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A Study of Solutions of Some Nonlinear Integral Equations in the Space of Functions of Bounded Second Variation in the Sense of Shiba

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Abstract. In this paper, we study the existence and uniqueness of solutions for nonlinear Hammerstein, Volterra–Hammerstein, and Volterra equations in the space of functions of bounded second variation in the sense of Shiba, $(\Lambda_p^2 BV([a,b]))$.

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1. Introduction

Integral equations and integro-differential equations arise in many branches of physics and engineering. For example, in potential theory, acoustics, elasticity, fluid mechanics, radiant transfer, and population theory. For the past decade, many attempts to solve the nonlinear Hammerstein, Hammerstein–Volterra, and Volterra integral equations were carried out by different studies using numerical methods, since these equations arise in many applications on physics, mathematics, and chemistry, such as stereology, heat conduction, crystal growth, and heat radiation from a semi-solid. Various methods have been used to approximate the solution of such integral equations. For example, in [1], a variation of the Nystrom method is introduced; in [2], they work with the classical method of successive approximations; in [3] and [4], some collocation methods were developed. On the other hand, the bounded variation functions have been suitable in the study of optimal control problems, as well as in the calculus of variations. Furthermore, these functions are useful for image retrieval problems and are well adapted to the study of parameter identification problems, such as the coefficients of an elliptical machine or a parabolic operator.

In [5] Ereú, Giménez, and Pérez studied the solutions of nonlinear Hammerstein and Volterra–Hammerstein integral equations in the space of functions of bounded variation in the sense of Shiba, denoted by $\Lambda_p BV$. The proofs of their results were based on the Banach Contraction Principle. They considered the Hammerstein equation defined as

$$x(t) = g(t) + \lambda \int_I K(t,s) f(x(s)) \mathrm{d}s,\ \lambda \in I, t \in I = [0,b], \tag{1.1}$$

and the Volterra–Hammerstein equation

$$x(t) = g(t) + \int_0^t K(t,s) f(x(s)) \mathrm{d}s,\ t \in I = [0,b], \tag{1.2}$$

where the integration is taken in the sense of Lebesgue, and they assumed the following hypotheses:

H_1) $g : I \to \mathbb{R}$ is a function of Λ_p-bounded variation.
H_2) $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function.
H_3) $K : I \times I \to \mathbb{R}$ is a function, such that

$$V_{\Lambda_p}(K(\cdot,s), I) \leq M(s), \quad \text{for a.e. } s \in I,$$

where $M : I \to \mathbb{R}$ is an L^p integrable function, and $K(t,\cdot)$ is Lebesgue integrable for each $t \in I = [0,b]$.

Together with some additional hypotheses, they showed that there exists a number $\tau > 0$, such that for every λ with $|\lambda| < \tau$, the Eq. (1.1) has a unique solution in $\Lambda_p BV([0,b])$.

In a similar way, in [5], they studied the solutions of Volterra equation

$$x(t) = g(t) + \int_a^t K(t,s) f(x(s)) ds,\ t \in I = [a,b], \tag{1.3}$$

in the space of functions $\Lambda_p BV$, and the proofs of their results were based on the Leray–Schauder Alternative theorem. Under the hypotheses H_1) and H_2), and the additional hypothesis

H_4) Let $K : \{(t,s) \in [a,b] \times [a,b] : s \leq t\} \to \mathbb{R}$ be a function, such that

$$\frac{|K(s,s)|}{(\lambda_1)^{\frac{1}{p}}} + V_{\Lambda_p}(K(\cdot,s) : [s,b]) \leq h(s),\ \text{for a.e. } s \in [a,b],$$

where $h : I \to \mathbb{R}_+$ is an L^p integrable function, and $K(t,\cdot)$ is Lebesgue integrable on $[a,t]$ for each $t \in [a,b]$,

they proved that there exists a unique solution $\widehat{x} \in \Lambda_p BV$ for the Eq. (1.3).

Motivated by the paper [5], in this work, we consider the Hammerstein Eq. (1.1), the Volterra–Hammerstein integral Eq. (1.2), and the Volterra Eq. (1.3). Under certain hypotheses, we study the solutions of these equations in the class of functions of bounded second variation in the sense of Shiba, $(\Lambda_p^2 BV)$ with $1 \leq p < \infty$.

This paper is structured as follows: Sect. 2 on preliminaries, which provides the necessary results for the proofs of the main theorems. Section 3 presents the existence and uniqueness results for Hammerstein equation and

Volterra–Hammerstein integral equations, whose proofs are based on the Banach Contraction Principle. In addition, the existence and uniqueness of solutions for Volterra equation are proved using the Leray–Schauder Alternative theorem. In Sect. 4, an application problem is presented; finally, some conclusions are drawn at the last section.

2. Preliminaries

This section is a review of the preliminary results which are fundamentals for the development of the main theorems. For the purpose of studying the solutions of the nonlinear integral Eqs. (1.1), (1.2), and (1.3), we consider the following hypotheses:

($\widehat{H}_1$) $f : I \to \mathbb{R}$ is a function of $(\Lambda, 2, p)-$th bounded second variation on $[a, b]$, that is, $f \in \Lambda_p^2 BV([a, b])$.

($\widehat{H}_2$) $K : I \times I \to \mathbb{R}$ is a function, such that

$$V_{\Lambda,2,p}(K(\cdot, s), I) \le M(s), \quad \text{for a.e. } s \in I,$$

where $M : I \to \mathbb{R}$ is an L^p integrable function, and $\{K(t, \cdot)\}_{t\in I}$ is bounded in $L^1(I)$, which is to say that there exists $\widehat{C} > 0$, such that $\|K(t, \cdot)\|_1 \le \widehat{C}$ for every $t \in I$. The variation $V_{\Lambda,2,p}(\cdot, I)$ will be defined below.

The following two theorems are used in the proof of the main results to guarantee the existence and uniqueness of solutions of the integral equations to be studied, and are stated here, so the paper is self-contained.

Theorem 2.1. {*Banach Contraction Principle.*} *Let $f : X \to X$ be a contraction mapping on a complete metric space and $B \subseteq X$ be a closed subset, such that $f(B) \subseteq B$. Then, f has a unique fixed point in B.*

Theorem 2.2. {*Leray–Schauder Alternative.*} *Let U be an open subset of a Banach space $(X, \|.\|)$ with $0 \in U$. Suppose there exists a nondecreasing continuous function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying $\phi(z) < z$ for $z > 0$. The function $H : \overline{U} \to X$ is such that verifies $\|H(x) - H(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \overline{U}$, where $\overline{U}$ is the closure of U in X. In addition, $H(U)$ is bounded, and $x \ne \lambda H(x)$ for all $x \in \partial U$ (boundary of U) and $\lambda \in (0, 1]$. Then, H has a fixed point in U.*

Next, we present the theory needed in this paper such as definitions, remarks, and lemmas that will be strongly used in the proofs of the main theorems.

In [6] is introduced the concept of the classes of functions of bounded second variation in the sense of Shiba, where it is shown that this class of functions, denoted by $\Lambda_p^2 BV$, is a normed vector space. In [7], it is proved in addition that this is a Banach space. The following definitions were considered in these papers.

Definition 2.3. A sequence of positive real numbers $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ is a $\mathcal{W}$-sequence if $\{\lambda_i\}_{i=1}^{\infty}$ is nondecreasing, and $\sum(1/\lambda_i) = +\infty$.

Definition 2.4. $\Pi_3([a,b])$ denote the set of partitions $\pi = \{x_i\}_{i=0}^n$ of the interval $[a,b]$ containing at least three points, $(\pi \in \Pi_3([a,b]))$.

Definition 2.5. Let $1 \le p < \infty$, and let $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ be a $\mathcal{W}$-sequence. The $(\Lambda, 2, p)$-th variation of f on $[a,b]$ is defined by

$$V_{\Lambda,2,p}(f;[a,b]) = V_{\Lambda,2,p}(f) = \sup_{\pi} \left(\sum_{i=0}^{n-2} \frac{|Q_1(f;x_{i+2},x_{i+1}) - Q_1(f;x_{i+1},x_i)|^p}{\lambda_i} \right)^{1/p},$$

where $Q_1(f;\beta,\alpha) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$, and the supremum is taken over all the partitions $\pi = \{x_i\}_{i=0}^n \in \Pi_3([a,b])$. The sum above is called an approximated sum for $V_{\Lambda,2,p}(f;[a,b])$.

We say that f has $(\Lambda, 2, p)-$th bounded second variation on $[a,b]$ whenever $V_{\Lambda,2,p}(f;[a,b]) < \infty$. We denote the space of such functions by $\Lambda_p^2 BV$ $([a,b])$.

Remark 2.6. $\Lambda_p^2 BV([a,b])$ together with the norm $\|f\|_{\Lambda,2,p} = \|f\|_\infty + V_{\Lambda,2,p}$ $(f;[a,b])$ is a Banach space. See [7].

The following three lemmas have been proved in [6], and are highly useful results for some of the main theorems.

Lemma 2.7. *Let* $1 \le p < \infty$, $f, g \in \Lambda_p^2 BV([a,b])$, *and* $\lambda \in \mathbb{R}$. *Then*

$$V_{\Lambda,2,p}(f + \lambda g) \le V_{\Lambda,2,p}(f) + |\lambda| V_{\Lambda,2,p}(g).$$

Lemma 2.8. *If* $1 \le p < \infty$ *and* $f \in \Lambda_p^2 BV([a,b])$, *then* $Q_1(f;\cdot,\cdot)$ *is bounded on* $[a,b] \times [a,b]$.

