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Singular Quasilinear Schrödinger Equations with Exponential Growth in Dimension Two

Uberlandio B. Severo, Manassés de Souza and Diogo de S. Germano

Abstract. In this work, we study the existence of positive solution for the following class of singular quasilinear Schrödinger equations:

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \frac{f(u)}{|x|^a} \quad \text{in} \quad \mathbb{R}^2,$$

where $a \in (0, 2)$, $g : \mathbb{R} \to \mathbb{R}_+$ is a continuously differentiable function, $V(x)$ is a positive potential and the nonlinearity $f(u)$ can exhibit critical exponential growth. In order to prove our existence result, we combine minimax methods with a singular version of the Trudinger-Moser inequality.

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1. Introduction and Main Result

In this paper, we consider quasilinear Schrödinger equations of the form

$$-\,\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \frac{f(u)}{|x|^a} \quad \text{in} \quad \mathbb{R}^2, \tag{1.1}$$

where $a \in (0, 2)$, $g : \mathbb{R} \to \mathbb{R}_+$ is a continuously differentiable function, $V : \mathbb{R}^2 \to \mathbb{R}$ is a positive potential and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that can exhibit critical exponential growth in sense of the Trudinger–Moser inequality (see (1.8)).

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The study of equation (1.1) is related with the existence of solitary wave solutions for the nonlinear Schrödinger equation

$$i\partial_t w = -\Delta w + W(x)w - \tilde{p}(x, |w|^2)w - \Delta[\rho(|w|^2)]\rho'(|w|^2)w \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where $N \geq 1$, $w : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is the unknown function, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $\rho : \mathbb{R}_+ \to \mathbb{R}$ and $\tilde{p} : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ are real functions satisfying appropriate conditions. Equation (1.2) is called in the current literature as *Generalized Quasilinear Schrödinger Equation* and it has been accepted as model in many physical phenomena depending on the function ρ. For instance, if $\rho(s) = 1$ then we have the classical semilinear Schrödinger equation, see [25]. When $\rho(s) = s$, the equation arises from fluid mechanics, plasma physics and dissipative quantum mechanics, see [23,27,31]. For $\rho(s) = (1+s)^{1/2}$, (1.2) models the propagation of a high-irradiance laser in a plasma as well as the self-channeling of a high-power ultrashort laser in matter, see [24]. For further physical applications, we quote [3,32].

When we consider standing wave solutions for (1.2), that is, solutions of type $w(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function, we know that w satisfies (1.2) if and only if the function $u(x)$ solves the elliptic equation (see [8])

$$-\Delta u + V(x)u - \Delta[\rho(u^2)]\rho'(u^2)u = p(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

where $V(x) := W(x) - E$ and $p(x,u) := \tilde{p}(x, u^2)$. Now, if we take

$$g^2(u) = 1 + \frac{[(\rho(u^2))']^2}{2},$$

then (1.3) turns into quasilinear elliptic equation (see [33])

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = p(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.4}$$

which becomes (1.1) when $N = 2$ and $p(x,u) = f(u)/|x|^a$. For example, when we have $g^2(s) = 1 + 2s^2$, that is, $\rho(s) = s$, we obtain the superfluid film equation in plasma physics

$$-\Delta u + V(x)u - \Delta(u^2)u = p(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.5}$$

which has been extensively studied, see for example [7,18,29,32]. More generally, if we put $g^2(s) = 1 + 2\gamma^2(s^2)^{2\gamma-1}$, $\gamma > 1/2$, that corresponds to $\rho(s) = s^\gamma$, we get the equation

$$-\Delta u + V(x)u - \gamma\Delta(|u|^{2\gamma})|u|^{2\gamma-2}u = p(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.6}$$

which was addressed for instance in [1,14,28,37]. Now, if we consider $\rho(s) = (1+s)^{1/2}$, that is, $g^2(s) = 1 + s^2/[2(1+s^2)]$ we obtain

$$-\Delta u + V(x)u - \Delta[(1+u^2)^{1/2}]\frac{u}{2(1+u^2)^{1/2}} = p(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.7}$$

which was studied for instance in [6,10].

Motivated by these physical aspects, Eq. (1.4) has attracted a lot of attention of many researchers and some existence and multiplicity results have been obtained (see [5,10–13,19,26,33–36]). In this work, more specifically, we intend to prove that equation (1.1) admits at least one positive solution. To achieve this goal, we shall apply variational methods in combination with a version singular of the Trudinger-Moser inequality.

