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Global Higher Integrability of the Gradient of Weak Solutions of a Quasilinear Elliptic Equation

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Abstract. In this paper, we establish global higher integrability of the gradient of the solution of the quasilinear elliptic equation $\Delta_A u = \text{div}\left(\frac{a(|F|)}{|F|}F\right)$ in $\mathbb{R}^n$, where Δ_A is the so called A-Laplace operator.

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1. Introduction

We are interested in higher integrability of the gradient of the solution of the following problem:

$$(P)\begin{cases} u \in W^{1,A}(\mathbb{R}^n), \\ \Delta_A u = \text{div}\,(\Theta(F)) \qquad \text{in } \mathbb{R}^n \end{cases}$$

where $n \geqslant 2$, $F = (F_1, \ldots, F_n) \in L^A(\mathbb{R}^n)$, $\Delta_A u = \text{div}\,(\Theta(\nabla u))$, $\Theta(X) = \frac{a(|X|)}{|X|}X$, and $A(t) = \int_0^t a(s)\text{d}s$, with a a function in $C^1((0,\infty)) \cap C^0([0,\infty))$ satisfying $a(0) = 0$, and the condition

$$a_0 \leqslant \frac{ta'(t)}{a(t)} \leqslant a_1 \quad \forall t > 0, \quad a_0, a_1 \text{ positive constants.} \tag{1.1}$$

Without loss of generality, we shall assume that $a_0 < 1 < a_1$.

We call a solution of problem (P) any function $u \in W^{1,A}(\mathbb{R}^n)$ that satisfies

$$\int_{\mathbb{R}^n} \Theta(\nabla u).\nabla\varphi \text{d}x = \int_{\mathbb{R}^n} \Theta(F).\nabla\varphi \text{d}x \quad \forall\varphi \in \mathcal{D}(\mathbb{R}^n)$$

We recall the definition of the Orlicz space $L^A(\mathbb{R}^n)$ and its norm (see [10])

$$L^A(\mathbb{R}^n) = \left\{ u \in L^1(\mathbb{R}^n) \, : \, \int_{\mathbb{R}^n} A(|u(x)|)\mathrm{d}x < \infty \right\},$$
$$||u||_A = \inf \left\{ k > 0 \, : \, \int_{\mathbb{R}^n} A\Big(\frac{|u(x)|}{k}\Big)\mathrm{d}x \leq 1 \right\}$$

The dual space of $L^A(\mathbb{R}^n)$ is the Orlicz space $L^{\widetilde{A}}(\mathbb{R}^n)$, where $\widetilde{A}(t) = \int_0^t a^{-1}(s)\mathrm{d}s$ and a^{-1} is the inverse function of a.

The Orlicz–Sobolev space $W^{1,A}(\mathbb{R}^n)$ and its norm are given by

$$W^{1,A}(\mathbb{R}^n) = \left\{ u \in L^A(\mathbb{R}^n) \, : \, |\nabla u| \in L^A(\mathbb{R}^n) \right\}, \quad ||u||_{1,A} = ||u||_A + ||\nabla u||_A.$$

Both $L^A(\mathbb{R}^n)$ and $W^{1,A}(\mathbb{R}^n)$ are reflexive Banach spaces.

The following useful inequalities can be easily deduced from (1.1) (see [9])

$$\frac{ta(t)}{1+a_1} \leq A(t) \leq ta(t) \quad \forall t \geq 0, \tag{1.2}$$
$$sa(t) \leq sa(s) + ta(t) \quad \forall s,t \geq 0, \tag{1.3}$$
$$\min(s^{a_0}, s^{a_1})a(t) \leq a(st) \leq \max(s^{a_0}, s^{a_1})a(t) \quad \forall s,t \geq 0, \tag{1.4}$$
$$\min(s^{1+a_0}, s^{1+a_1})\frac{A(t)}{1+a_1} \leq A(st) \leq (1+a_1)\max(s^{1+a_0}, s^{1+a_1})A(t) \quad \forall s,t \geq 0. \tag{1.5}$$

Using (1.5) and the convexity of A, we obtain

$$\begin{aligned} A(s+t) &= A\left(2.\left(\frac{s+t}{2}\right)\right) \leq (1+a_1)2^{1+a_1}A\left(\frac{s+t}{2}\right) \\ &\leq (1+a_1)2^{a_1}(A(s)+A(t)) \quad \forall s,t \geq 0 \end{aligned} \tag{1.6}$$

We also recall the following monotonicity inequality (see [3])

$$\Big(\Theta(X) - \Theta(Y)\Big).(X-Y) \geqslant C(A,n)|X-Y|^2 \frac{a\big((|X|^2+|Y|^2)^{1/2}\big)}{(|X|^2+|Y|^2)^{1/2}} \quad \forall (X,Y) \in \mathbb{R}^{2n}\setminus\{0\} \tag{1.7}$$

where $C(A,n)$ is a positive constant depending only on A and n.

There is a wide range of functions $a(t)$ satisfying (1.1). In particular, we observe that we have $a_0 = a_1 = p-1$ if and only if $a(t) = ct^{a_0}$ for some positive constant c. In this case, we have $A(t) = \frac{ct^p}{p}$ and $\Delta_A = c\Delta_p$, where Δ_p is the $p-$Laplace operator. We refer to [4] for more examples of these functions.

Definition 1.1. We say that a function $A : [0,\infty) \to [0,\infty)$ is an $N-$function if $A' \in C^1((0,\infty)) \cap C^0([0,\infty))$, $A'(0) = 0$, and A' satisfies (1.1).

Here is the main result of this paper.

Theorem 1.1. *Let $B : [0,\infty) \to [0,\infty)$ be a function such that $B \circ A^{-1}$ is an $N-$function and let $u \in W^{1,A}(\mathbb{R}^n)$ be a solution of problem (P). If $F \in L^B(\mathbb{R}^n)$, then we have $|\nabla u| \in L^B(\mathbb{R}^n)$ with*

$$\int_{\mathbb{R}^n} B(|\nabla u|)\mathrm{d}x \le C \int_{\mathbb{R}^n} B(|F|)\mathrm{d}x$$

where C is a positive constant depending only on n, A, and B.

Remark 1.1. We would like to mention that this regularity is well known for the Laplace equation (see [8]). For the p-Laplace equation, we refer to [8] and to [5] for systems.

