{d}\mu(u)\right|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \\
&\le \sum_{i=1}^{12} \left\|\int_{G_m^2} K_{i,n}(u)(f(.+u) - f(.))\mathrm{d}\mu(u)\right\|_p \\
&=: \sum_{i=1}^{12} I_{i,n}.
\end{aligned}$$

Using generalized Minkowski's inequality [32, vol. 1, p. 19] and inequality

$$\begin{aligned}
&|f(x+u) - f(x)| \\
&\quad \le |f(x^1+u^1, x^2+u^2) - f(x^1+u^1, x^2)| + |f(x^1+u^1, x^2) - f(x^1, x^2)|,
\end{aligned}$$

we write that

$$\left\|\int_{G_m^2} D_{M_j}(u^1)D_{M_j}(u^2)(f(.+u) - f(.))\mathrm{d}\mu(u)\right\|_p \le \tag{18}$$
$$\le \int_{G_m^2} D_{M_j}(u^1)D_{M_j}(u^2)\left(\int_{G_m^2} |f(x+u) - f(x)|^p \,\mathrm{d}\mu(x)\right)^{\frac{1}{p}} \mathrm{d}\mu(u)$$
$$\le \omega_p^1(f, 1/M_j) + \omega_p^2(f, 1/M_j).$$

Applying inequality (18), Lemmas 2 and 3 for the expressions $I_{1,n}$, $I_{5,n}$ and $I_{9,n}$, we obtain that

$$I_{1,n} \leq c \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=0}^{M_j-1} t_{lM_j+k,n}(\omega_p^1(f,1/M_j) + \omega_p^2(f,1/M_j)),$$

$$\begin{aligned} I_{5,n} &\leq c \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n}(\omega_p^1\left(f,1/M_{|n|}\right) + \omega_p^2\left(f,1/M_{|n|}\right)) \\ &\leq c \sum_{k=M_{|n|}}^{n} t_{k,n}(\omega_p^1\left(f,1/M_{|n|}\right) + \omega_p^2\left(f,1/M_{|n|}\right)). \end{aligned}$$

and

$$I_{9,n} \leq c \sum_{k=n_{|n|}M_{|n|}}^{n} t_{k,n}(\omega_p^1\left(f,1/M_{|n|}\right) + \omega_p^1\left(f,1/M_{|n|}\right)).$$

In case a.), we write that

$$I_{1,n} \leq c \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} M_j t_{(l+1)M_j-1,n}(\omega_p^1(f,1/M_j) + \omega_p^2(f,1/M_j)).$$

In case b.), we have that

$$I_{1,n} \leq c \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} M_j t_{lM_j,n}(\omega_p^1(f,1/M_j) + \omega_p^2(f,1/M_j)).$$

Inequalities (8), (18), Lemma 3 and (15) give

$$\begin{aligned} I_{2,n} \leq& \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s^1=0}^{l-1} \\ &\times \left\| \int_{G_m^2} r_j^{s^1}(u^1) r_j^l(u^2) D_{M_j}(u^1) W_{l,j,n}(u^2)(f(.+u) - f(.)) \mathrm{d}\mu(u) \right\|_p \\ =& \; c \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{s^1=0}^{l-1} \sum_{k=1}^{M_j-2} \left|\Delta t_{lM_j+k,n}\right| k\omega_p^2(f,1/M_j) \\ &+c \sum_{j=0}^{|n|-1} (M_j-1) \sum_{l=1}^{m_j-1} \sum_{s^1=0}^{l-1} t_{(l+1)M_j-1,n}\omega_p^2(f,1/M_j) \\ =:& \; I_{2,n}^1 + I_{2,n}^2. \end{aligned}$$

We write in case (a)

$$\begin{aligned} \sum_{k=1}^{M_j-2} |\Delta t_{lM_j+k,n}|k &= \sum_{k=1}^{M_j-2} (t_{lM_j+k+1,n} - t_{lM_j+k,n})k \\ &= (M_j-2)t_{(l+1)M_j-1,n} - \sum_{k=1}^{M_j-2} t_{lM_j+k,n} \\ &\leq M_j t_{(l+1)M_j-1,n} \end{aligned} \tag{19}$$

and

$$I_{2,n}^1 \leq c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n} \omega_p^2 \left(f, 1/M_j\right). \tag{20}$$