Remark 2.9. It can be easily shown from the Lemma above that if $f \in \Lambda_p^2 BV([a,b])$, then f is bounded.

Lemma 2.10. *If* $f \in \Lambda_p^2 BV([a,b])$, *where* $1 \le p < \infty$, *then* f *is Lipschitz and hence is continuous on* $[a,b]$.

Remark 2.11. Denote $L_a^b(f)$ the Lipschitz constant of a function $f : [a,b] \to \mathbb{R}$, that is

$$L_a^b(f) := \sup \left\{ \left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| : x_1, x_2 \in [a,b], x_1 \neq x_2 \right\}.$$

The following lemma is proved in the same way as Lemma 2.8 in [6].

Lemma 2.12. *Let* $K : I \times I \to \mathbb{R}$. *If* $K(\cdot, s) \in \Lambda_p^2 BV([s,b])$, *then there exists a function* $C(s)$, *such that* $|Q_1(K(\cdot,s);\cdot,\cdot)|^p \le C(s)$ *on* $[s,b] \times [s,b]$.

In [7], it is shown that if $g \in \Lambda_p^2 BV([a,b])$ is strictly increasing, and if $f \in \Lambda_p^2 BV([g(a), g(b)])$, then the composition $f \circ g \in \Lambda_p^2 BV([a,b])$. Here, we prove this result by removing the requirement that g is strictly increasing.

Lemma 2.13. *If* $g \in \Lambda_p^2 BV([a,b])$, *and* $f \in \Lambda_p^2 BV([g(a), g(b)])$, *then*

$$f \circ g \in \Lambda_p^2 BV([a,b]).$$

Proof. Let $\Lambda = \{\lambda_i\}_{i\geq 0}$ be a W-sequence. Let us consider a partition $\pi = \{t_i\}_{i=0}^{n} \in \Pi_3([a,b])$ such that $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. We need to consider several cases for the proof.

- Suppose $g(t_{i+2}) = g(t_{i+1})$ and $g(t_{i+1}) = g(t_i)$, and thus

$$\frac{|Q_1(f\circ g; t_{i+2}, t_{i+1}) - Q_1(f\circ g; t_{i+1}, t_i)|^p}{\lambda_i} = \frac{\left|\frac{0}{t_{i+2}-t_{i+1}} - \frac{0}{t_{i+1}-t_i}\right|^p}{\lambda_i} = 0.$$

- Suppose $g(t_{i+2}) \neq g(t_{i+1})$ and $g(t_{i+1}) = g(t_i)$, then

$$\begin{aligned}
&\frac{|Q_1(f\circ g; t_{i+2}, t_{i+1}) - Q_1(f\circ g; t_{i+1}, t_i)|^p}{\lambda_i}\\
&= \frac{\left|\frac{f(g(t_{i+2}))-f(g(t_{i+1}))}{t_{i+2}-t_{i+1}}\right|^p}{\lambda_i} = \frac{\left|\frac{f(g(t_{i+2}))-f(g(t_{i+1}))}{g(t_{i+2})-g(t_{i+1})} \times \frac{g(t_{i+2})-g(t_{i+1})}{t_{i+2}-t_{i+1}}\right|^p}{\lambda_i}\\
&= \frac{\left|\frac{f(g(t_{i+2}))-f(g(t_{i+1}))}{g(t_{i+2})-g(t_{i+1})}\right|^p \left|\frac{g(t_{i+2})-g(t_{i+1})}{t_{i+2}-t_{i+1}}\right|^p}{\lambda_i}\\
&= |Q_1(f; g(t_{i+2}), g(t_{i+1}))|^p \times \frac{\left|\frac{g(t_{i+2})-g(t_{i+1})}{t_{i+2}-t_{i+1}} - \frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}\right|^p}{\lambda_i}\\
&\leq (M_1)^p \times \frac{\left|\frac{g(t_{i+2})-g(t_{i+1})}{t_{i+2}-t_{i+1}} - \frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}\right|^p}{\lambda_i},
\end{aligned}$$

  where $M_1 = \sup\{Q_1(f;\beta,\alpha);\ \alpha,\beta \in [g(a), g(b)]\}$ is assured by Lemma 2.8.
- A similar argument to above applies when $g(t_{i+2}) = g(t_{i+1})$ and $g(t_{i+1}) \neq g(t_i)$

$$\begin{aligned}
&\frac{|Q_1(f\circ g; t_{i+2}, t_{i+1}) - Q_1(f\circ g; t_{i+1}, t_i)|^p}{\lambda_i}\\
&\leq (M_1)^p \times \frac{\left|\frac{g(t_{i+2})-g(t_{i+1})}{t_{i+2}-t_{i+1}} - \frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}\right|^p}{\lambda_i},
\end{aligned}$$

  where $M_1 = \sup\{Q_1(f;\beta,\alpha);\ \alpha,\beta \in [g(a), g(b)]\}$.
- For the case $g(t_{i+2}) \neq g(t_{i+1})$ and $g(t_{i+1}) \neq g(t_i)$, the proof is identical to that of Theorem 5 in [7]

$$\begin{aligned}
&\frac{|Q_1(f\circ g; t_{i+2}, t_{i+1}) - Q_1(f\circ g; t_{i+1}, t_i)|^p}{\lambda_i}\\
&\leq 2^p\ (M_1)^p \times \frac{\left|\frac{g(t_{i+2})-g(t_{i+1})}{t_{i+2}-t_{i+1}} - \frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}\right|^p}{\lambda_i}\\
&\quad +2^p\ (M_2)^p \times \frac{\left|\frac{f(g(t_{i+2}))-f(g(t_{i+1}))}{g(t_{i+2})-g(t_{i+1})} - \frac{f(g(t_{i+1}))-f(g(t_i))}{g(t_{i+1})-g(t_i)}\right|^p}{\lambda_i}.
\end{aligned}$$

Therefore

$$V_{\Lambda,2,p}(f\circ g; [a,b]) \leq 3M_1\, V_{\Lambda,2,p}(g; [a,b]) + M_2\, V_{\Lambda,2,p}(f; [g(a), g(b)]) < \infty,$$

with $M_2 = \sup\{Q_1(g;\beta,\alpha);\ \alpha,\beta \in [a,b]\}$, so we can conclude that

$$f \circ g \in \Lambda_p^2 BV([a,b]).$$

□

Lemma 2.14. *Suppose that f and K satisfy hypotheses $\widehat{H}_1$ and $\widehat{H}_2$, respectively. Let F be defined as $F(x)(t) = \int_I K(t,s)f(x(s))\mathrm{d}s$ for every $x \in \Lambda_p^2 BV(I)$, with $I = [0,b]$. Then*

$$V_{\Lambda,2,p}(F(x)) \leq b^{\frac{p-1}{p}} \|f\|_\infty \left(\int_I (M(s))^p \,\mathrm{d}s\right)^{\frac{1}{p}} < +\infty.$$

Proof. By Lemma 2.13, we have that $f \circ x \in \Lambda_p^2 BV([0,b])$. Even more, f is continuous on $[0,b]$, because f is Lipschitz, and hence f is Lebesgue integrable. Since $K(t,.)$ is Lebesgue integrable for every $t \in I$, we have that $K(t,\cdot)f(x(\cdot))$ is Lebesgue integrable for every $t \in I$. Thus, the function $F(x)$ is well defined. Let $\Lambda = \{\lambda_i\}_{i\geq 0}$ be a W-sequence, and $\pi = \{t_i\}_{i=0}^n \in \Pi_3([0,b])$ a partition, such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Let us study

$$V_{\Lambda,2,p}(F(x)) = \sup_\pi \left(\sum_{i=0}^{n-2} \frac{|Q_1(F(x);t_{i+2},t_{i+1}) - Q_1(F(x);t_{i+1},t_i)|^p}{\lambda_i}\right)^{1/p}.$$

Assume that

$$\begin{aligned}
A_i &= |Q_1(F(x);t_{i+2},t_{i+1}) - Q_1(F(x);t_{i+1},t_i)|^p \\
&= \left|\frac{F(x)(t_{i+2}) - F(x)(t_{i+1})}{t_{i+2}-t_{i+1}} - \frac{F(x)(t_{i+1}) - F(x)(t_i)}{t_{i+1}-t_i}\right|^p \\
&= \left|\int_I \left[\frac{K(t_{i+2},s) - K(t_{i+1},s)}{(t_{i+2}-t_{i+1})} - \frac{K(t_{i+1},s) - K(t_i,s)}{(t_{i+1}-t_i)}\right] f(x(s))\mathrm{d}s\right|^p \\
&\leq \sup_{s\in I} |f(x(s))|^p \left|\int_I \frac{K(t_{i+2},s) - K(t_{i+1},s)}{(t_{i+2}-t_{i+1})} - \frac{K(t_{i+1},s) - K(t_i,s)}{(t_{i+1}-t_i)}\mathrm{d}s\right|^p.
\end{aligned}$$

Now, since $p \geq 1$, and the function x^p is convex on $[0,+\infty)$, we can use the normalization and Jensen's inequality to get

$$A_i \leq b^{p-1} \sup_{s\in I} |f(x(s))|^p \int_I |Q_1(K(\cdot,s);t_{i+2},t_{i+1}) - Q_1(K(\cdot,s);t_{i+1},t_i)|^p \,\mathrm{d}s.$$