Example 1.1. Assume that Φ is an $N-$function with $\Phi' = \varphi$ and let $B = \Phi \circ A$. Then, B is an $N-$function as a compose of two $N-$functions, namely Φ and A. Moreover, $B \circ A^{-1} = \Phi$ is also an $N-$function. Hence, the following functions satisfy the assumption of Theorem 1.1:

- $B(t) = (A(t))^p$ with $p > 1$, $(\Phi(t) = t^p)$.
- $B(t) = (A(t))^\alpha \ln(\beta A(t) + \gamma)$ with $\alpha, \beta, \gamma > 0$, $(\Phi(t) = t^\alpha \ln(\beta t + \gamma))$.
- $B(t) = (A(t))^{p(A(t))}$, $(\Phi(t) = t^{p(t)})$, provided $p(t)$ satisfies for all $t > 0$: $\alpha_0 \leqslant t\ln(t)p'(t) + p(t) - 1 \leqslant \alpha_1$, for some positive constants α_0 and α_1.

Corollary 1.1. *Under the assumption of Theorem 1.1, if moreover, the integral $\int_1^\infty \frac{B^{-1}(t)}{t^{\frac{n+1}{n}}}\mathrm{d}t$ is finite, then we have $u \in C_\mu(\mathbb{R}^n)$, where $\mu(t) = \int_t^\infty \frac{B^{-1}(s)}{s^{\frac{n+1}{n}}}\mathrm{d}s$ and $C_\mu(\mathbb{R}^n)$ is the space of continuous functions with μ as a modulus of continuity.*

Proof. From Theorem 1.1, we know that $u \in W^{1,B}(\mathbb{R}^n)$. Since the integral $\int_1^\infty \frac{B^{-1}(t)}{t^{\frac{n+1}{n}}}\mathrm{d}t$ is finite, the result follows from the embedding $W^{1,B}(\mathbb{R}^n) \subset C_\mu(\mathbb{R}^n)$ (see [1, Theorem 8.40]). $\square$

Remark 1.2. Since $(B \circ A^{-1})'$ satisfies (1.1), using (1.5) with $t = 1$ and $s = \frac{1}{t}$, we see that there exist two positive constants $\mu > 1$ and K such that

$$0 \le B \circ A^{-1}\left(\frac{1}{t}\right) \le \frac{K}{t^\mu} \quad \text{for all} \quad t \ge 1.$$

Therefore, the integral $\int_1^\infty B \circ A^{-1}\left(\frac{1}{t}\right) dt$ is finite. This property will be used in the proof of Theorem 1.1 in Sect. 3.

In the sequel, we will denote by u a solution of problem (P). In Sect. 2, we recall some well known results about $A-$harmonic functions and establish a few Lemmas to pave the way for the proof of Theorem 1.1 which will be given in Sect. 3.

2. Some Auxiliary Lemmas

Let $x_0 \in \mathbb{R}^n$ and $R > 0$. For each open ball $B_R(x_0)$ in $\mathbb{R}^n$ of center x_0 and radius R, let v be the unique solution of the following problem:

$$(P_0)\begin{cases} v \in W^{1,A}(B_R(x_0)), \\ \Delta_A v = 0 \quad \text{in } B_R(x_0), \\ v = u \quad \text{in } \partial B_R(x_0) \end{cases}$$

First, we recall some properties of the solution of problem (P_0).

Lemma 2.1.

$$\int_{B_R(x_0)} a(|\nabla v|)|\nabla v| \leq 2^{a_1+2} \int_{B_R(x_0)} a(|\nabla u|)|\nabla u| \mathrm{d}x$$

Proof. See [3], Proof of Lemma 3.1. □

Remark 2.1. Using (1.2) and Lemma 2.1, we obtain

$$\begin{aligned} \int_{B_R(x_0)} A(|\nabla v|)\mathrm{d}x &\leq \int_{B_R(x_0)} a(|\nabla v|)|\nabla v| \mathrm{d}x \\ &\leq 2^{a_1+2} \int_{B_R(x_0)} a(|\nabla v|)|\nabla v| \mathrm{d}x \\ &\leq (1+a_1)2^{a_1+2} \int_{B_R(x_0)} A(|\nabla u|)\mathrm{d}x \end{aligned}$$

Lemma 2.2. *There exists a positive constant $C_1 = C_1(n, a_1)$ such that*

$$\sup_{B_{R/2}(x_0)} A(|\nabla v|) \leq \frac{C_1}{R^n} \int_{B_R(x_0)} A(|\nabla v|)\mathrm{d}x$$

Proof. See [9], Lemma 5.1.

For each function $f : \mathbb{R}^n \to \mathbb{R}\ (\mathbb{R}^n)$, let $(f)_{x_0,r} = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} f dx$ and $(f)_r = (f)_{0,r}$. Then we have the following property of the function v. □

Lemma 2.3. *There exist two positive constants $\alpha = \alpha(n, a_1) < 1$ and $C_2 = C_2(n, a_1)$ such that we have for any $r \in (0, R)$*

$$\fint_{B_r(x_0)} A(|\nabla v - (\nabla v)_{x_0,r}|)\mathrm{d}x \leq C_2 \left(\frac{r}{R}\right)^\alpha \fint_{B_R(x_0)} A(|\nabla v - (\nabla v)_{x_0,R}|)\mathrm{d}x$$

Proof. See [9], Lemma 5.1. □

Remark 2.2. Using (1.6), Remark 2.1, and Jensen's inequality, we obtain

$$\begin{aligned} \fint_{B_R(x_0)} A(|\nabla v - (\nabla v)_{x_0,R}|)\mathrm{d}x \ &\leq \ \fint_{B_R(x_0)} A(|\nabla v| + |(\nabla v)_{x_0,R}|)\mathrm{d}x \\ &\leq (1+a_1)2^{a_1}\left(\fint_{B_R(x_0)} A(|\nabla v|)\mathrm{d}x + A(|(\nabla v)_{x_0,R}|)\right) \\ &= (1+a_1)2^{a_1}\left(\fint_{B_R(x_0)} A(|\nabla v|)\mathrm{d}x + A\left(\fint_{B_R(x_0)} |\nabla v|\mathrm{d}x\right)\right) \end{aligned}$$

$$\leq (1+a_1)2^{a_1}\left(\fint_{B_R(x_0)} A(|\nabla v|)\mathrm{d}x + \fint_{B_R(x_0)} A(|\nabla v|)\mathrm{d}x\right)$$
$$= (1+a_1)2^{a_1+1}\fint_{B_R(x_0)} A(|\nabla v|)\mathrm{d}x$$
$$\leq (1+a_1)^2 2^{2a_1+3}\fint_{B_R(x_0)} A(|\nabla u|)\mathrm{d}x \tag{2.1}$$

Combining (2.1) and Lemma 2.3, we obtain

$$\fint_{B_r(x_0)} A(|\nabla v-(\nabla v)_{x_0,r}|)\mathrm{d}x \leq C_2(1+a_1)^2 2^{2a_1+3}\left(\frac{r}{R}\right)^{\alpha}\fint_{B_R(x_0)} A(|\nabla u|)\mathrm{d}x$$

The next lemma is the key tool in the proof of Theorem 1.1.