We have in case (b)

$$\begin{aligned} \sum_{k=1}^{M_j-2} \left|\Delta t_{lM_j+k,n}\right| k &= \sum_{k=1}^{M_j-2} t_{lM_j+k,n} - (M_j-2)t_{(l+1)M_j-1,n} \\ &\leq \sum_{k=1}^{M_j-2} t_{lM_j+k,n} \leq M_j t_{lM_j,n} \end{aligned} \tag{21}$$

and

$$I_{2,n}^1 \leq c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n} \omega_p^2 \left(f, 1/M_j\right).$$

Let us discuss the expression $I_{2,n}^2$. In case (a), we are ready. That is

$$I_{2,n}^2 \leq c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n} \omega_p^2 \left(f, 1/M_j\right),$$

In case (b), we get that

$$I_{2,n}^2 \leq c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n} \omega_p^2 \left(f, 1/M_j\right),$$

We apply analogical method for the expression $I_{3,n}$. In case a.), we have

$$I_{3,n} \leq c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n} \omega_p^1 \left(f, 1/M_j\right).$$

In case (b), we get

$$I_{3,n} \leq c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n} \omega_p^1 \left(f, 1/M_j\right).$$

It follows from Lemma 3, equality (10) and (15) that:

$$\begin{aligned}
I_{6,n} \le & \sum_{l=1}^{n_{|n|}-1} \sum_{s^1=0}^{l-1} \\
& \times \left\| \int_{G_m^2} (r_{|n|}(u^1))^{s^1} (r_{|n|}(u^2))^l D_{M_{|n|}}(u^1) Q_{l,n}(u^2) \left(f(.+u) - f(.)\right) \mathrm{d}\mu(u) \right\|_p \\
\le & \sum_{l=1}^{n_{|n|}-1} \sum_{s^1=0}^{l-1} \sum_{k=1}^{M_{|n|}-2} \left| \Delta t_{lM_{|n|}+k,n} \right| k \times \\
& \times \left\| \int_{G_m^2} (r_{|n|}(u^1))^{s^1} (r_{|n|}(u^2))^l D_{M_{|n|}}(u^1) K_k(u^2) (|f(.+u) - f(.)) \mathrm{d}\mu(u) \right\|_p \\
& +(M_{|n|}-1) \sum_{l=1}^{n_{|n|}-1} \sum_{s^1=0}^{l-1} t_{(l+1)M_{|n|}-1,n} \times \\
& \times \left\| \int_{G_m^2} (r_{|n|}(u^1))^{s^1} (r_{|n|}(u^2))^l D_{M_{|n|}}(u^1) K_{M_{|n|}-1}(u^2) (f(.+u) - f(.)) \mathrm{d}\mu(u) \right\|_p \\
= & c \sum_{l=1}^{n_{|n|}-1} \sum_{s^1=0}^{l-1} \sum_{k=1}^{M_{|n|}-2} \left| \Delta t_{lM_{|n|}+k,n} \right| k \omega_p^2(f, 1/M_{|n|}) \\
& +c(M_{|n|}-1) \sum_{l=1}^{n_{|n|}-1} \sum_{s^1=0}^{l-1} t_{(l+1)M_{|n|}-1,n} \omega_p^2(f, 1/M_{|n|}) \\
=: & I_{6,n}^1 + I_{6,n}^2.
\end{aligned}$$

In case (a), applying (19) for $j = |n|$, we obtain

$$\begin{aligned}
I_{6,n}^1 &\le c M_{|n|} \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n} \omega_p^2\left(f, 1/M_{|n|}\right) \\
&\le c M_{|n|} t_{n_{|n|}M_{|n|}-1,n} \omega_p^2\left(f, 1/M_{|n|}\right) \\
&\le cnt_{n,n} \omega_p^2\left(f, 1/M_{|n|}\right).
\end{aligned}$$

In case (b), using inequality (21) for $j = |n|$, we have that

$$\sum_{k=0}^{M_{|n|}-2} \left| \Delta t_{lM_{|n|}+k,n} \right| k \le \sum_{k=0}^{M_{|n|}-2} t_{lM_{|n|}+k,n}$$

and

$$\begin{aligned}
I_{6,n}^1 &\le c \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-2} t_{lM_{|n|}+k,n} \omega_p^2\left(f, 1/M_{|n|}\right) \\
&\le c \sum_{k=M_{|n|}}^{n} t_{k,n} \omega_p^2\left(f, 1/M_{|n|}\right).
\end{aligned}$$