As in the papers [10,12,33], we assume the following assumptions on the function $g(s)$:

(g_0) $g \in C^1(\mathbb{R}, \mathbb{R}_+)$ is even, $g'(s) \geq 0$ for all $s \geq 0$ and $g(0) = 1$;
(g_1) there exists $\alpha \geq 1$ such that $(\alpha - 1)g(s) \geq g'(s)s$ for all $s \geq 0$;
(g_2) $\lim_{s\to+\infty} \frac{g(s)}{s^{\alpha-1}} =: \beta > 0$.

Typical examples satisfying (g_0)−(g_2) are given by the functions:

(a) $g(s) \equiv 1$ ($\alpha = 1$ and $\beta = 1$);
(b) $g(s) = (1 + 2s^2)^{1/2}$ ($\alpha = 2$ and $\beta = \sqrt{2}$);
(c) $g(s) = (1 + 2\gamma^2(s^2)^{2\gamma-1})^{1/2}$ ($\alpha = 2\gamma$ and $\beta = \sqrt{2}\gamma$),

which appear in the context of mathematical physics as indicated previously.

As it is known, the main difficulties in dealing with problem (1.4) is the lack of compactness, which is inherent to elliptic problems defined in unbounded domains and the fact that the energy functional associated to (1.4) is not generally well defined in the usual Sobolev space, because the presence of the integral $\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2$ (see more details in Sect. 2). Hence, a direct variational approach is not possible.

To the best of our knowledge, the first existence result for generalized quasilinear elliptic problem of the type (1.4) in unbounded domains involving variational methods was due to [33]. The authors have used a change of variables and the Mountain-Pass Theorem to obtain positive solutions for (1.4) when $p(x, u)$ is superlinear and has subcritical growth . Later on, by using change of variable, many authors proposed the critical problem when $p(x, u)$ is the form $|u|^{\alpha 2^*-2}u + f(u)$, see for instance [12,13]. In [12], by using the semilinear dual equation, the authors postulated that the number $\alpha 2^* = 2\alpha N/(N - 2)$ must be the critical exponent for an equation of type (1.4) in $\mathbb{R}^N (N \geq 3)$. In the paper [26], Li and Wu studied the existence, multiplicity and concentration of solutions for the critical case ($N \geq 3$).

In the subcritical case, through change of variable, the authors in [19] studied problem (1.4) by using Orlicz space framework and proved the existence of positive solutions via minimax methods. Moreover, they considered the nonlinearity $p(x, t)$ behaving like t at the origin and t^3 at infinity. Recently, by using the non-Nehari manifold method, Chen et al. in [5] proved that (1.4) admits a ground state solution under a monotonicity condition and some standard growth conditions on $p(x, u)$. In [10], Deng and Huang proved the existence of ground state solutions for (1.4) by using Jeanjean's monotonicity trick (see [21]).

Next, we assume that $V : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying the condition

(V) there exists a constant $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^2$.

Unlike the articles cited above, this is the only condition imposed on the potential V. Here, we do not need another condition on V in order to guarantee some compactness result. Instead we exploit the fact that the embedding

$$X := \left\{ v \in H^1(\mathbb{R}^2); \int_{\mathbb{R}^2} V(x)v^2 \mathrm{d}x < \infty \right\} \hookrightarrow L^p(\mathbb{R}^2, |x|^{-a}\mathrm{d}x)$$

is compact (see Section 2).

About the nonlinearity $f(u)$, we introduce the notion of criticality in dimension two for this class of problems. More precisely, we say that $f : \mathbb{R} \to \mathbb{R}$ has critical exponential growth at $+\infty$ if there exists $\varsigma_0 > 0$ such that

$$\lim_{s \to +\infty} f(s)e^{-\varsigma s^{2\alpha}} = \begin{cases} 0, & \text{for all } \varsigma > \varsigma_0, \\ +\infty, & \text{for all } \varsigma < \varsigma_0. \end{cases} \tag{1.8}$$

As far as we know, this is the first work dealing with this class of quasilinear Schrödinger equations and involving exponential critical growth with singularity. We point out that (1.8) extends the definition founded in the papers [15,17,30]. Since the exponent 2α can be bigger than 2, the growth (1.8) is better than the usual growth $e^{\varsigma s^2}$. This is possible due to the properties of the function $g(s)$. Moreover, we assume that f satisfies the following conditions:

(f_1) $f(s) = o(s)$ as $s \to 0^+$ and $f(s) = 0$, for all $s \in (-\infty, 0]$;

(f_2) there exist $\theta > \alpha$ such that

$$0 < 2\theta F(s) := 2\theta \int_0^s f(t)\mathrm{d}t \le sf(s), \quad \text{for all } s \in (0, +\infty);$$

(f_3) there exist constants $s_0, M_0 > 0$ such that

$$F(s) \le M_0 f(s), \quad \text{for all } \ s \ge s_0;$$

(f_4) there exists $\xi_0 > 0$ such that

$$\liminf_{s \to +\infty} sf(s)e^{-\varsigma_0 s^{2\alpha}} \ge \xi_0.$$

An elementary example of function satisfying (f_1) − (f_4) is given by $f(s) = F'(s)$, where $F(s) = s^{3\alpha}e^{s^{2\alpha}}$ for $s \ge 0$ and $F(s) = 0$ for $s < 0$, with constant $\varsigma_0 = 1$.

Now, let $C_0^\infty(\mathbb{R}^2)$ be the space of infinitely differentiable functions with compact support and $H^1(\mathbb{R}^2)$ the usual Sobolev space with the norm

$$\|u\|_{1,2} = \left[\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)\mathrm{d}x \right]^{1/2}.$$

In this context, we say that a function $u : \mathbb{R}^2 \to \mathbb{R}$ is a weak solution of problem (1.1) if $u \in H^1(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}^2)$ and for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ it holds

$$\int_{\mathbb{R}^2} g^2(u)\nabla u \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^2} g(u)g'(u)|\nabla u|^2 \varphi + \int_{\mathbb{R}^2} V(x)u\varphi \mathrm{d}x - \int_{\mathbb{R}^2} f(u)\varphi \mathrm{d}x = 0. \tag{1.9}$$

Now, we may state our main result.

Theorem 1.1. *Suppose that* (g_0)−(g_2), (V), (1.8) *and* (f_1)−(f_4) *are satisfied. Then, problem* (1.1) *has a positive solution.*

As already mentioned, the main difficulty in treating this class of Schrödinger equations in $\mathbb{R}^2$ is the possible lack of compactness as well as the critical exponential growth. Our result extends and improves the papers [14, 15, 17, 30] in the sense that we are considering a broader class of operators. In order, to prove Theorem 1.1, we use a change of variables and we transform equation (1.1) into a semilinear one. The functional energy, denoted by I, associated to this semilinear problem is well defined and it is differentiable in the subspace X of $H^1(\mathbb{R}^2)$ (for details see Sect. 2). Therefore, we justify that critical points of I provide weak solutions to problem (1.1).

The hypotheses (f_1) and (f_2) are sufficient conditions to guarantee the geometry of a suitable version of the Mountain-Pass Theorem (see Theorem 2.7). Moreover, (f_2) is important to prove that Cerami sequences are bounded (see Lemma 4.1). With respect to the hypothesis (f_4), it is fundamental to prove an estimate for the minimax level of I, see Proposition 5.1. Furthermore, (f_4) is more general than a similar condition found in [15], because here we do not require a lower bounded for the constant ξ_0. The hypothesis (f_3) is central for the proof of the convergence in Lemma 4.4. These last two results allows us to obtain the estimate

$$\left(\frac{\alpha}{\beta}\right)^2 \varsigma_0 \|\nabla v_n\|_2^2 < 4\pi$$

for n sufficiently large, where (v_n) is a Cerami sequence at the minimax level. This estimate is fundamental for applying Corollary 4.7 and consequently is used in the proof of Theorem 1.1. The conditions of the type (f_3) and (f_4) were considered in the pioneering work due to de Figueiredo et al. [9].
The outline of the paper is as follows: in the forthcoming section is the reformulation of the problem and some preliminary results, including the appropriate variational setting to study the quasilinear problem, the regularity of the dual energy functional and properties of its critical points. Moreover, we present the singular Trudinger-Moser inequality due to [4]. In Sect. 3, we prove that the energy functional satisfies the geometric conditions of Theorem 2.7. Section 4 is dedicated to the proof of some technical results involving the Cerami sequences associated to the energy functional. In Sect. 5, we derive an important estimate for the mountain pass level and Sect. 6 is devoted to the proof of the main result of the work.