Lemma 2.4. *There exist two positive constants $\gamma = \gamma(n, a_0, a_1)$ and $m = m(\alpha, n, a_0, a_1)$ such that for each $\delta \in (0,1)$, we have for any $x_0 \in \mathbb{R}^n$, $R > 0$ and $r \in (0, R)$:*

$$\fint_{B_r(x_0)} |A(|\nabla u|) - (A(|\nabla u|))_r|\mathrm{d}x \;\leq\; \frac{\gamma}{\delta^{a_1(1+a_1)}}\left(\frac{R}{r}\right)^m \fint_{B_R(x_0)} A(|F|)\mathrm{d}x$$
$$+\gamma\left(\delta^{a_0+1} + \frac{1}{\delta^{a_1(1+a_1)}}\left(\frac{r}{R}\right)^{\alpha}\right)\fint_{B_R(x_0)} A(|\nabla u|)\mathrm{d}x$$

The proof of Lemma 2.4 requires several lemmas. □

Lemma 2.5. *Let $G : \mathbb{R}^{2n}\backslash\{0\} \to \mathbb{R}$ defined by*

$$G(\xi,\zeta) = \int_0^1 \frac{a(|\theta_t|)}{|\theta_t|}dt, \quad \theta_t = t\xi + (1-t)\zeta.$$

Then there exists two positive constants $c_{a_1,n}$ depending only on n and a_1, and $C_{a_0,n}$ depending only on n and a_0 such that:

$$\forall(\xi,\zeta) \in \mathbb{R}^{2n}\backslash\{0\}, \quad c_{a_1,n}\frac{a(|\xi|+|\zeta|)}{|\xi|+|\zeta|} \leq G(\xi,\zeta) \leq C_{a_0,n}\frac{a(|\xi|+|\zeta|)}{|\xi|+|\zeta|}$$

The proof of Lemma 2.5 is based on the following lemma proved in [2] for $n = 2$ and whose proof extends easily to $n \geq 3$.

Lemma 2.6. *Let $n \geqslant 2$, $p > 1$ and let $F_p : \mathbb{R}^{2n}\backslash\{0\} \to \mathbb{R}$ defined by*

$$F_p(\xi,\zeta) = \int_0^1 |t\xi + (1-t)\zeta|^{p-2}dt$$

Then, there exists two positive constants $c(p,n) < C(p,n)$ depending only on p and n such that:

$$\forall(\xi,\zeta) \in \mathbb{R}^{2n}\backslash\{0\}, \quad c(p,n)\big(|\xi|^2+|\zeta|^2\big)^{\frac{p-2}{2}} \leq F_p(\xi,\zeta) \leq C(p,n)\big(|\xi|^2+|\zeta|^2\big)^{\frac{p-2}{2}}$$

Proof of Lemma 2.5. For $(\xi,\zeta) \in \mathbb{R}^{2n}\backslash\{0\}$, we set

$$X_0 = \frac{\xi}{(|\xi|^2+|\zeta|^2)^{1/2}}, \quad X_1 = \frac{\zeta}{(|\xi|^2+|\zeta|^2)^{1/2}}.$$

It follows that

$$\theta_t = (|\xi|^2+|\zeta|^2)^{1/2}(tX_0+(1-t)X_1), \quad |\theta_t| = (|\xi|^2+|\zeta|^2)^{1/2}|tX_0+(1-t)X_1|.$$

Then, we have by inequality (1.4)

$$a(|\theta_t|) \le \max\big(|tX_0+(1-t)X_1|^{a_0}, |tX_0+(1-t)X_1|^{a_1}\big)a\big((|\xi|^2+|\zeta|^2)^{1/2}\big)$$
$$\min\big(|tX_0+(1-t)X_1|^{a_0}, |tX_0+(1-t)X_1|^{a_1}\big)a\big((|\xi|^2+|\zeta|^2)^{1/2}\big) \le a(|\theta_t|)$$

Note that since $|tX_0+(1-t)X_1| \le t|X_0|+(1-t)|X_1| \le t+(1-t) = 1$ and $0 < a_0 \le a_1$, we have $|tX_0+(1-t)X_1|^{a_1} \le |tX_0+(1-t)X_1|^{a_0}$. Hence we get

$$|tX_0+(1-t)X_1|^{a_1}a\big((|\xi|^2+|\zeta|^2)^{1/2}\big) \le a(|\theta_t|)$$
$$\le |tX_0+(1-t)X_1|^{a_0}a\big((|\xi|^2+|\zeta|^2)^{1/2}\big)$$

which leads to

$$F_{a_1+1}(X_0,X_1)\frac{a\big((|\xi|^2+|\zeta|^2)^{1/2}\big)}{(|\xi|^2+|\zeta|^2)^{1/2}} \le G(\xi,\zeta) \le F_{a_0+1}(X_0,X_1)\frac{a\big((|\xi|^2+|\zeta|^2)^{1/2}\big)}{(|\xi|^2+|\zeta|^2)^{1/2}}.$$

Now, if we apply Lemma 2.6 with $p = a_0+1$ and $p = a_1+1$, we get since $|X_0|^2+|X_1|^2 = 1$

$$F_{a_0+1}(X_0,X_1) \le C(a_0+1,n)\big(|X_0|^2+|X_1|^2\big)^{\frac{a_0-1}{2}} = C(a_0+1,n)$$
$$F_{a_1+1}(X_0,X_1) \ge c_{a_1+1,n}\big(|X_0|^2+|X_1|^2\big)^{\frac{a_1-1}{2}} = c(a_1+1,n)$$

Finally, we obtain

$$c(a_1+1,n)\frac{a\big((|\xi|^2+|\zeta|^2)^{1/2}\big)}{(|\xi|^2+|\zeta|^2)^{1/2}} \leqslant G(\xi,\zeta) \leqslant C(a_0+1,n)\frac{a\big((|\xi|^2+|\zeta|^2)^{1/2}\big)}{(|\xi|^2+|\zeta|^2)^{1/2}}$$

Since the two norms $|\xi|+|\zeta|$ and $(|\xi|^2+|\zeta|^2)^{1/2}$ are equivalent, the lemma follows using (1.4) □