We discuss the expression $I_{6,n}^2$

$$I_{6,n}^2 \le c(M_{|n|}-1) \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n} \omega_p^2\left(f, 1/M_{|n|}\right).$$

In case (a), we write that

$$I_{6,n}^2 \leq c(M_{|n|}-1)t_{n_{|n|}M_{|n|}-1,n}\omega_p^2\left(f,1/M_{|n|}\right) \leq cnt_{n,n}\omega_p^2\left(f,1/M_{|n|}\right).$$

In case (b), we have that

$$I_{6,n}^2 \leq c\sum_{k=M_{|n|}}^{n} t_{k,n}\omega_p^2\left(f,1/M_{|n|}\right).$$

We apply analogical method for the expression $I_{7,n}$.

Let us discuss the expression $I_{10,n}$. By the equality (12), Lemma 3 and (15), we may write that

$$\begin{aligned}
I_{10,n} \leq & \sum_{s^1=0}^{n_{|n|}-1}\left\|\int_{G_m^2}(r_{|n|}(u^1))^{s^1}(r_{|n|}(u^2))^{n_{|n|}}D_{M_{|n|}}(u^1)R_n(u^2)(f(.+u)-f(.))\mathrm{d}\mu(u)\right\|_p \\
\leq & \sum_{s^1=0}^{n_{|n|}-1}\sum_{k=1}^{n-n_{|n|}M_{|n|}-1}\left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right|k \\
& \times\left\|\int_{G_m^2}(r_{|n|}(u^1))^{s^1}(r_{|n|}(u^2))^{n_{|n|}}D_{M_{|n|}}(u^1)K_k(u^2)(f(x+u)-f(x))\mathrm{d}\mu(u)\right\|_p \\
& +(n-n_{|n|}M_{|n|})\sum_{s^1=0}^{n_{|n|}-1}t_{n,n} \\
& \times\left\|\int_{G_m^2}(r_{|n|}(u^1))^{s^1}(r_{|n|}(u^2))^{n_{|n|}}D_{M_{|n|}}(u^1)K_{M_{|n|}-1}(u^2)(f(x+u)-f(x))\mathrm{d}\mu(u)\right\|_p \\
\leq & \; c\sum_{s^1=0}^{n_{|n|}-1}\sum_{k=1}^{n-n_{|n|}M_{|n|}-1}\left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right|k\omega_p^2\left(f,1/M_{|n|}\right) \\
& +c(n-n_{|n|}M_{|n|})\sum_{s^1=0}^{n_{|n|}-1}t_{n,n}\omega_p^2\left(f,1/M_{|n|}\right) \\
=: & \; I_{10,n}^1+I_{10,n}^2.
\end{aligned}$$

First, we discuss $I_{10,n}^1$. In case (a), we calculate that

$$\begin{aligned}
& \sum_{k=1}^{n-n_{|n|}M_{|n|}-1}\left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right|k \\
& = \sum_{k=1}^{n-n_{|n|}M_{|n|}-1}(t_{n_{|n|}M_{|n|}+k+1,n}-t_{n_{|n|}M_{|n|}+k,n})k \\
& = (n-n_{|n|}M_{|n|}-1)t_{n,n}-\sum_{k=1}^{n-n_{|n|}M_{|n|}-1}t_{n_{|n|}M_{|n|}+k,n} \\
& \leq nt_{n,n}
\end{aligned}$$

and

$$I_{10,n}^1 \leq cnt_{n,n}\omega_p^2\left(f,1/M_{|n|}\right).$$

In case (b), we have that

$$\begin{aligned}&\sum_{k=1}^{n-n_{|n|}M_{|n|}-1}\left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right|k\\&=\sum_{k=1}^{n-n_{|n|}M_{|n|}-1}t_{n_{|n|}M_{|n|}+k,n}-(n-n_{|n|}M_{|n|}-1)t_{n,n}\\&\leq\sum_{k=1}^{n-n_{|n|}M_{|n|}-1}t_{n_{|n|}M_{|n|}+k,n}\end{aligned}$$

and

$$I^1_{10,n}\leq c\sum_{k=n_{|n|}M_{|n|}}^{n}t_{k,n}\omega_p^2\left(f,1/M_{|n|}\right).$$

Now, we discuss the expression $I^2_{10,n}$. In case (a), we write

$$I^2_{10,n}\leq cnt_{n,n}\omega_p^2\left(f,1/M_{|n|}\right).$$

In case (b), we have

$$I^2_{10,n}\leq c\sum_{k=n_{|n|}M_{|n|}}^{n}t_{k,n}\omega_p^2\left(f,1/M_{|n|}\right).$$

We apply similar method for the expression $I_{11,n}$.