Thus

$$\begin{aligned}
&\sum_{i=0}^{n-2} \frac{A_i}{\lambda_i} \\
&\quad\leq \sum_{i=0}^{n-2} \frac{b^{p-1}\sup_{s\in I}|f(x(s))|^p \int_I |Q_1(K(\cdot,s);t_{i+2},t_{i+1}) - Q_1(K(\cdot,s);t_{i+1},t_i)|^p \,\mathrm{d}s}{\lambda_i} \\
&\quad= b^{p-1} \sup_{s\in I} |f(x(s))|^p \int_I \sum_{i=0}^{n-2} \frac{|Q_1(K(\cdot,s);t_{i+2},t_{i+1}) - Q_1(K(\cdot,s);t_{i+1},t_i)|^p}{\lambda_i}\mathrm{d}s.
\end{aligned}$$

Raising both sides of the inequality to the power $\dfrac{1}{p}$ yields

$$\left(\sum_{i=0}^{n-2}\frac{A_i}{\lambda_i}\right)^{\frac{1}{p}} \leq b^{\frac{p-1}{p}}\sup_{s\in I}|f(x(s))| \times\left(\int_I\sum_{i=0}^{n-2}\frac{|Q_1(K(\cdot,s);t_{i+2},t_{i+1})-Q_1(K(\cdot,s);t_{i+1},t_i)|^p}{\lambda_i}\mathrm{d}s\right)^{\frac{1}{p}},$$

taking supremum in the above inequality, and by Hypothesis $\widehat{H}_2$, it follows that $V_{\Lambda,2,p}(K(\cdot,s):I)\leq M(s)$, where M is L^p integrable, so we conclude that:

$$V_{\Lambda,2,p}(F(x))\leq b^{\frac{p-1}{p}}\|f\|_\infty\left(\int_I(M(s))^p\,\mathrm{d}s\right)^{\frac{1}{p}}<+\infty.$$

This completes the proof. □

Lemma 2.15. *Suppose that f and K satisfy hypotheses $\widehat{H}_1$ and $\widehat{H}_2$, respectively. Assume $F(x)$ is the integral function for every $x\in\Lambda_p^2BV(I)$, with $I=[0,b]$, defined as in the previous lemma. Then, for every $x,y\in\Lambda_p^2BV(I)$, the inequality*

$$V_{\Lambda,2,p}(\lambda\,(F(x)-F(y))) \leq b^{\frac{p-1}{p}}L_0^b(f)\,|\lambda|\,\|x-y\|_{\Lambda,2,p}\left(\int_I(M(s))^p\,\mathrm{d}s\right)^{\frac{1}{p}},\quad \lambda\in I,$$

holds, where $L_0^b(f)$ is the Lipschitz constant associated with f restricted to the interval I.

Proof. Let $x,y\in\Lambda_p^2BV(I)$, $\Lambda=\{\lambda_i\}_{i\geq 0}$ be a W-sequence, and $\pi=\{t_i\}_{i=0}^n\in\Pi_3([0,b])$ a partition, such that $0=t_0<t_1<\cdots<t_{n-1}<t_n=b$. By the definition, we have

$$V_{\Lambda,2,p}(\lambda\,(F(x)-F(y))) =\sup_\pi\left(\sum_{i=0}^{n-2}\frac{|Q_1(\lambda\,(F(x)-F(y))\,;t_{i+2},t_{i+1})-Q_1(\lambda\,(F(x)-F(y))\,;t_{i+1},t_i)|^p}{\lambda_i}\right)^{1/p}.$$

Thus

$$\begin{aligned}Q_1(\lambda\,(F(x)-F(y))\,;t_{i+2},t_{i+1}) &=\frac{\lambda\,(F(x)-F(y))\,(t_{i+2})-\lambda\,(F(x)-F(y))\,(t_{i+1})}{t_{i+2}-t_{i+1}}\\ &=\frac{\lambda\left[\int_I[K(t_{i+2},s)-K(t_{i+1},s)]\,f(x(s))\mathrm{d}s-\int_I[K(t_{i+2},s)-K(t_{i+1},s)]\,f(y(s))ds\right]}{t_{i+2}-t_{i+1}}\\ &=\frac{\lambda\left[\int_I[K(t_{i+2},s)-K(t_{i+1},s)]\,[f(x(s))-f(y(s))]\,\mathrm{d}s\right]}{t_{i+2}-t_{i+1}}.\end{aligned}\tag{2.1}$$

Analogously, we have that

$$Q_1(\lambda\,(F(x)-F(y))\,;t_{i+1},t_i) = \frac{\lambda\left[\int_I\left[K(t_{i+1},s)-K(t_i,s)\right]\left[f(x(s))-f(y(s))\right]\mathrm{d}s\right]}{t_{i+1}-t_i}. \tag{2.2}$$

We proceed now as in the proof of Lemma 2.14. By replacing (2.1) and (2.2), we have

$$\begin{aligned} B_i &= |Q_1(\lambda\,(F(x)-F(y))\,;t_{i+2},t_{i+1}) - Q_1(\lambda\,(F(x)-F(y))\,;t_{i+1},t_i)|^p \\ &= \left|\frac{\lambda\left[\int_I\left[K(t_{i+2},s)-K(t_{i+1},s)\right]\left[f(x(s))-f(y(s))\right]\mathrm{d}s\right]}{t_{i+2}-t_{i+1}}\right. \\ &\qquad \left. -\frac{\lambda\left[\int_I\left[K(t_{i+1},s)-K(t_i,s)\right]\left[f(x(s))-f(y(s))\right]\mathrm{d}s\right]}{t_{i+1}-t_i}\right|^p \\ &= \left|\frac{\lambda\int_I\, H(t_{i+2},t_{i+1},t_i,s)\left[f(x(s))-f(y(s))\right]\mathrm{d}s}{(t_{i+2}-t_{i+1})(t_{i+1}-t_i)}\right|^p, \end{aligned}$$

where

$$\begin{aligned} H(t_{i+2},t_{i+1},t_i,s) &= (t_{i+1}-t_i)\left[K(t_{i+2},s)-K(t_{i+1},s)\right] \\ &\quad -(t_{i+2}-t_{i+1})\left[K(t_{i+1},s)-K(t_i,s)\right]. \end{aligned}$$

By Lemma 2.10, f is Lipschitz, and thus

$$\begin{aligned} B_i &\le |\lambda|^p\left(L_0^b(f)\right)^p \sup_{s\in I}|x(s)-y(s)|^p \\ &\qquad \int_I\left|\frac{K(t_{i+2},s)-K(t_{i+1},s)}{(t_{i+2}-t_{i+1})}-\frac{K(t_{i+1},s)-K(t_i,s)}{(t_{i+1}-t_i)}\right|^p \mathrm{d}s \\ &= |\lambda|^p\left(L_0^b(f)\right)^p \sup_{s\in I}|x(s) \\ &\qquad -y(s)|^p\int_I|Q_1(K(\cdot,s);t_{i+2},t_{i+1})-Q_1(K(\cdot,s);t_{i+1},t_i)|^p\,\mathrm{d}s. \end{aligned}$$

Again, since $p\ge 1$, and x^p is convex on $[0,\infty)$, we can normalize and use the Jensen's inequality to have

$$\begin{aligned} &\sum_{i=0}^{n-2}\frac{|Q_1(\lambda\,(F(x)-F(y))\,;t_{i+2},t_{i+1})-Q_1(\lambda\,(F(x)-F(y))\,;t_{i+1},t_i)|^p}{\lambda_i} \\ &\le \sum_{i=0}^{n-2}\frac{|\lambda|^p\left(L_0^b(f)\right)^p\sup_{s\in I}|x(s)-y(s)|^p\int_I|Q_1(K(\cdot,s);t_{i+2},t_{i+1})-Q_1(K(\cdot,s);t_{i+1},t_i)|^p\,\mathrm{d}s}{\lambda_i} \\ &\le b^{p-1}\left(L_0^b(f)|\lambda|\sup_{s\in I}|x(s)-y(s)|\right)^p \\ &\quad\times\int_I\sum_{i=0}^{n-2}\frac{|Q_1(K(\cdot,s);t_{i+2},t_{i+1})-Q_1(K(\cdot,s);t_{i+1},t_i)|^p\,\mathrm{d}s}{\lambda_i}\mathrm{d}s. \end{aligned}$$

Raising to the power $\dfrac{1}{p}$, and by Hypothesis $\widehat{H}_2$, we get

$$\begin{aligned}&\left(\sum_{i=0}^{n-2}\frac{|Q_1(\lambda\left(F(x)-F(y)\right);t_{i+2},t_{i+1})-Q_1(\lambda\left(F(x)-F(y)\right);t_{i+1},t_i)|^p}{\lambda_i}\right)^{\frac{1}{p}}\\&\quad\leq b^{\frac{p-1}{p}}L_0^b(f)|\lambda|\left\|x-y\right\|_{\Lambda,2,p}\\&\qquad\times\left(\int_I\sum_{i=0}^{n-2}\frac{|Q_1(K(\cdot,s);t_{i+2},t_{i+1})-Q_1(K(\cdot,s);t_{i+1},t_i)|^p\,\mathrm{d}s}{\lambda_i}\mathrm{d}s\right)^{\frac{1}{p}}\\&\quad\leq b^{\frac{p-1}{p}}L_0^b(f)|\lambda|\left\|x-y\right\|_{\Lambda,2,p}\left(\int_I(M(s))^p\,\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

Taking supremum yields

$$V_{\Lambda,2,p}\left(\lambda(F(x)-F(y))\right)\leq b^{\frac{p-1}{p}}L_0^b(f)|\lambda|\left\|x-y\right\|_{\Lambda,2,p}\left(\int_I(M(s))^p\,\mathrm{d}s\right)^{\frac{1}{p}},$$

and the proof is complete. □

The following lemma is a special case of the triangle inequality. The proof is a consequence of the Lemmas 2.14 and 2.7.