Lemma 2.7. *For any $X, Y \in \mathbb{R}^n$, we have:*

$$A(|X|) \ge A(|Y|) + \langle\Theta(Y), X-Y\rangle$$

Proof. Let $X, Y \in \mathbb{R}^n$. First, we have

$$A(|X|)-A(|Y|) = \int_0^1 \frac{d}{dt}[A(|tX+(1-t)Y|)]dt$$
$$= \int_0^1 A'(|tX+(1-t)Y|).\frac{< tX+(1-t)Y, X-Y >}{|tX+(1-t)Y|}dt$$
$$= \int_0^1 a(|tX+(1-t)Y|).\frac{< tX+(1-t)Y, X-Y >}{|tX+(1-t)Y|}dt$$

$$= \int_0^1 \langle \Theta(tX + (1-t)Y), X - Y \rangle \, dt$$

Next, using (1.7), this leads to

$$\begin{aligned} A(|X|) - A(|Y|) &= \int_0^1 \frac{1}{t} \langle \Theta(tX + (1-t)Y), t(X - Y) \rangle \, dt \\ &\geq \int_0^1 \frac{1}{t} \langle \Theta(Y), t(X - Y) \rangle \, dt \\ &= \langle \Theta(Y), X - Y \rangle \end{aligned}$$

which achieves the proof. □

Lemma 2.8. *For any $\delta \in (0,1)$, $X, Y \in \mathbb{R}^n$, we have:*

$$|A(|X|) - A(|Y|)| \leq \frac{C_3}{\delta^{a_1(1+a_1)}} A(|X - Y|) + C_4 \delta^{a_0+1} (A(|X|) + A(|Y|))$$

where $C_3 = (1 + a_1) C_{a_0,n}$ and $C_4 = (1 + a_1) 2^{a_1} C_3$

Proof. Let $\delta \in (0,1)$, $X, Y \in \mathbb{R}^n$. First, we have

$$\begin{aligned} |A(|X|) - A(|Y|)| &= \left| \int_0^1 \frac{d}{dt} [A(|tX + (1-t)Y|)] dt \right| \\ &= \left| \int_0^1 A'(|tX + (1-t)Y|) . \frac{< tX + (1-t)Y, X - Y >}{|tX + (1-t)Y|} dt \right| \\ &= \left| \int_0^1 a(|tX + (1-t)Y|) . \frac{< tX + (1-t)Y, X - Y >}{|tX + (1-t)Y|} dt \right| \\ &\leq |X - Y| . (|X| + |Y|) . \int_0^1 \frac{a(|tX + (1-t)Y|)}{|tX + (1-t)Y|} dt \end{aligned}$$

Next, we get by using Lemma 2.5

$$\begin{aligned} |A(|X|) - A(|Y|)| &\leq C_{a_0,n} |X - Y| . (|X| + |Y|) . \frac{a(|X| + |Y|)}{|X| + |Y|} \\ &= C_{a_0,n} |X - Y| . a(|X| + |Y|) \end{aligned} \tag{2.2}$$

We observe that we have by (1.2)–(1.4), for $s, t \geq 0$ since $\delta \in (0,1)$

$$\begin{aligned} sa(t) &= \frac{s}{\delta^{a_1}} . \delta^{a_1} a(t) \leq \frac{s}{\delta^{a_1}} . a(\delta t) \leq \frac{s}{\delta^{a_1}} . a\left(\frac{s}{\delta^{a_1}} .\right) + \delta t a(\delta t) \\ &\leq \frac{s}{\delta^{a_1}} . \frac{1}{(\delta^{a_1})^{a_1}} a(s) + \delta t \delta^{a_0} a(t) = \frac{sa(s)}{\delta^{a_1(1+a_1)}} + \delta^{a_0+1} t a(t) \\ &\leq \frac{1 + a_1}{\delta^{a_1(1+a_1)}} A(s) + (1 + a_1) \delta^{a_0+1} A(t) \end{aligned} \tag{2.3}$$

Using (2.2) and (2.3), with $s = |X - Y|$ and $t = |X| + |Y|$, we get

$$\begin{aligned} |A(|X|) - A(|Y|)| \leq \frac{(1 + a_1) C_{a_0,n}}{\delta^{a_1(1+a_1)}} A(|X - Y|) + (1 + a_1) C_{a_0,n} \delta^{a_0+1} \\ A(|X| + |Y|) \end{aligned} \tag{2.4}$$

Using (1.6), we get from (2.4)

$$|A(|X|)-A(|Y|)| \le \frac{C_3}{\delta^{a_1(1+a_1)}} A(|X-Y|)+(1+a_1)^2 2^{a_1} C_{a_0,n}\delta^{a_0+1} (A(|X|)+A(|Y|))$$

Hence, the lemma follows. □

Lemma 2.9. *There exist two positive constants* $C_5=\frac{(1+a_1)2^{\frac{a_1}{2}}}{C(A,n)}$ *and* $C_6=(1+a_1)2^{a_1}\left(1+(1+a_1)2^{a_1+2}\right)\left(C_5+\frac{1+a_1}{4}\right)$ *such that we have for each* $\eta\in(0,1)$, $x_0\in\mathbb{R}^n$, $R>0$ *and* $r\in(0,R)$*:*

$$⨍_{B_r(x_0)} A(|\nabla u-\nabla v|)\mathrm{d}x \le \left(\frac{R}{r}\right)^n \Big(\frac{C_5}{\eta^{1+a_1(1+a_1)}} ⨍_{B_R(x_0)} A(|F|)\mathrm{d}x + C_6\eta^{a_0} ⨍_{B_R(x_0)} A(|\nabla u|)\mathrm{d}x\Big)$$

Proof. We observe that by translation, it is enough to prove the lemma for $x_0=0$. Using $w=(u-v)\chi_{B_R}$ as a test function for problems (P) and (P_0) and subtracting the two equations and using (2.3), we get for $\eta\in(0,1)$:

$$\begin{aligned}\int_{B_R} &\big(\Theta(\nabla u)-\Theta(\nabla v)\big)(\nabla u-\nabla v)\mathrm{d}x\\ &= \int_{B_R}\frac{a(|F|)}{|F|}F.\nabla w\mathrm{d}x \le \int_{B_R} a(|F|).|\nabla w|\mathrm{d}x\\ &\le \frac{1+a_1}{\eta^{a_1(1+a_1)}}\int_{B_R}A(|F|)\mathrm{d}x+(1+a_1)\eta^{a_0+1}\int_{B_R}A(|\nabla w|)\mathrm{d}x \quad (2.5)\end{aligned}$$