For the expression $I_{4,n}$, equality (9) and Minkowski's inequality yield

$$\begin{aligned}I_{4,n}&\leq\left\|\sum_{j=0}^{|n|-1}\sum_{l=1}^{m_j-1}\int_{G_m^2}(r_j(u^1))^l(r_j(u^2))^l\mathcal{W}_{l,j,n}(u^1,u^2)(f(.+u)-f(.))\mathrm{d}\mu(u)\right\|_p\\&\leq\sum_{j=0}^{|n|-1}\sum_{l=1}^{m_j-1}\sum_{k=1}^{M_j-2}\left|\Delta t_{lM_j+k,n}\right|k\\&\quad\times\left\|\int_{G_m^2}(r_j(u^1))^l(r_j(u^2))^l\mathcal{K}_k(u^1,u^2)(f(.+u)-f(.))\mathrm{d}\mu(u)\right\|_p\\&\quad+\sum_{j=0}^{|n|-1}(M_j-1)\sum_{l=1}^{m_j-1}t_{(l+1)M_j-1,n}\\&\quad\times\left\|\int_{G_m^2}(r_j(u^1))^l(r_j(u^2))^l\mathcal{K}_{M_j-1}(u^1,u^2)(f(.+u)-f(.))\mathrm{d}\mu(u)\right\|_p\\&=:I^1_{4,n}+I^2_{4,n}.\end{aligned}$$

We apply Lemma 4 and (16)

$$
\begin{aligned}
I_{4,n}^{1} &\le c \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=1}^{M_j-2} \left|\Delta t_{lM_j+k,n}\right| k\omega_p^{1,2}\left(f, 1/M_j, 1/M_j\right) \|\mathcal{K}_k\|_1 \\
&\le c \sum_{j=0}^{|n|-1} \sum_{l=1}^{m_j-1} \sum_{k=1}^{M_j-2} \left|\Delta t_{lM_j+k,n}\right| k\omega_p^{1,2}\left(f, 1/M_j, 1/M_j\right).
\end{aligned}
$$

In case (a), we write that

$$
\begin{aligned}
\sum_{k=1}^{M_j-2} |\Delta t_{lM_j+k,n}| k &= \sum_{k=1}^{M_j-2} (t_{lM_j+k+1,n} - t_{lM_j+k,n})k \\
&= (M_j-2)t_{(l+1)M_j-1,n} - \sum_{k=1}^{M_j-2} t_{lM_j+k,n} \\
&\le M_j t_{(l+1)M_j-1,n} \qquad (22)
\end{aligned}
$$

and

$$
I_{4,n}^{1} \le c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n}\omega_p^{1,2}\left(f, 1/M_j, 1/M_j\right).
$$

In case (b), we have

$$
\begin{aligned}
\sum_{k=1}^{M_j-2} \left|\Delta t_{lM_j+k,n}\right| k &= \sum_{k=1}^{M_j-2} t_{lM_j+k,n} - (M_j-2)t_{(l+1)M_j-1,n} \\
&\le \sum_{k=1}^{M_j-2} t_{lM_j+k,n} \le M_j t_{lM_j,n} \qquad (23)
\end{aligned}
$$

and

$$
I_{4,n}^{1} \le c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n}\omega_p^{1,2}\left(f, 1/M_j, 1/M_j\right).
$$

Now, we estimate the expression $I_{4,n}^{2}$. Lemma 4 and (16) yield

$$
\begin{aligned}
I_{4,n}^{2} &\le c \sum_{j=0}^{|n|-1} (M_j-1) \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n}\|\mathcal{K}_{M_j-1}\|_1 \omega_p^{1,2}\left(f, 1/M_j, 1/M_j\right) \\
&\le c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{(l+1)M_j-1,n}\omega_p^{1,2}\left(f, 1/M_j, 1/M_j\right).
\end{aligned}
$$

Therefore, we are ready in case (a). In case (b), we may write that

$$
I_{4,n}^{2} \le c \sum_{j=0}^{|n|-1} M_j \sum_{l=1}^{m_j-1} t_{lM_j,n}\omega_p^{1,2}\left(f, 1/M_j, 1/M_j\right).
$$