Lemma 2.16. *Suppose that f and K satisfy hypotheses $\widehat{H}_1$ and $\widehat{H}_2$, respectively. Let $G:\Lambda_p^2BV(I)\to\Lambda_p^2BV(I)$ be defined by $G(x)(t):=g(t)+\lambda F(x)(t)$, where $F(x)$ is defined just as in Lemma 2.14, and $\lambda\in I=[0,b]$. Then*

$$\left\|G(x)\right\|_{\Lambda,2,p}\leq\left\|g\right\|_{\Lambda,2,p}+|\lambda|\left\|F(x)\right\|_{\Lambda,2,p}.$$

Lemma 2.17. *Let $T=\{(t,s):0\leq t\leq b,0\leq s\leq t\}$, and $K:T\to\mathbb{R}$ be a function, such that $K(\cdot,s)\in\Lambda_p^2BV([s,b])$. If $k(s,s)=0$ for every $s\in[0,b]$ or if there exists a function $L_1:[0,b]\to[0,+\infty)$, such that $\left|\frac{K(t,s)}{t-s}\right|\leq L_1(s)$ for every $s,t\in[0,b]$, with $t\neq s$, then for*

$$\widehat{K}(t,s)=\begin{cases}K(t,s), & 0\leq s\leq t\\ 0 & t<s\leq b;\end{cases}$$

we have that

$$\begin{aligned}&V_{\Lambda,2,p}\left(\widehat{K}(\cdot,s),[0,b]\right)\\&\quad\leq\left(\frac{(L_1(s))^p+\left(1+2^{p+1}\right)C(s)}{\lambda_0}\right)^{\frac{1}{p}}+V_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right),\end{aligned}\tag{2.3}$$

where $C(s)$ is guaranteed by the Lemma 2.12. In the case that $K(s,s)=0$, we consider $L_1(s)=0$ in (2.3).

Proof. Let $\Lambda=\{\lambda_i\}_{i\geq0}$ be a W-sequence, $s\in[0,b]$ and $\pi=\{t_i\}_{i=0}^n\in\Pi_3([0,b])$ be a partition, such that $0=t_0<t_1<\cdots<t_{n-1}<t_n=b$, then $s\in[t_r,t_{r+1}]$ for some r, with $0\leq r\leq n-1$. Thus

$$\begin{aligned}
&\sum_{i=0}^{n-2}\frac{\left|\frac{\widehat{K}(t_{i+2},s)-\widehat{K}(t_{i+1},s)}{t_{i+2}-t_{i+1}}-\frac{\widehat{K}(t_{i+1},s)-\widehat{K}(t_{i},s)}{t_{i+1}-t_{i}}\right|^p}{\lambda_i}\\
&=\frac{\left|\frac{\widehat{K}(t_{2},s)-\widehat{K}(t_{1},s)}{t_{2}-t_{1}}-\frac{\widehat{K}(t_{1},s)-\widehat{K}(t_{0},s)}{t_{1}-t_{0}}\right|^p}{\lambda_0}\\
&\quad+\frac{\left|\frac{\widehat{K}(t_{3},s)-\widehat{K}(t_{2},s)}{t_{3}-t_{2}}-\frac{\widehat{K}(t_{2},s)-\widehat{K}(t_{1},s)}{t_{2}-t_{1}}\right|^p}{\lambda_1}\\
&\quad+\cdots+\frac{\left|\frac{\widehat{K}(t_{r+1},s)-\widehat{K}(t_{r},s)}{t_{r+1}-t_{r}}-\frac{\widehat{K}(t_{r},s)-\widehat{K}(t_{r-1},s)}{t_{r}-t_{r-1}}\right|^p}{\lambda_{r-1}}\\
&\quad+\frac{\left|\frac{\widehat{K}(t_{r+2},s)-\widehat{K}(t_{r+1},s)}{t_{r+2}-t_{r+1}}-\frac{\widehat{K}(t_{r+1},s)-\widehat{K}(t_{r},s)}{t_{r+1}-t_{r}}\right|^p}{\lambda_r}\\
&\quad+\cdots+\frac{\left|\frac{\widehat{K}(t_{r+3},s)-\widehat{K}(t_{r+2},s)}{t_{r+3}-t_{r+2}}-\frac{\widehat{K}(t_{r+2},s)-\widehat{K}(t_{r+1},s)}{t_{r+2}-t_{r+1}}\right|^p}{\lambda_{r+1}}\\
&\quad+\cdots+\frac{\left|\frac{\widehat{K}(t_{n},s)-\widehat{K}(t_{n-1},s)}{t_{n}-t_{n-1}}-\frac{\widehat{K}(t_{n-1},s)-\widehat{K}(t_{n-2},s)}{t_{n-1}-t_{n-2}}\right|^p}{\lambda_{n-2}}\\
&=\frac{\left|\frac{K(t_{r+1},s)}{t_{r+1}-t_{r}}\right|^p}{\lambda_{r-1}}+\frac{\left|\frac{K(t_{r+2},s)-K(t_{r+1},s)}{t_{r+2}-t_{r+1}}-\frac{K(t_{r+1},s)}{t_{r+1}-t_{r}}\right|^p}{\lambda_r}\\
&\quad+\frac{\left|\frac{K(t_{r+3},s)-K(t_{r+2},s)}{t_{r+3}-t_{r+2}}-\frac{K(t_{r+2},s)-K(t_{r+1},s)}{t_{r+2}-t_{r+1}}\right|^p}{\lambda_{r+1}}\\
&\quad+\cdots+\frac{\left|\frac{K(t_{n},s)-K(t_{n-1},s)}{t_{n}-t_{n-1}}-\frac{K(t_{n-1},s)-K(t_{n-2},s)}{t_{n-1}-t_{n-2}}\right|^p}{\lambda_{n-2}}.
\end{aligned}$$

Now, from Lemma 2.12, there exists a function $C(s)$, such that

$$|Q_1(K(\cdot,s);\cdot,\cdot)|^p \leq C(s) \quad \text{on } [s,b]\times[s,b].$$

Hence, by the case $\left|\frac{K(t,s)}{t-s}\right| \leq L_1(s)$ for every $s,t\in[0,b]$, we have

$$\begin{aligned}
&\sum_{i=0}^{n-2}\frac{\left|\frac{\widehat{K}(t_{i+2},s)-\widehat{K}(t_{i+1},s)}{t_{i+2}-t_{i+1}}-\frac{\widehat{K}(t_{i+1},s)-\widehat{K}(t_{i},s)}{t_{i+1}-t_{i}}\right|^p}{\lambda_i}\\
&\leq \frac{(L_1(s))^p}{\lambda_0}+\frac{2^{p+1}C(s)}{\lambda_0}+V^p_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right)\\
&\leq \frac{(L_1(s))^p}{\lambda_0}+\frac{(1+2^{p+1})C(s)}{\lambda_0}+V^p_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right).
\end{aligned}$$

For the case in which $K(s,s)=0$ for every $s\in[0,b]$, we have

$$\sum_{i=0}^{n-2}\frac{\left|\frac{\widehat{K}(t_{i+2},s)-\widehat{K}(t_{i+1},s)}{t_{i+2}-t_{i+1}}-\frac{\widehat{K}(t_{i+1},s)-\widehat{K}(t_i,s)}{t_{i+1}-t_i}\right|^p}{\lambda_i}$$
$$\leq \frac{\left|\frac{K(t_{r+1},s)-K(s,s)}{t_{r+1}-s}\right|^p}{\lambda_0}+\frac{2^p\left|\frac{K(t_{r+2},s)-K(t_{r+1},s)}{t_{r+2}-t_{r+1}}\right|}{\lambda_0}+\frac{2^p\left|\frac{K(t_{r+1},s)-K(s,s)}{t_{r+1}-s}\right|^p}{\lambda_0}$$
$$+\frac{\left|\frac{K(t_{r+3},s)-K(t_{r+2},s)}{t_{r+3}-t_{r+2}}-\frac{K(t_{r+2},s)-K(t_{r+1},s)}{t_{r+2}-t_{r+1}}\right|^p}{\lambda_{r+1}}$$
$$+\cdots+\frac{\left|\frac{K(t_n,s)-K(t_{n-1},s)}{t_n-t_{n-1}}-\frac{K(t_{n-1},s)-K(t_{n-2},s)}{t_{n-1}-t_{n-2}}\right|^p}{\lambda_{n-2}}$$
$$\leq \frac{(1+2^{p+1})C(s)}{\lambda_0}+V^p_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right).$$