Using (2.5) and (1.7), we get

$$\frac{C(A,n)}{2^{\frac{a_1}{2}}}\int_{B_R}\frac{a(|\nabla u|+|\nabla v|)}{|\nabla u|+|\nabla v|}.|\nabla w|^2\mathrm{d}x \le \frac{1+a_1}{\eta^{a_1(1+a_1)}}\int_{B_R}A(|F|)\mathrm{d}x +(1+a_1)\eta^{a_0+1}\int_{B_R}A(|\nabla w|)\mathrm{d}x$$

or

$$\int_{B_R}\frac{a(|\nabla u|+|\nabla v|)}{|\nabla u|+|\nabla v|}.|\nabla w|^2\mathrm{d}x \le \frac{C_5}{\eta^{a_1(1+a_1)}}\int_{B_R}A(|F|)\mathrm{d}x+C_5\eta^{a_0+1}\int_{B_R}A(|\nabla w|)\mathrm{d}x \quad (2.6)$$

By (1.2), we can write

$$\begin{aligned}\int_{B_r}A(|\nabla w|)\mathrm{d}x &\le \int_{B_r}|\nabla w|a(|\nabla w|)\mathrm{d}x\\ &= \int_{E_{1,r}}|\nabla w|a(|\nabla w|)\mathrm{d}x+\int_{E_{2,r}}|\nabla w|a(|\nabla w|)\mathrm{d}x \quad (2.7)\end{aligned}$$

where $E_{1,r}=B_r\cap\{\ (|\nabla u|+|\nabla v|)a(|\nabla w|)\le|\nabla w|a(|\nabla u|+|\nabla v|)\ \}$ and $E_{2,r}=B_r\cap\{\ (|\nabla u|+|\nabla v|)a(|\nabla w|)>|\nabla w|a(|\nabla u|+|\nabla v|)\ \}$.

From the definition of $E_{1,r}$, we see that

$$\int_{E_{1,r}} |\nabla w| a(|\nabla w|) \mathrm{d}x \leq \int_{E_{1,r}} \frac{a(|\nabla u| + |\nabla v|)}{|\nabla u| + |\nabla v|} . |\nabla w|^2 \mathrm{d}x \tag{2.8}$$

Using the monotonicity of a, the definition of $E_{2,r}$, and Young's inequality, we get

$$\begin{aligned}
\int_{E_{2,r}} |\nabla w| a(|\nabla w|) \mathrm{d}x \ &\leq \ \int_{E_{2,r}} a(|\nabla u| + |\nabla v|). |\nabla w| \mathrm{d}x \\
&\leq \int_{E_{2,r}} \left(\frac{a(|\nabla u| + |\nabla v|)}{|\nabla u| + |\nabla v|} . |\nabla w|^2 \right)^{\frac{1}{2}} . ((|\nabla u| + |\nabla v|) a(|\nabla u| + |\nabla v|))^{\frac{1}{2}} \mathrm{d}x \\
&\leq \frac{1}{\eta} \int_{E_{2,r}} \frac{a(|\nabla u| + |\nabla v|)}{|\nabla u| + |\nabla v|} . |\nabla w|^2 \mathrm{d}x + \frac{\eta}{4} \int_{E_{2,r}} (|\nabla u| + |\nabla v|) a(|\nabla u| + |\nabla v|) \mathrm{d}x \\
&\leq \frac{1}{\eta} \int_{E_{2,r}} \frac{a(|\nabla u| + |\nabla v|)}{|\nabla u| + |\nabla v|} . |\nabla w|^2 \mathrm{d}x + \frac{(1+a_1)\eta}{4} \int_{B_r} A(|\nabla u| + |\nabla v|) \mathrm{d}x
\end{aligned} \tag{2.9}$$

Combing (2.6), (2.7), (2.8), and (2.9), and using (1.6) and Remark 2.1, we arrive at

$$\begin{aligned}
&\int_{B_r} A(|\nabla w|) \mathrm{d}x \\
&\quad \leq \frac{1}{\eta} \int_{B_r} \frac{a(|\nabla u| + |\nabla v|)}{|\nabla u| + |\nabla v|} . |\nabla w|^2 \mathrm{d}x + \frac{(1+a_1)\eta}{4} \int_{B_R} A(|\nabla u| + |\nabla v|) \mathrm{d}x \\
&\quad \leq \frac{C_5}{\eta^{a_1(1+a_1)+1}} \int_{B_R} A(|F|) \mathrm{d}x + C_5 \eta^{a_0} \int_{B_R} A(|\nabla w|) \mathrm{d}x \\
&\qquad + \frac{(1+a_1)\eta}{4} \int_{B_R} A(|\nabla u| + |\nabla v|) \mathrm{d}x \\
&\quad \leq \frac{C_5}{\eta^{a_1(1+a_1)+1}} \int_{B_R} A(|F|) \mathrm{d}x + \left(C_5 \eta^{a_0} + \frac{(1+a_1)\eta}{4} \right) \int_{B_R} A(|\nabla u| + |\nabla v|) \mathrm{d}x \\
&\quad \leq (1+a_1) 2^{a_1} \left(C_5 \eta^{a_0} + \frac{(1+a_1)\eta}{4} \right) \left(\int_{B_R} A(|\nabla u|) \mathrm{d}x + \int_{B_R} A(|\nabla v|) \mathrm{d}x \right) \\
&\qquad + \frac{C_5}{\eta^{a_1(1+a_1)+1}} \int_{B_R} A(|F|) \mathrm{d}x \\
&\quad \leq (1+a_1) 2^{a_1} \left(C_5 + \frac{1+a_1}{4} \right) (1 + (1+a_1) 2^{a_1+2}) \eta^{a_0} \int_{B_R} A(|\nabla u|) \mathrm{d}x \\
&\qquad + \frac{C_5}{\eta^{a_1(1+a_1)+1}} \int_{B_R} A(|F|) \mathrm{d}x
\end{aligned}$$

Hence, the lemma follows. □

Proof of Lemma 2.4. Let $\delta \in (0,1)$, $R > 0$ and $r \in (0, R)$. First, we have by Lemma 2.8 used for $X = \nabla u(x)$ and $Y = \nabla u(y)$

$$\begin{aligned}
&\fint_{B_r} |A(|\nabla u|) - (A(|\nabla u|))_r|)\mathrm{d}x \\
&\quad = \fint_{B_r} \left| \fint_{B_r} (A(|\nabla u(x)|) - A(|\nabla u(y)|))\mathrm{d}y \right| \mathrm{d}x \\
&\quad \le \fint_{B_r \times B_r} |A(|\nabla u(x)|) - A(|\nabla u(y)|)|\mathrm{d}x\mathrm{d}y \\
&\quad \le \frac{C_3}{\delta^{a_1(1+a_1)}} \fint_{B_r \times B_r} A(|\nabla u(x) - \nabla u(y)|)\mathrm{d}x\mathrm{d}y \\
&\qquad + C_4 \delta^{a_0+1} \left(\fint_{B_r \times B_r} A(|\nabla u(x)|)\mathrm{d}x\mathrm{d}y + \fint_{B_r \times B_r} A(|\nabla u(y)|)\mathrm{d}x\mathrm{d}y \right) \\
&\quad = \frac{C_3}{\delta^{a_1(1+a_1)}} \fint_{B_r \times B_r} A(|\nabla u(x) - \nabla u(y)|)\mathrm{d}x\mathrm{d}y \\
&\qquad + 2C_4 \delta^{a_0+1} \fint_{B_r} A(|\nabla u(x)|)\mathrm{d}x \\
&\quad \le \frac{C_3}{\delta^{a_1(1+a_1)}} \fint_{B_r \times B_r} A(|\nabla u(x) - (\nabla u)_r| + |\nabla u(y) - (\nabla u)_r|)\mathrm{d}x\mathrm{d}y \\
&\qquad + 2C_4 \delta^{a_0+1} \fint_{B_r} A(|\nabla u|)\mathrm{d}x
\end{aligned}$$