Let us discuss the expression $I_{8,n}$. Equality (11) and the usual Minkowski's inequality yield

$$\begin{aligned} I_{8,n} &\leq \left\| \sum_{l=1}^{n_{|n|}-1} \int_{G_m^2} (r_{|n|}(u^1))^l (r_{|n|}(u^2))^l \mathcal{Q}_{l,n}(u^1,u^2)(f(.+u)-f(.))\mathrm{d}\mu(u) \right\|_p \\ &\leq \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_j-2} \left|\Delta t_{lM_{|n|}+k,n}\right| k \\ &\quad \times \left\| \int_{G_m^2} (r_{|n|}(u^1))^l (r_{|n|}(u^2))^l \mathcal{K}_k(u^1,u^2)(f(.+u)-f(.))\mathrm{d}\mu(u) \right\|_p \\ &\quad +(M_{|n|}-1) \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n} \\ &\quad \times \left\| \int_{G_m^2} (r_{|n|}(u^1))^l (r_{|n|}(u^2))^l \mathcal{K}_{M_{|n|}-1}(u^1,u^2)(f(.+u)-f(.)\mathrm{d}\mu(u) \right\|_p \\ &=: I_{8,n}^1 + I_{8,n}^2. \end{aligned}$$

Let us turn our attention to the expression $I_{8,n}^1$. Applying Lemma 4 and (16) again, we may write that

$$\begin{aligned} I_{8,n}^1 &\leq c \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-2} \left|\Delta t_{lM_{|n|}+k,n}\right| k \|\mathcal{K}_k\|_1 \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right) \\ &\leq c \sum_{l=1}^{n_{|n|}-1} \sum_{k=1}^{M_{|n|}-2} \left|\Delta t_{lM_{|n|}+k,n}\right| k \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right). \end{aligned}$$

In case (a), taking into account inequality (22) for $j=|n|$, we get

$$\sum_{k=1}^{M_{|n|}-2} \left|\Delta t_{lM_{|n|}+k,n}\right| k \leq M_{|n|} t_{(l+1)M_{|n|}-1,n}$$

and

$$\begin{aligned} I_{8,n}^1 &\leq cM_{|n|} \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right) \\ &\leq cM_{|n|} t_{n_{|n|}M_{|n|}-1,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right) \\ &\leq cnt_{n,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right). \end{aligned}$$

In case (b), applying inequality (23) for $j=|n|$, we have that

$$\sum_{k=0}^{M_{|n|}-2} \left|\Delta t_{lM_{|n|}+k,n}\right| k \leq \sum_{k=0}^{M_{|n|}-2} t_{lM_{|n|}+k,n}$$

and

$$I_{8,n}^{1} \leq c \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-2} t_{lM_{|n|}+k,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right)$$
$$\leq c \sum_{k=M_{|n|}}^{n} t_{k,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right).$$

For the expression $I_{8,n}^2$, Lemma 4 and (16) yield that

$$I_{8,n}^{2} \leq c(M_{|n|}-1) \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n} \|\mathcal{K}_{M_{|n|}-1}\|_1 \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right)$$
$$\leq cM_{|n|} \sum_{l=1}^{n_{|n|}-1} t_{(l+1)M_{|n|}-1,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right).$$

In case (a), we immediately write

$$I_{8,n}^{2} \leq cM_{|n|} t_{n_{|n|}M_{|n|}-1,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right)$$
$$\leq cnt_{n,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right).$$

In case (b), we have

$$I_{8,n}^{2} \leq c \sum_{l=1}^{n_{|n|}-1} \sum_{k=0}^{M_{|n|}-1} t_{lM_{|n|}+k,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right)$$
$$\leq c \sum_{k=M_{|n|}}^{n} t_{k,n} \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right).$$