Therefore, in both cases, the inequality

$$\sum_{i=0}^{n-2}\frac{\left|\frac{\widehat{K}(t_{i+2},s)-\widehat{K}(t_{i+1},s)}{t_{i+2}-t_{i+1}}-\frac{\widehat{K}(t_{i+1},s)-\widehat{K}(t_i,s)}{t_{i+1}-t_i}\right|^p}{\lambda_i}$$
$$\leq \frac{(L_1(s))^p}{\lambda_0}+\frac{\left(1+2^{p+1}\right)C(s)}{\lambda_0}+V^p_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right),$$

holds. Raising to the power $\frac{1}{p}$, and taking supremum, we have

$$V_{\Lambda,2,p}\left(\widehat{K}(\cdot,s),[0,b]\right)$$
$$\leq\left(\frac{(L_1(s))^p}{\lambda_0}+\frac{\left(1+2^{p+1}\right)C(s)}{\lambda_0}+V^p_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right)\right)^{\frac{1}{p}}.$$

Since $p>1$, $(a+b)^{\frac{1}{p}}\leq a^{\frac{1}{p}}+b^{\frac{1}{p}}$ holds. Then

$$V_{\Lambda,2,p}\left(\widehat{K}(\cdot,s),[0,b]\right)$$
$$\leq\left(\frac{(L_1(s))^p+\left(1+2^{p+1}\right)C(s)}{\lambda_0}\right)^{\frac{1}{p}}+V_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right),$$

and this complete the proof. □

3. Main Theorems

In this section, the main theorems of this paper are proved, which guarantee the existence and uniqueness of continuous solutions of the Eqs. (1.1), (1.2), and (1.3) in the space of functions of $(\lambda,2,p)$-th bounded second variation, $\Lambda_p^2BV(I)$. In addition to the hypotheses considered in Sect. 2, we consider

$\widehat{H}_3$) Let $T = \{(t,s) : 0 \leq t \leq b, 0 \leq s \leq t\}$, $K : T \to \mathbb{R}$ be such that $K(t,\cdot) \in L^1([0,t])$, $\|K(t,\cdot)\|_1 \leq \widehat{C}$ for each $t \in [0,b]$, and $K(\cdot,s)$ satisfy the hypotheses of Lemma 2.17, such that

$$\left(\frac{(L_1(s))^p + \left(1+2^{p+1}\right)C(s)}{\lambda_0}\right)^{\frac{1}{p}} + V_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right) \leq m(s),$$

where $m : I \to \mathbb{R}$ is L^p integrable.

$\widehat{H}_4$) Let $T = \{(t,s) \in [a,b]\times[a,b] : s \leq t\}$, $K : T \to \mathbb{R}$ be such that $K(t,\cdot) \in L^1([a,t])$, $\|K(t,\cdot)\|_1 \leq \widehat{C}$ for each $t \in [a,b]$, and $K(\cdot,s)$ satisfy the hypotheses of Lemma 2.17 for every $s \in [a,b]$, such that

$$\left(\frac{(L_1(s))^p + \left(1+2^{p+1}\right)C(s)}{\lambda_0}\right)^{\frac{1}{p}} + V_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right) \leq h(s),$$

where $h : [a,b] \to \mathbb{R}$ is L^p integrable.

Theorem 3.1. *Suppose that f, g satisfy Hypothesis $\widehat{H}_1$, and that K satisfies Hypothesis $\widehat{H}_2$. Then, there exists a number $\tau > 0$ such that for every λ satisfying $|\lambda| < \tau$, the Eq. (1.1) has a unique solution in $\Lambda_p^2 BV(I)$, defined on $I = [0,b]$.*

Proof. Let $\Lambda = \{\lambda_i\}_{i\geq 0}$ be a W-sequence, and $\pi = \{t_i\}_{i=0}^n \in \Pi_3([0,b])$ a partition, such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Take $r > 0$, such that $\|g\|_{\Lambda,2,p} < r$. Choose a number $\tau > 0$, such that

$$\|g\|_{\Lambda,2,p} + \tau\|f\|_\infty \left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_I (M(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right] < r \quad \text{and} \tag{3.1}$$

$$\tau L_0^b(f)\left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_I (M(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right] < 1, \tag{3.2}$$

where $\widehat{C}$ is assured by Hypothesis $\widehat{H}_2$, and by Lemma 2.10, there exists the Lipschitz constant $L_0^b(f)$ restricted to the interval I. Define the function $G : \Lambda_p^2 BV(I) \to \Lambda_p^2 BV(I)$ by $G(x)(t) = g(t) + \lambda\int_I K(t,s) f(x(s))\mathrm{d}s$. For the proof, we use Theorem 2.1, the Banach Contraction Principle. For this purpose, we denote the closed ball of center 0 and radius r in the space $\Lambda_p^2 BV(I)$ by $\overline{B}_r$. We show first that $G(\overline{B}_r) \subset \overline{B}_r$. Indeed, for every $x \in \overline{B}_r$, it follows from Lemma 2.16 that:

$$\|G(x)\|_{\Lambda,2,p} \leq \|g\|_{\Lambda,2,p} + |\lambda|\,\|F(x)\|_{\Lambda,2,p}. \tag{3.3}$$

However

$$\begin{aligned}
\|F(x)\|_{\Lambda,2,p} &= \|F(x)\|_\infty + V_{\Lambda,2,p}(F(x)) \\
&= \sup_{t\in I}|F(x)(t)| + V_{\Lambda,2,p}(F(x)) \\
&= \sup_{t\in I}\left|\int_I K(t,s) f(x(s))\mathrm{d}s\right| + V_{\Lambda_p}(F(x)) \\
&\leq \|f\|_\infty \sup_{t\in I}\left(\int_I |K(t,s)|\,\mathrm{d}s\right) + V_{\Lambda,2,p}(F(x))
\end{aligned}$$

$$\leq \|f\|_\infty \widehat{C} + V_{\Lambda,2,p}\left(F(x)\right). \tag{3.4}$$

Thus, by Lemma 2.14, the inequality (3.4) yields

$$\|F(x)\|_{\Lambda,2,p} \leq \|f\|_\infty \left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_I (M(s))^p\, \mathrm{d}s\right)^{\frac{1}{p}}\right].$$

Replacing the above inequality in (3.3), and by the inequality (3.1), we obtain

$$\begin{aligned}\|G(x)\|_{\Lambda,2,p} &\leq \|g\|_{\Lambda,2,p} + |\lambda|\,\|f\|_\infty \left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_I (M(s))^p\, \mathrm{d}s\right)^{\frac{1}{p}}\right]\\ &\leq \|g\|_{\Lambda,2,p} + \tau\,\|f\|_\infty \left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_I (M(s))^p\, \mathrm{d}s\right)^{\frac{1}{p}}\right]\\ &< r.\end{aligned}$$

Thus, $G(\overline{B}_r) \subset \overline{B}_r$. Now, we prove that G is a contraction mapping. Take $x, y \in \overline{B}_r$, then

$$\begin{aligned}&\|G(x) - G(y)\|_{\Lambda,2,p} = \|G(x) - G(y)\|_\infty + V_{\Lambda_p}\left(G(x) - G(y)\right)\\ &= \left\|\lambda \int_I K(t,s)[f(x(s)) - f(y(s))]\mathrm{d}s\right\|_\infty + V_{\Lambda,2,p}\left(\lambda\left(F(x) - F(y)\right)\right)\\ &= \sup_{t\in I}\left|\lambda \int_I K(t,s)[f(x(s)) - f(y(s))]\mathrm{d}s\right| + V_{\Lambda,2,p}\left(\lambda\left(F(x) - F(y)\right)\right).\end{aligned}$$

By Lipschitz condition of f, hypothesis $\widehat{H}_2$), and Lemma 2.15 we have that

$$\begin{aligned}&\|G(x) - G(y)\|_{\Lambda,2,p}\\ &\quad\leq L_0^b(f)|\lambda|\,\|x-y\|_\infty \sup_{t\in I}\left(\int_I |K(t,s)|\,\mathrm{d}s\right)\\ &\qquad + b^{\frac{p-1}{p}} L_0^b(f)|\lambda|\,\|x-y\|_{\Lambda,2,p}\left(\int_I (M(s))^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\quad\leq \tau L_0^b(f)\left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_I (M(s))^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right]\|x-y\|_{\Lambda,2,p}.\end{aligned}$$

Therefore, by (3.2), G is a contraction mapping, and by Theorem 2.1, G has a unique fixed point in $\overline{B}_r$, which is to say that there exists a unique $x \in \overline{B}_r$, such that

$$g(t) + \lambda \int_I K(t,s) f(x(s))\mathrm{d}s = x(t).$$

Therefore, x is the only solution of the Eq. (1.1). □

Remark 3.2. By the previous theorem, the solution x of the Eq. (1.1) belongs to space $\Lambda_p^2 BV(I)$, this implies that x is continuous, by Lemma 2.10.

Theorem 3.3. *Suppose that g satisfies the Hypothesis $\widehat{H}_1$, K satisfies the Hypothesis $\widehat{H}_3$ and that f satisfies the Hypothesis $\widehat{H}_1$, such that*

$$L_0^b(f)\left[\widehat{C}+b^{\frac{p-1}{p}}\left(\int_0^b (m(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right]<1,$$

where $L_0^b(f)$ is the Lipschitz constant associated with f restricted to the interval $[0,b]$ and $\widehat{C}$ is assured by Hypothesis $\widehat{H}_3$. Then, the Eq. (1.2) has a unique solution in $\Lambda_p^2 BV$, defined on $I=[0,b]$.