This leads by (1.6) to

$$\begin{aligned}
&\fint_{B_r} |A(|\nabla u|) - (A(|\nabla u|))_r|\mathrm{d}x \\
&\quad \le \frac{C_3(1+a_1)2^{a_1}}{\delta^{a_1(1+a_1)}} \left(\fint_{B_r \times B_r} A(|\nabla u(x) - (\nabla u)_r|)\mathrm{d}x\mathrm{d}y \right. \\
&\qquad \left. + \fint_{B_r \times B_r} A(|\nabla u(y) - (\nabla u)_r|)\mathrm{d}x\mathrm{d}y \right) \\
&\qquad + 2C_4 \delta^{a_0+1} \fint_{B_r} A(|\nabla u|)\mathrm{d}x \\
&\quad = \frac{2C_3(1+a_1)2^{a_1}}{\delta^{a_1(1+a_1)}} \fint_{B_r} A(|\nabla u - (\nabla u)_r|)\mathrm{d}x + 2C_4 \delta^{a_0+1} \fint_{B_r} A(|\nabla u|)\mathrm{d}x
\end{aligned} \tag{2.10}$$

Using (1.6) again, we get

$$\begin{aligned}
&\fint_{B_r} A(|\nabla u - (\nabla u)_r|)\mathrm{d}x \\
&\quad \le \fint_{B_r} A(|\nabla u - \nabla v| + |\nabla v - (\nabla v)_r| + |(\nabla u)_r - (\nabla v)_r|)\mathrm{d}x \\
&\quad \le (1+a_1)2^{a_1} \left(\fint_{B_r} A(|\nabla u - \nabla v|)\mathrm{d}x \right. \\
&\quad \left. + \fint_{B_r} A(|\nabla v - (\nabla v)_r| + |(\nabla u)_r - (\nabla v)_r|)\mathrm{d}x \right)
\end{aligned}$$

$$\leq (1+a_1)2^{a_1} \fint_{B_r} A(|\nabla u - \nabla v|)\mathrm{d}x$$
$$+(1+a_1)^2 2^{2a_1} \left(\fint_{B_r} A(|\nabla v - (\nabla v)_r|)\mathrm{d}x + \fint_{B_r} A(|(\nabla u)_r - (\nabla v)_r|)\mathrm{d}x \right) \tag{2.11}$$

Using Jensen's inequality, we derive

$$\fint_{B_r} A(|(\nabla u)_r - (\nabla v)_r|)\mathrm{d}x = A\left(\left| \fint_{B_r} (\nabla u - \nabla v)\mathrm{d}x \right| \right)$$
$$\leq \fint_{B_r} A(|\nabla u - \nabla v|)\mathrm{d}x \tag{2.12}$$

From (2.11) and (2.12), we obtain

$$\fint_{B_r} A(|\nabla u - (\nabla u)_r|)\mathrm{d}x$$
$$\leq (1+a_1)2^{a_1}(1+(1+a_1)2^{a_1}) \fint_{B_r} A(|\nabla u - \nabla v|)\mathrm{d}x$$
$$+(1+a_1)^2 2^{2a_1} \fint_{B_r} A(|\nabla v - (\nabla v)_r|)\mathrm{d}x \tag{2.13}$$

Combining (2.10) and (2.13), we obtain

$$\fint_{B_r} |A(|\nabla u|) - (A(|\nabla u|))_r|\mathrm{d}x \leq \frac{C_7}{\delta^{a_1(1+a_1)}} \fint_{B_r} A(|\nabla u - \nabla v|)\mathrm{d}x$$
$$+\frac{C_8}{\delta^{a_1(1+a_1)}} \fint_{B_r} A(|\nabla v - (\nabla v)_r|)\mathrm{d}x + 2C_4\delta^{a_0+1} \fint_{B_r} A(|\nabla u|)\mathrm{d}x \tag{2.14}$$

where $C_7 = 2C_3(1+a_1)^2 2^{2a_1}(1+(1+a_1)2^{a_1})$ and $C_8 = 2C_3(1+a_1)^3 2^{3a_1}$.

Finally, by taking into account Remark 2.2 and Lemma 2.8, we obtain from (2.14) for any $\eta \in (0,1)$

$$\fint_{B_r} |A(|\nabla u|) - (A(|\nabla u|))_r|\mathrm{d}x \leq \frac{C_7}{\delta^{a_1(1+a_1)}} \left(\frac{R}{r}\right)^n$$
$$\left(\frac{C_5}{\eta^{1+a_1(1+a_1)}} \fint_{B_R} A(|F|)\mathrm{d}x + C_6\eta^{a_0} \fint_{B_R(x_0)} A(|\nabla u|)\mathrm{d}x \right)$$
$$+\frac{C_8 C_2}{\delta^{a_1(1+a_1)}}(1+a_1)^2 2^{2a_1+3} \left(\frac{r}{R}\right)^\alpha \fint_{B_R} A(|\nabla u|)\mathrm{d}x$$
$$+2C_4\delta^{a_0+1} \fint_{B_r} A(|\nabla u|)\mathrm{d}x$$

To conclude the proof, we let $\gamma = \max(C_5C_7, C_6C_7 + C_2C_8(1+a_1)^2 2^{2a_1+3}, 2C_4)$, $m = n + \frac{(\alpha+n)(1+a_1(1+a_1))}{a_0}$, and choose η such that $\eta^{a_0} = \left(\frac{r}{R}\right)^{\alpha+n}$ or $\eta = \left(\frac{r}{R}\right)^{\frac{\alpha+n}{a_0}}$. Then we obtain

$$\fint_{B_r} |A(|\nabla u|) - (A(|\nabla u|))_r| \mathrm{d}x \ \le \ \frac{\gamma}{\delta^{a_1(1+a_1)}} \left(\frac{R}{r}\right)^m \fint_{B_R} A(|F|)\mathrm{d}x$$
$$+\gamma\left(\delta^{a_0+1} + \frac{1}{\delta^{a_1(1+a_1)}}\left(\frac{r}{R}\right)^\alpha\right) \fint_{B_R} A(|\nabla u|)\mathrm{d}x$$