At last, we discuss the expression $I_{12,n}$. Usual Minkowski's inequality, equality (13), and Lemma 4 yield that

$$I_{12,n} = \left\| \int_{G_m^2} (r_{|n|}(u^1))^{n_{|n|}} (r_{|n|}(u^2))^{n_{|n|}} \mathcal{R}_n(u^1,u^2)(f(.+u)-f(.))\mathrm{d}\mu(u) \right\|_p$$
$$\leq \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} \left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right| k$$
$$\times \left\| \int_{G_m^2} (r_{|n|}(u^1))^{l} (r_{|n|}(u^2))^{l} \mathcal{K}_k(u^1,u^2)(f(.+u)-f(.))\mathrm{d}\mu(u) \right\|_p$$
$$+t_{n,n}(n-n_{|n|}M_{|n|})$$
$$\times \left\| \int_{G_m^2} (r_{|n|}(u^1))^{l} (r_{|n|}(u^2))^{l} \mathcal{K}_{n-n_{|n|}M_{|n|}}(u^1,u^2)(f(.+u)-f(.))\mathrm{d}\mu(u) \right\|_p$$
$$\leq c \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} \left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right| k \|\mathcal{K}_k\|_1 \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right)$$
$$+c(n-n_{|n|}M_{|n|}) t_{n,n} \|\mathcal{K}_{n-n_{|n|}M_{|n|}}\|_1 \omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right)$$
$$=: I_{12,n}^1 + I_{12,n}^2.$$

In case (a), we obtain that

$$\begin{aligned}
&\sum_{k=1}^{n-n_{|n|}M_{|n|}-1} \left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right| k \\
&= \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} (t_{n_{|n|}M_{|n|}+k+1,n} - t_{n_{|n|}M_{|n|}+k,n})k \\
&= (n - n_{|n|}M_{|n|} - 1)t_{n,n} - \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} t_{n_{|n|}M_{|n|}+k,n} \\
&\leq nt_{n,n}
\end{aligned}$$

and

$$I^1_{12,n} \leq cnt_{n,n}\omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right).$$

In case (b), we have that

$$\begin{aligned}
&\sum_{k=1}^{n-n_{|n|}M_{|n|}-1} \left|\Delta t_{n_{|n|}M_{|n|}+k,n}\right| k \\
&= \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} t_{n_{|n|}M_{|n|}+k,n} - (n - n_{|n|}M_{|n|} - 1)t_{n,n} \\
&\leq \sum_{k=1}^{n-n_{|n|}M_{|n|}-1} t_{n_{|n|}M_{|n|}+k,n}
\end{aligned}$$

and

$$I^1_{12,n} \leq c \sum_{k=n_{|n|}M_{|n|}}^{n} t_{k,n}\omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right).$$

Now, we discuss the expression $I^2_{12,n}$.

In case (a), we write

$$I^2_{12,n} \leq cnt_{n,n}\omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right).$$

In case (b), we have

$$\begin{aligned}
I^2_{12,n} &\leq c(n - n_{|n|}M_{|n|})t_{n,n}\omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right) \\
&\leq c \sum_{k=n_{|n|}M_{|n|}}^{n} t_{k,n}\omega_p^{1,2}\left(f, 1/M_{|n|}, 1/M_{|n|}\right).
\end{aligned}$$

The well-known inequality

$$\omega_p^{1,2}\left(f, 1/M_j, 1/M_j\right) \leq \omega_p^1\left(f, 1/M_j\right) + \omega_p^2\left(f, 1/M_j\right)$$

$(j \in \mathbb{N})$ completes the proof of the main Theorem. □

At last, we apply our theorem for Lipschitz functions, we present the two-dimensional version of the results in the papers [5,19,20] and we generalize the result in [21], as well.

Theorem 2. *Let $f \in Lip(\alpha, p, G_m^2)$ for some $\alpha > 0$ and $1 \le p \le \infty$. For matrix transform σ_n of Vilenkin–Fourier series, we suppose that the conditions in Theorem 1 are satisfied.*

(a) The next estimate holds

$$\|\sigma_n^T(f) - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if } 0 < \alpha < 1, \\ O(\log n/n), & \text{if } \alpha = 1, \\ O(1/n), & \text{if } \alpha > 1. \end{cases}$$

(b) The equality

$$\|\sigma_n^T(f) - f\|_p = O\left(\sum_{j=0}^{|n|-1} t_{M_j,n} M_j^{1-\alpha} + M_{|n|}^{-\alpha}\right)$$

holds.

Proof. In both cases, we use that fact the expression $c\sum_{k=M_{|n|}}^{n} t_{k,n}\omega_p^k\left(f, 1/M_{|n|}\right)$ can be replaced by $O\left(\omega_p^k\left(f, 1/M_{|n|}\right)\right)$, where $k \in \{1, 2\}$.

The proof is analogous to the proof of the one-dimensional theorem in [5]. Therefore, we omit it. □

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