Proof. Take $r>0$, such that

$$\|g\|_{\Lambda,2,p}+\|f\|_\infty\left[\widehat{C}+b^{\frac{p-1}{p}}\left(\int_0^b (m(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right]<r, \tag{3.5}$$

where $\widehat{C}$ is assured by Hypothesis $\widehat{H}_3$. Let $\Lambda=\{\lambda_i\}_{i\geq 0}$ be a W-sequence, and $\pi=\{t_i\}_{i=0}^n\in\Pi_3([0,b])$ a partition, such that $0=t_0<t_1<\cdots<t_{n-1}<t_n=b$. Define the functions $\widetilde{G}(x)(t)=g(t)+\widetilde{F}(x)(t)$, where $\widetilde{F}(x)(t)=\int_0^t K(t,s)f(x(s))\mathrm{d}s$ with $t\in[0,b]$, $x\in\Lambda_p^2 BV$, and

$$\widehat{K}(t,s)=\begin{cases}K(t,s), & 0\leq s\leq t\\ 0 & t<s\leq b.\end{cases} \tag{3.6}$$

To prove the theorem, we use the Banach Contraction Principle. We begin by showing that $\widetilde{G}(\overline{B}_r)\subset\overline{B}_r$. For all $x\in\overline{B}_r$ by Lemma 2.16, we have that

$$\left\|\widetilde{G}(x)\right\|_{\Lambda,2,p}\leq\|g\|_{\Lambda,2,p}+\left\|\widetilde{F}(x)\right\|_{\Lambda,2,p}, \tag{3.7}$$

but

$$\begin{aligned}\left\|\widetilde{F}(x)\right\|_{\Lambda,2,p}&=\|\widetilde{F}(x)\|_\infty+V_{\Lambda,2,p}(\widetilde{F}(x))\\&=\sup_{t\in I}\left|\int_0^t K(t,s)f(x(s))\mathrm{d}s\right|+V_{\Lambda,2,p}(\widetilde{F}(x))\\&\leq\|f\|_\infty\sup_{t\in I}\left(\int_0^t |K(t,s)|\mathrm{d}s\right)+V_{\Lambda,2,p}(\widetilde{F}(x))\\&\leq\widehat{C}\,\|f\|_\infty+V_{\Lambda,2,p}(\widetilde{F}(x)).\end{aligned} \tag{3.8}$$

To compute $V_{\Lambda,2,p}(\widetilde{F}(x))$, we apply a similar argument to that in the proof of Lemma 2.14

$$V_{\Lambda,2,p}(\widetilde{F}(x))=\sup_\pi\left(\sum_{i=0}^{n-2}\frac{|Q_1(\widetilde{F}(x);t_{i+2},t_{i+1})-Q_1(\widetilde{F}(x);t_{i+1},t_i)|^p}{\lambda_i}\right)^{1/p}.$$

Consider

$$\begin{aligned}A_i&=\left|Q_1(\widetilde{F}(x);t_{i+2},t_{i+1})-Q_1(\widetilde{F}(x);t_{i+1},t_i)\right|^p\\&=\left|\frac{\widetilde{F}(x)(t_{i+2})-\widetilde{F}(x)(t_{i+1})}{t_{i+2}-t_{i+1}}-\frac{\widetilde{F}(x)(t_{i+1})-\widetilde{F}(x)(t_i)}{t_{i+1}-t_i}\right|^p\end{aligned}$$

$$= \left| \frac{\int_0^{t_{i+2}} K(t_{i+2},s)f(x(s))\mathrm{d}s - \int_0^{t_{i+1}} K(t_{i+1},s)f(x(s))\mathrm{d}s}{t_{i+2}-t_{i+1}} \right.$$
$$\left. - \frac{\int_0^{t_{i+1}} K(t_{i+i},s)f(x(s))\mathrm{d}s - \int_0^{t_i} K(t_i,s)f(x(s))\mathrm{d}s}{t_{i+1}-t_i} \right|^p$$
$$= \left| \frac{\int_0^b \left[\widehat{K}(t_{i+2},s) - \widehat{K}(t_{i+1},s)\right] f(x(s))\mathrm{d}s}{t_{i+2}-t_{i+1}} \right.$$
$$\left. - \frac{\int_0^b \left[\widehat{K}(t_{i+1},s) - \widehat{K}(t_i,s)\right] f(x(s))\mathrm{d}s}{t_{i+1}-t_i} \right|^p$$
$$\leq \sup_{s\in I} |f(x(s))|^p \int_0^b \left| \frac{\widehat{K}(t_{i+2},s) - \widehat{K}(t_{i+1},s)}{t_{i+1}-t_{i+1})} - \frac{\widehat{K}(t_{i+1},s) - \widehat{K}(t_i,s)}{(t_{i+1}-t_i)} \right|^p \mathrm{d}s$$
$$= \sup_{s\in I} |f(x(s))|^p \int_0^b \left| Q_1(\widehat{K}(\cdot,s);t_{i+2},t_{i+1}) - Q_1(\widehat{K}(\cdot,s);t_{i+1},t_i) \right|^p \mathrm{d}s.$$

By normalizing and applying the Jensen's inequality, we obtain

$$A_i \leq b^{\frac{p-1}{p}} \sup_{s\in I} |f(x(s))|^p \int_0^b \left| Q_1(\widehat{K}(\cdot,s);t_{i+2},t_{i+1}) - Q_1(\widehat{K}(\cdot,s);t_{i+1},t_i) \right|^p \mathrm{d}s.$$

Now, proceeding as in the proof of Lemma 2.14, by Lemma 2.17, and Hypothesis $\widehat{H}_3$, we have

$$V_{\Lambda,2,p}(\widetilde{F}(x)) \leq b^{\frac{p-1}{p}} \|f\|_\infty \left(\int_0^b (m(s))^p \,\mathrm{d}s \right)^{\frac{1}{p}}. \tag{3.9}$$

Replacing (3.8) and (3.9) in the inequality (3.7) yields

$$\left\|\widetilde{G}(x)\right\|_{\Lambda,2,p} \leq \|g\|_{\Lambda,2,p} + \|f\|_\infty \left[\widehat{C} + b^{\frac{p-1}{p}} \left(\int_0^b (m(s))^p \mathrm{d}s \right)^{\frac{1}{p}} \right]$$
$$< r.$$

To show that $\widetilde{G}$ is a contraction mapping, we argue as in the proof of Theorem 3.1. For $x, y \in \overline{B}_r$, we have that

$$\left\|\widetilde{G}(x) - \widetilde{G}(y)\right\|_{\Lambda,2,p} = \left\|\widetilde{F}(x) - \widetilde{F}(y)\right\|_\infty + V_{\Lambda,2,p}\left(\widetilde{F}(x) - \widetilde{F}(y)\right)$$
$$\leq L_0^b(f) \left[\widehat{C} + b^{\frac{p-1}{p}} \left(\int_0^b (m(s))^p \,\mathrm{d}s \right)^{\frac{1}{p}} \right] \|x-y\|_{\Lambda,2,p}.$$

By hypothesis, we have that $\widetilde{G}$ is a contraction mapping, by Theorem 2.1, $\widetilde{G}$ has a unique fixed point in $\overline{B}_r$, which is to say that there exists a unique $\widetilde{x} \in \overline{B}_r$, such that

$$g(t)+\int_0^t K(t,s)f(\widetilde{x}(s))\mathrm{d}s=\widetilde{x}(t).$$

Therefore, $\widetilde{x}$ is a unique solution of Eq. (1.2). □

Theorem 3.4. *Suppose that g satisfies the Hypothesis $\widehat{H}_1$, K satisfies the Hypothesis $\widehat{H}_4$ and that f satisfies the Hypothesis $\widehat{H}_1$, such that*

$$L_a^b(f)\left[\widehat{C}+b^{\frac{p-1}{p}}\left(\int_a^b (m(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right]<1,$$

where $L_a^b(f)$ is the Lipschitz constant associated with f restricted to the interval $[a,b]$ and $\widehat{C}$ is assured by Hypothesis $\widehat{H}_4$. Then, the Eq. (1.3) has a unique solution in $\Lambda_p^2 BV$, defined on $I=[a,b]$.

Proof. The proof is analogous to the proof of the previous theorem, using Banach Contraction Principle. □

For a different proof, the following theorems are proved, using the alternative Leray–Schauder Theorem.

Theorem 3.5. *Suppose that g satisfies the Hypothesis $\widehat{H}_1$, K satisfies the Hypothesis $\widehat{H}_4$ and that f satisfies the Hypothesis $\widehat{H}_1$, such that*

$$L_a^b(f)\left[\widehat{C}+b^{\frac{p-1}{p}}\left(\int_a^b (m(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right]<1,$$

where $L_a^b(f)$ is the Lipschitz constant associated with f restricted to the interval $[a,b]$ and $\widehat{C}$ is assured by Hypothesis $\widehat{H}_4$. Then, there exists a solution $\widehat{x}\in\Lambda_p^2 BV$ for the Eq. (1.3).