□

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. First, we recall that for each function $f \in L^1(\mathbb{R}^n)$, the Hardy–Littlewood maximal function associated with f is given by

$$M[f](x_0) = \sup_{r>0} \fint_{B_r(x_0)} |f(x)|\mathrm{d}x, \quad x_0 \in \mathbb{R}^n$$

and the sharp maximal function associated with f is defined by

$$f^\sharp(x_0) = \sup_{r>0} \fint_{B_r(x_0)} |f - (f)_{x_0,r}|\mathrm{d}x, \quad x_0 \in \mathbb{R}^n$$

Proof of Theorem 1.1. We deduce from Lemma 2.4 that we have for every $\delta \in (0,1)$, $r > 0$, $R = \frac{r}{\delta^{\frac{a_1(1+a_1)+a_0+1}{\alpha}}}$, $\kappa = a_1(1+a_1) + \frac{m(a_1(1+a_1)+a_0+1)}{\alpha}$, and $x_0 \in \mathbb{R}^n$

$$\fint_{B_r(x_0)} |A(|\nabla u|) - (A(|\nabla u|))_r|)\mathrm{d}x \le \frac{\gamma}{\delta^\kappa} \fint_{B_R x_0)} A(|F|)\mathrm{d}x$$
$$+2\gamma\delta^{a_0+1} \fint_{B_R x_0)} A(|\nabla u|)\mathrm{d}x$$

which leads to

$$(A(|\nabla u|))^\sharp(x_0) \ \le \ \frac{\gamma}{\delta^\kappa} M[A(|F|)](x_0) + 2\gamma\delta^{a_0+1} M[A(|\nabla u|)](x_0) \tag{3.1}$$

Now, let B be a function such that $B \circ A^{-1}$ is an $N-$function and assume that $F \in C_0^\infty(\mathbb{R}^n)$ and $|\nabla u| \in L^B(\mathbb{R}^n)$. Then by using (1.6), we obtain from (3.1) (see [6]) for $\gamma' = \gamma(1+c_1)2^{c_1}$, where c_1 is the equivalent to the constant a_1 for the function $B \circ A^{-1}$

$$\int_{\mathbb{R}^n} B(|\nabla u|)\mathrm{d}x \le \int_{\mathbb{R}^n} B\left(A^{-1}(A(|\nabla u|))^\sharp(x)\right)\mathrm{d}x$$
$$\le 2\gamma'\delta^{1+a_0} \int_{\mathbb{R}^n} B\left(A^{-1} M[A(|\nabla u|)](x)\right)\mathrm{d}x$$
$$+\frac{\gamma'}{\delta^\kappa} \int_{\mathbb{R}^n} B\left(A^{-1} M[A(|F|)](x)\right)\mathrm{d}x \tag{3.2}$$

By the Hardy–Littlewood maximal theorem (see [7]), we have for some positive constant C_9 depending only upon n, A and B

$$\int_{\mathbb{R}^n} B\left(A^{-1} M[A(|F|)](x)\right)\mathrm{d}x \le C_9 \int_{\mathbb{R}^n} B(|F|)\mathrm{d}x \tag{3.3}$$

$$\int_{\mathbb{R}^n} B\left(A^{-1} M[A(|\nabla u|)](x)\right)\mathrm{d}x \le C_9 \int_{\mathbb{R}^n} B(|\nabla u|)\mathrm{d}x \tag{3.4}$$

We deduce then from (3.2), (3.3) and (3.4) that

$$\int_{\mathbb{R}^n} B(|\nabla u|)\mathrm{d}x \leq 2\gamma' C_9 \delta^{1+a_0} \int_{\mathbb{R}^n} B(|\nabla u|)\mathrm{d}x + \frac{\gamma' C_9}{\delta^\kappa} \int_{\mathbb{R}^n} B(|F|)\mathrm{d}x \tag{3.5}$$

Now, if we choose δ such that $\delta \in \left(0, \min\left(1, (2\gamma' C_9)^{-\frac{1}{1+a_0}}\right)\right)$, we get

$$\int_{\mathbb{R}^n} B(|\nabla u|)\mathrm{d}x \leq \frac{\gamma' C_9}{\delta^\kappa (1 - 2\gamma' C_9 \delta^{1+a_0})} \int_{\mathbb{R}^n} B(|F|)\mathrm{d}x \tag{3.6}$$

This completes the proof when $F \in C_0^\infty(\mathbb{R}^n)$ and $|\nabla u| \in L^B(\mathbb{R}^n)$. Next, we will establish it when $F \in L^B(\mathbb{R}^n)$. To do that, we consider a sequence $(F_k)_k$ of vector functions in $C_0^\infty(\mathbb{R}^n)$ that converges to F in $L^B(\mathbb{R}^n)$. We denote by $(u_k)_k$ the sequence of unique solutions of the following problem

$$(P_k) \begin{cases} u_k \in W^{1,A}(\mathbb{R}^n), \\ \operatorname{div}\big(\Theta(\nabla u_k)\big) = \operatorname{div}\left(\Theta(F_k)\right) \qquad \text{in } \mathbb{R}^n \end{cases}$$

Since F_k has compact support in $\mathbb{R}^n$, there exists $l_k > 0$ such that $\operatorname{suppt}(F_k) \subset B_{l_k}$, and therefore we have for all k

$$\operatorname{div}\big(\Theta(\nabla u_k)\big) = 0 \quad \text{in } \mathbb{R}^n \backslash B_{l_k} \tag{3.7}$$

We will prove that $|\nabla u_k| \in L^B(\mathbb{R}^n)$. Since $u_k \in C^1(\mathbb{R}^n)$, it is enough to show that $\int_{\{|x|>2l_k\}} B(|\nabla u_k(x)|)\mathrm{d}x < \infty$. We observe that because of (3.7), we can apply Lemma 2.3 in $B_{|x|-l_k}(x)$ for each $x \in \mathbb{R}^n \backslash B_{2l_k}$

$$A(|\nabla u_k(x)|) \leq \frac{C_1}{(|x| - l_k)^n} \int_{B_{|x|-l_k}(x)} A(|\nabla u_k|)\mathrm{d}y \tag{3.8}$$