Proof. The idea of the proof is to verify the hypothesis of Theorem 2.2, Leray–Schauder Alternative. To this end, we use an analogous argument to that of the proof of Theorem 3.3. Let $r>0$, such that

$$\|g\|_{\Lambda,2,p}+\|f\|_\infty\left[\widehat{C}+(b-a)^{\frac{p-1}{p}}\left(\int_a^b (h(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right]<r. \tag{3.10}$$

Take $\Lambda=\{\lambda_i\}_{i\geq 0}$ be a W-sequence, and $\pi=\{t_i\}_{i=0}^n\in\Pi_3([a,b])$ a partition, such that $a=t_0<t_1<\cdots<t_{n-1}<t_n=b$. Define the functions $H(x)(t)=g(t)+\widehat{F}(x)(t)$, where $\widehat{F}(x)(t)=\int_a^t K(t,s)f(x(s))\mathrm{d}s$ with $t\in[a,b]$, $x\in\Lambda_p^2BV$, and

$$\widehat{K}(t,s)=\begin{cases}K(t,s), & a\leq s\leq t\\ 0 & t<s\leq b;\end{cases} \tag{3.11}$$

by the technique used to prove Theorem 3.3, for all $x\in\overline{B}_r$, we can show that

$$\|H(x)\|_{\Lambda,2,p} \leq \|g\|_{\Lambda,2,p} + \|f\|_\infty \left[\widehat{C} + (b-a)^{\frac{p-1}{p}} \left(\int_a^b (h(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right] < r, \quad (3.12)$$

$$\|H(x)-H(y)\|_{\Lambda,2,p} \leq L_a^b(f) \left[\widehat{C} + (b-a)^{\frac{p-1}{p}} \left(\int_a^b (h(s))^p \,\mathrm{d}s\right)^{\frac{1}{p}}\right] \|x-y\|_{\Lambda,2,p}. \quad (3.13)$$

Define the function $\phi : [0,+\infty) \to [0,+\infty)$ as

$$\phi(z) = \left[\widehat{C} + (b-a)^{\frac{p-1}{p}} \left(\int_a^b (h(s))^p \,\mathrm{d}s\right)^{\frac{1}{p}}\right] z.$$

It is clear that $\phi(z) < z$, by the hypothesis. Now, let us assume that there is $x \in \overline{B}_r$, such that $x = \lambda H(x)$ for some $\lambda \in (0,1]$. We wish to show that x is an interior point of the ball. Indeed

$$\|x\|_{\Lambda,2,p} = \lambda \|H(x)\|_{\Lambda,2,p} \leq \|H(x)\|_{\Lambda,2,p}.$$

From the preceding expression and the inequality (3.12), it follows that $\|x\|_{\Lambda,2,p} < r$, so x is an interior point of the ball. On the other hand, $H(x)$ is in $\Lambda_p^2 BV$, because both g and $\widehat{F}(x)$ belong to the space, so $H(x)$ is bounded by Remark 2.9, and by Theorem 2.2, Leray–Shauder Alternative, H has a fixed point, which is to say that there exists $\widehat{x} \in \Lambda_p^2 BV$, such that $H(\widehat{x})(t) = \widehat{x}(t)$. Thus, $\widehat{x}$ is a solution of the Volterra Eq. (1.3). □

Theorem 3.6. *Suppose that g satisfies the Hypothesis $\widehat{H}_1$, K satisfies the Hypothesis $\widehat{H}_4$ and that f satisfies the Hypothesis $\widehat{H}_1$, such that*

$$L_a^b(f) \left[\widehat{C} + b^{\frac{p-1}{p}} \left(\int_a^b (m(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right] < 1, \quad \textit{and}$$

$$L_a^b(f) \left[2(b-a) \sup_{s\in[a,b]} |K(s,s)| + 4(b-a)^{2-\frac{1}{p}} \left(\int_a^b C(s)\mathrm{d}s\right)^{\frac{1}{p}} + 4(\lambda_0)^{\frac{1}{p}} (b-a)^{2-\frac{1}{p}} \left(\int_a^b (h(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right] < 1,$$

where $L_a^b(f)$ is the Lipschitz constant associated with f, $\widehat{C}$ is assured by Hypothesis $\widehat{H}_4$, $\sup_{s\in[a,b]} |K(s,s)| < +\infty$, and $C(s)$ is as in the Lemma 2.12. Then, there exists a unique solution $\widehat{x} \in \Lambda_p^2 BV$ of the equation (1.3), defined on $I = [a,b]$.

Proof. Let $\Lambda = \{\lambda_i\}_{i\geq 0}$ be a W-sequence, $s \in [a,b]$, $\pi = \{t_i\}_{i=0}^n \in \Pi_3([a,b])$ a partition, such that $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, then $s \in [t_i, t_{i+1}]$ for some i, $0 \leq i \leq n-2$. Theorem 3.5 guarantees the existence of a solution of the Volterra Eq. (1.3). So assume for the sake of contradiction that there exist two different solutions of the Eq. (1.3), say $\widehat{x}$, $\widehat{y}$. First, notice that

$$\begin{aligned}|\widehat{y}(t)-\widehat{x}(t)|^p &= \left|\int_a^t K(t,s)[f(\widehat{y}(s))-f(\widehat{x}(s))]\mathrm{d}s\right|^p\\ &\leq (t-a)^{p-1}\int_a^t |K(t,s)|^p\,|f(\widehat{y}(s))-f(\widehat{x}(s))|^p\,\mathrm{d}s\\ &\leq (b-a)^{p-1}\int_a^b |K(t,s)|^p\,(L_a^b(f))^p\,|\widehat{y}(s)-\widehat{x}(s)|^p\,\mathrm{d}s.\end{aligned}$$

Raising both sides of this inequality to the power $\frac{1}{p}$ yields

$$\begin{aligned}|\widehat{y}(t)-\widehat{x}(t)| &\leq (b-a)^{\frac{p-1}{p}}L_a^b(f)\left(\int_a^b |K(t,s)|^p\,|\widehat{y}(s)-\widehat{x}(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\leq (b-a)^{\frac{p-1}{p}}L_a^b(f)\|\widehat{y}-\widehat{x}\|_\infty\left(\int_a^b |K(t,s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

Taking supremum on $t\in[a,b]$

$$\|\widehat{y}-\widehat{x}\|_\infty \leq (b-a)^{\frac{p-1}{p}}L_a^b(f)\|\widehat{y}-\widehat{x}\|_\infty\sup_{t\in[a,b]}\left(\int_a^b |K(t,s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.$$

Then, it follows that:

$$1\leq (b-a)^{\frac{p-1}{p}}L_a^b(f)\sup_{t\in[a,b]}\left(\int_a^b |K(t,s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}. \tag{3.14}$$

On the other hand, notice that, for $\tau\in[s,b]$

$$\begin{aligned}&\frac{\left|Q_1(\widehat{K}(\cdot,s);b,\tau)-Q_1(\widehat{K}(\cdot,s);\tau,s)\right|^p}{\lambda_0}\\ &\quad\leq\sum_{i=0}^{n-2}\frac{\left|Q_1(\widehat{K}(\cdot,s);t_{i+2},t_{i+1})-Q_1(\widehat{K}(\cdot,s);t_{i+1},t_i)\right|^p}{\lambda_i}\\ &\quad\leq V^p_{\Lambda,2,p}\left(\widehat{K}(\cdot,s),[a,b]\right).\end{aligned}$$

Thus, by Hypothesis $\widehat{H}_4$, we obtain

$$\begin{aligned}&\left|Q_1(\widehat{K}(\cdot,s);\tau,s)-Q_1(\widehat{K}(\cdot,s);b,\tau)\right|\\ &\quad\leq(\lambda_0)^{\frac{1}{p}}V_{\Lambda,2,p}\left(\widehat{K}(\cdot,s),[a,b]\right)\\ &\quad\leq(\lambda_0)^{\frac{1}{p}}\left[\left(\frac{(L_1(s))^p+\left(1+2^{p+1}\right)C(s)}{\lambda_0}\right)^{\frac{1}{p}}+V_{\Lambda,2,p}\left(K(\cdot,s),[s,b]\right)\right]\\ &\quad\leq(\lambda_0)^{\frac{1}{p}}h(s).\end{aligned}$$

Then, by Lemma 2.12, we have

$$\left|Q_1(\widehat{K}(\cdot,s);\tau,s)\right| \le (C(s))^{\frac{1}{p}} + (\lambda_0)^{\frac{1}{p}} h(s).$$

Thus

$$\begin{aligned}\left|\widehat{K}(\tau,s)-\widehat{K}(s,s)\right| &\le \left[(C(s))^{\frac{1}{p}} + (\lambda_0)^{\frac{1}{p}} h(s)\right]|\tau-s| \\ &= (C(s))^{\frac{1}{p}}|\tau-s| + (\lambda_0)^{\frac{1}{p}}|\tau-s|h(s) \\ &\le (C(s))^{\frac{1}{p}}|b-a| + (\lambda_0)^{\frac{1}{p}}|b-a|h(s),\end{aligned}$$

and hence

$$|K(\tau,s)| \le |K(s,s)| + (C(s))^{\frac{1}{p}}|b-a| + (\lambda_0)^{\frac{1}{p}}|b-a|h(s).$$

Therefore, we have that

$$|K(\tau,s)|^p \le 2^p\,|K(s,s)|^p + C(s)\,(4|b-a|)^p + \lambda_0\,(4|b-a|h(s))^p\,.$$