As observed in Remark 1.2, there exists two positive constants μ and K such that

$$0 \leq B \circ A^{-1}(st) \leq K s^\mu B \circ A^{-1}(t) \qquad \forall s, t \geq 0. \tag{3.9}$$

Using (3.8) and (3.9), and taking into account Remark 1.2, we get

$$\begin{aligned}
&\int_{\{|x|>2l_k\}} B(|\nabla u_k(x)|)\mathrm{d}x \\
&\leq K \left(C_1 \int_{B_{|x|-l_k}(x)} A(|\nabla u_k|)\mathrm{d}y\right)^\mu . \int_{\{|x|>2l_k\}} B \circ A^{-1}\left(\frac{1}{(|x| - l_k)^n}\right)\mathrm{d}x \\
&\leq K\omega_n \left(C_1 \int_{\mathbb{R}^n} A(|\nabla u_k|)\mathrm{d}y\right)^\mu . \int_{2l_k}^\infty r^{n-1} B \circ A^{-1}\left(\frac{1}{(r - l_k)^n}\right) dr \\
&\leq K\omega_n 2^{n-1} \left(C_1 \int_{\mathbb{R}^n} A(|\nabla u_k|)\mathrm{d}y\right)^\mu . \int_{2l_k}^\infty (r - l_k)^{n-1} B \circ A^{-1}\left(\frac{1}{(r - l_k)^n}\right) dr \\
&= \frac{K\omega_n 2^{n-1}}{n} \left(C_1 \int_{\mathbb{R}^n} A(|\nabla u_k|)\mathrm{d}y\right)^\mu . \int_{l_k^n}^\infty B \circ A^{-1}\left(\frac{1}{t}\right) dt < \infty
\end{aligned}$$

It follows that $|\nabla u_k| \in L^B(\mathbb{R}^n)$. Therefore, (3.6) is valid for u_k and we have

$$\int_{\mathbb{R}^n} B(|\nabla u_k|)\mathrm{d}x \leq \frac{\gamma' C_9}{\delta^\kappa (1 - 2\gamma' C_9 \delta^{1+a_0})} \int_{\mathbb{R}^n} B(|F_k|)\mathrm{d}x \quad \forall k \tag{3.10}$$

Since $F_k \to F$ strongly in $L^B(\mathbb{R}^n)$, we deduce from (3.10) that we have for some positive integer k_0

$$\int_{\mathbb{R}^n} B(|\nabla u_k|)\mathrm{d}x \le \frac{2\gamma' C_9}{\delta^\kappa(1-2\gamma' C_9\delta^{1+a_0})}\int_{\mathbb{R}^n} B(|F|)\mathrm{d}x \quad \forall k \ge k_0 \tag{3.11}$$

Therefore, u_k is uniformly bounded in $W^{1,B}(\mathbb{R}^n)$ and consequently has a weakly convergent subsequence to some function v in $W^{1,B}(\mathbb{R}^n)$. Using Lemma 2.7 with function B, we get

$$\int_{\mathbb{R}^n} B(|\nabla u_k|)\mathrm{d}x \ge \int_{\mathbb{R}^n} B(|\nabla v|)\mathrm{d}x + \int_{\mathbb{R}^n}\left\langle \frac{b(|\nabla v|)}{|\nabla v|}\nabla v, \nabla u_k - \nabla v\right\rangle \mathrm{d}x \tag{3.12}$$

Passing to the limit in (3.11) and taking into account (3.12) and the convergence for the weak topology, we infer that

$$\int_{\mathbb{R}^n} B(|\nabla v|)\mathrm{d}x \le \frac{2\gamma' C_9}{\delta^\kappa(1-2\gamma' C_9\delta^{1+a_0})}\int_{\mathbb{R}^n} B(|F|)\mathrm{d}x \tag{3.13}$$

The proof of the theorem will be complete if we prove that v is a solution of problem (P). To this end, we observe that it is enough to show that $\Theta(\nabla u_k)$ has a weakly convergent subsequence in $L^{\widetilde{A}}(\mathbb{R}^n)$ to $\Theta(\nabla v)$. The proof is well known for monotone and continuous operators. We give it here for the sake of completeness.

Using $w = u_k - v$ as a test function for problem (P_k), we get

$$\int_{\mathbb{R}^n} \Theta(\nabla u_k).(\nabla u_k - \nabla v)\mathrm{d}x = \int_{\mathbb{R}^n} \Theta(F_k).(\nabla u_k - \nabla v)\mathrm{d}x \tag{3.14}$$

Now, given that ∇u_k is uniformly bounded in $L^A(\mathbb{R}^n)$, $\Theta(\nabla u_k)$ is uniformly bounded in $L^{\widetilde{A}}(\mathbb{R}^n)$, and, therefore, has a weakly convergent subsequence to some vector function V in $L^{\widetilde{A}}(\mathbb{R}^n)$. Moreover, $u_k \rightharpoonup v$ weakly in $W^{1,B}(\mathbb{R}^n)$ and $F_k \to F$ strongly in $L^B(\mathbb{R}^n)$. Therefore, in particular, $u_k \rightharpoonup v$ weakly in $W^{1,A}(\mathbb{R}^n)$ and $F_k \to F$ strongly in $L^A(\mathbb{R}^n)$. Hence, we obtain from (3.14)

$$\limsup_{k\to\infty}\int_{\mathbb{R}^n} |\nabla u_k| a(|\nabla u_k|)\mathrm{d}x = \int_{\mathbb{R}^n} V.\nabla v \mathrm{d}x \tag{3.15}$$

At this point, we use (1.7) for $X = \nabla u_k$ and a vector function $Y \in L^A(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \big(\Theta(\nabla u_k) - \Theta(Y)\big).(\nabla u_k - Y)\mathrm{d}x \ge 0.$$

Letting $k \to \infty$ and taking into account (3.15), one can check that

$$\int_{\mathbb{R}^n} \big(V - \Theta(Y)\big).(\nabla v - Y)\mathrm{d}x \ge 0.$$

Now choosing $Y = \nabla v - \lambda\vartheta$, where λ is a positive number and ϑ is an arbitrary vector function in $\mathcal{D}(\mathbb{R}^n)$, we obtain

$$\int_{\mathbb{R}^n} \big(V - \Theta(\nabla v - \lambda\vartheta)\big).\vartheta \mathrm{d}x \ge 0.$$

Letting $\lambda \to 0$, we get $\int_{\mathbb{R}^n} \big(V - \Theta(\nabla v)\big).\vartheta \mathrm{d}x \geq 0$. Since ϑ is arbitrary in $\mathcal{D}(\mathbb{R}^n)$, it follows that $V = \Theta(\nabla v)$ in $\mathbb{R}^n$. This achieves the proof. □

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