Thus

$$\begin{aligned}&\int_a^t |K(t,s)|^p\,\mathrm{d}s \\ &\quad\le \int_a^t \sup_{\tau\in[s,b]} |K(\tau,s)|^p\,\mathrm{d}s \\ &\quad\le \int_a^b 2^p\left[\sup_{s\in[a,b]}|K(s,s)|^p + C(s)\,(2|b-a|)^p + \lambda_0 2^p|b-a|^p\,(h(s))^p\right]\mathrm{d}s \\ &\quad= 2^p(b-a)\sup_{s\in[a,b]}|K(s,s)|^p + 4^p(b-a)^p \\ &\qquad\times\int_a^b C(s)\mathrm{d}s + \lambda_0 4^p(b-a)^p\int_a^b (h(s))^p\mathrm{d}s.\end{aligned}$$

It follows that:

$$\begin{aligned}&(b-a)^{\frac{p-1}{p}}L_a^b(f)\sup_{t\in[a,b]}\left(\int_a^t |K(t,s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\ &\quad\le L_a^b(f)\left[2(b-a)\sup_{s\in[a,b]}|K(s,s)| + 4(b-a)^{2-\frac{1}{p}}\left(\int_a^b C(s)\mathrm{d}s\right)^{\frac{1}{p}}\right. \\ &\qquad\left. +4(\lambda_0)^{\frac{1}{p}}(b-a)^{2-\frac{1}{p}}\left(\int_a^b (h(s))^p\mathrm{d}s\right)^{\frac{1}{p}}\right].\end{aligned}$$

Therefore, from the hypothesis and the inequality (3.14), we get a contradiction. Therefore, it is guaranteed the uniqueness of the solution of the equation (1.3), which completes the proof. □

4. Application

In this section, we present an application problem, where the nonlinear Hammerstein–Volterra integral equation is solved by means of numerical methods. Here, it guaranteed the uniqueness and continuity of the solution in the space of functions of bounded second variation in the sense of Shiba.

Example. (Application) Since several problems in mathematics, physics, and chemistry are modeled by the integral Eqs. (1.1), (1.2), and (1.3), diverse numerical methods have been used to approximate solutions of nonlinear Hammerstein–Volterra integral equations, such as the successive approximation method introduced in [2], a collocation-type method developed in [4]. In [3], Brunner proposed a collocation-type method for the nonlinear integral Eq. (1.2), and discussed its connection with the iterated collocation method; in [8], it is studied an approximation by means of the fixed point method for the nonlinear Hammerstein–Volterra integral equation. Consider the Hammerstein–Volterra integral equation

$$x(t) = \tfrac{2}{15}t^6 - \tfrac{1}{3}t^4 + t^2 - 1 + \textstyle\int_0^t (t^2 - s^2)x^2(s)\mathrm{d}s, \quad \text{con } 0 \le t \le 1. \tag{4.1}$$

Let us show that there is a continuous unique solution in the space of bounded second variation functions in the Shiba sense, considering $0 \le t \le \frac{1}{2}$.

We verify the hypotheses of Theorem 3.3.

1. In [7], it is shown that $f(t) = t^2 \in \Lambda_p^2 BV([0, \frac{1}{2}])$.
2. In [6], it is proved that $\Lambda_p^2 BV([a, b])$ contains all affine functions, and in [7], it is proved in addition that $\Lambda_p^2 BV([a, b])$ is a Banach algebra. Thus, we have that $g(t) = \left(\dfrac{2}{15}t^6 - \dfrac{1}{3}t^4 + t^2 - 1\right) \in \Lambda_p^2 BV([0, \frac{1}{2}])$.
3. Define

$$\widehat{K}(t,s) = \begin{cases} K(t,s), & 0 \le s \le t \\ 0, & t < s \le \frac{1}{2}, \end{cases} \tag{4.2}$$

where $K(t, s) = t^2 - s^2$. It is clear that $K(t, \cdot)$ is Lebesgue integrable, and

$$\int_0^t |K(t,s)|\mathrm{d}s = \int_0^t |t^2 - s^2|\mathrm{d}s \le \int_0^t (t^2 + s^2)\mathrm{d}s = \frac{4}{3}t^3 \le \frac{1}{6} = \widehat{C}.$$

On the other hand, by Lemma 2.17, we have

$$V_{\Lambda,2,p}\left(\widehat{K}(\cdot, s), \left[0, \frac{1}{2}\right]\right) \le \left(\frac{(L_1(s))^p + \left(1 + 2^{p+1}\right)C(s)}{\lambda_0}\right)^{\frac{1}{p}} + V_{\Lambda,2,p}\left(K(\cdot, s), \left[s, \frac{1}{2}\right]\right).$$

To compute $V_{\Lambda,2,p}\left(K(\cdot,s),[s,\frac{1}{2}]\right)$, consider

$$\begin{aligned} B_i &= \frac{|Q_1(K(\cdot,s);t_{i+2},t_{i+1}) - Q_1(K(\cdot,s);t_{i+1},t_i)|^p}{\lambda_i} \\ &= \frac{\left|\frac{(t_{i+2}^2 - s^2) - (t_{i+1}^2 - s^2)}{t_{i+2}-t_{i+1}} - \frac{(t_{i+1}^2 - s^2) - (t_i^2 - s^2)}{t_{i+1}-t_i}\right|^p}{\lambda_i} \\ &= \frac{|t_{i+2}-t_i|^p}{\lambda_i}, \end{aligned}$$

so $V_{\Lambda,2,p}\left(K(\cdot,s),[s,\frac{1}{2}]\right) \le \dfrac{2\left(\frac{1}{2}-s\right)}{(\lambda_0)^{\frac{1}{p}}}$, See example 1 in [7]. However

$$|Q_1\left(K(\cdot,s);x,y\right)|^p = \left|\frac{K(x,s)-K(y,s)}{x-y}\right|^p = |x+y|^p \le 1,$$

$$\text{and} \quad \left|\frac{K(t,s)}{t-s}\right| = |t+s| \le \frac{1}{2} + s.$$

Then, $C(s) = 1$, and $L_1(s) = \frac{1}{2} + s$. Consequently

$$\begin{aligned} V_{\Lambda_p}\left(\widehat{K}(\cdot,s),[0,1]\right) &\le \frac{\left(\left(\frac{1}{2}+s\right)^p + (1+2^{p+1})\right)^{\frac{1}{p}} + 2\left(\frac{1}{2}-s\right)}{(\lambda_0)^{\frac{1}{p}}} \\ &\le \frac{\left(\frac{1}{2}+s\right) + \left(1+2^{p+1}\right)^{\frac{1}{p}} + 2\left(\frac{1}{2}-s\right)}{(\lambda_0)^{\frac{1}{p}}} = M(s). \end{aligned}$$

Therefore, it is evident that $M(s)$ is L^p integrable.

4. Considering $p = 1$ and for any W-sequence with $\lambda_0 > 3.75$, in particular for $\lambda_0 = 4$, we have

$$\int_0^{\frac{1}{2}} (M(s))^p \,\mathrm{d}s = \int_0^{\frac{1}{2}} \left(\frac{\left(\frac{1}{2}+s\right) + 5 + 2\left(\frac{1}{2}-s\right)}{4}\right) \mathrm{d}s = 0.78125;$$

therefore, $L_0^{\frac{1}{2}}(f)\left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_0^{\frac{1}{2}} (M(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right] = 1\left[\frac{1}{6} + 0.78125\right] = 0.94791 < 1$. Note that considering any other value of p, for example $p = 2$ and for any W-sequence with $\lambda_0 > 6.510024$, in particular for $\lambda_0 = 7$, we have

$$\int_0^{\frac{1}{2}} (M(s))^p \,\mathrm{d}s = \int_0^{\frac{1}{2}} \left(\frac{\left(\frac{1}{2}+s\right) + 3 + 2\left(\frac{1}{2}-s\right)}{(7)^{\frac{1}{2}}}\right)^2 \mathrm{d}s = 1.2917,$$

so $L_0^{\frac{1}{2}}(f)\left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_0^{\frac{1}{2}} (m(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right] = 1\left[\frac{1}{6} + \sqrt{\frac{1}{2}}\sqrt{1.2917}\right] = 0.97031 < 1$. In general, considering any value of $p \ge 1$ and a suitable λ_0, it is guaranteed that $L_0^{\frac{1}{2}}(f)\left[\widehat{C} + b^{\frac{p-1}{p}}\left(\int_0^{\frac{1}{2}} (M(s))^p \mathrm{d}s\right)^{\frac{1}{p}}\right] < 1$. Hence, all conditions of Theorem 3.3 are satisfied, and thus, the equation (4.1) has a unique continuous solution in $\Lambda_p^2 BV$, which is defined on $[0, \frac{1}{2}]$. □

5. Conclusion

In this paper, we proved existence-uniqueness theorems for nonlinear Hammerstein, Hammerstein–Volterra, and Volterra integral equations in the space of functions of bounded second variation in the sense of Shiba. In addition, we proved that solutions not only exist but are continuous. For the proofs of the theorems for nonlinear Hammerstein, and Hammerstein–Volterra equations, we used the Banach Contraction Principle and for nonlinear Volterra equation, we used the Leray–Schauder theorem. We also presented an application problem. We hope that the ideas and techniques used in this paper may be an inspiration to readers that are interested in studying these nonlinear integral equations in some new spaces of generalized bounded variation, and that these results may be also a contribution to different areas whose applications are modeled by this type of integral equations.

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