2. Variational Setting and Preliminaries

We begin this section by defining the following subspace X of $H^1(\mathbb{R}^2)$:

$$X = \left\{ v \in H^1(\mathbb{R}^2); \int_{\mathbb{R}^2} V(x)v^2 \mathrm{d}x < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_X = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv)\mathrm{d}x, \quad u, v \in X \tag{2.1}$$

and its corresponding norm $\|v\|_X = \langle v, v\rangle^{1/2}$. It is clear that the hypothesis (V) implies the continuity of the embedding $X \hookrightarrow H^1(\mathbb{R}^2)$. Furthermore, by considering the weighted Lebesgue space

$$L^p(\mathbb{R}^2, |x|^{-a}\mathrm{d}x) = \left\{u : \mathbb{R}^2 \to \mathbb{R} : u \text{ is mensurable and } \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^a}\mathrm{d}x < \infty\right\},$$

we have the following compactness lemma:

Lemma 2.1. *Suppose $p \geq 2$ and $a \in (0, 2)$. Then, the embedding $X \hookrightarrow L^p(\mathbb{R}^2, |x|^{-a}\mathrm{d}x)$ is compact.*

Proof. See Theorem 1.2 in [38]. □

Now, we are going to introduce our variational structure. As observed in the Introduction, formally (1.1) is the Euler-Lagrange equation associated to the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^2} g^2(u)|\nabla u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(u)}{|x|^a}\mathrm{d}x. \tag{2.2}$$

The first difficulty that we have to deal with is to find an appropriate variational setting in order to apply variational methods to study the existence of critical points for J, because $g^2(u)|\nabla u|^2$ is not necessary in $L^1(\mathbb{R}^2)$ if $u \in H^1(\mathbb{R}^2)$. To overcome this difficulty, we follow ideas introduced in [33] (see also [13]), that is, we make use of the change of variables

$$v = G(u) = \int_0^u g(s)\mathrm{d}s.$$

Hence, after this change of variables, we obtain the new functional

$$I(v) = J(G^{-1}(u)) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla v|^2 + V(x)[G^{-1}(v)]^2)\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(G^{-1}(v))}{|x|^a}\mathrm{d}x, \tag{2.3}$$

which is well defined in the space X, under the conditions on g, V and f. For an easy reference, we list below the main properties of the function G^{-1}.

Lemma 2.2. *Under conditions $(g_0) - (g_2)$, we have the following properties:*

1. *G^{-1} is increasing; also G e G^{-1} are odd functions;*
2. *$0 < [G^{-1}(t)]' = \frac{1}{g(G^{-1}(t))} \leq 1 = \frac{1}{g(0)}$ for all $t \in \mathbb{R}$;*
3. *$|G^{-1}(t)| \leq |t|$ for all $t \in \mathbb{R}$;*
4. *$\frac{G^{-1}(t)}{\alpha} \leq \frac{t}{g(G^{-1}(t))} \leq G^{-1}(t)$ for all $t \geq 0$ and $\frac{[G^{-1}(t)]^2}{\alpha} \leq \frac{G^{-1}(t)t}{g(G^{-1}(t))} \leq [G^{-1}(t)]^2$ for all $t \in \mathbb{R}$;*
5. *$\frac{|G^{-1}(t)|^{\alpha-1}}{g(G^{-1}(t))} \leq \frac{1}{\beta}$ for all $t \in \mathbb{R}$;*
6. *$|G^{-1}(t)|^\alpha \leq \frac{\alpha}{\beta}|t|$ for all $t \in \mathbb{R}$;*
7. *$\frac{G^{-1}(t)}{t^{1/\alpha}} \to \left(\frac{\alpha}{\beta}\right)^{1/\alpha}$ as $t \to +\infty$;*
8. *there exists a positive constant C such that*

$$|G^{-1}(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{1/\alpha}, & |t| \geq 1. \end{cases}$$

Proof. The item (1) follows from the monotonicity of G and since g is even. To prove (2), just to derive the equality $G(G^{-1}(t)) = t$. For item (3), we use the Mean Value Theorem and (2) to conclude that $|G^{-1}(t)| = |G^{-1}(t) - G^{-1}(0)| = [G^{-1}(\xi)]'|t| \leq |t|$ for some ξ between 0 and t. Therefore this item is proved.

In order to show (4), consider $\sigma_1(t) := \alpha t - g(G^{-1}(t))G^{-1}(t)$ and $\sigma_2(t) := g(G^{-1}(t))G^{-1}(t) - t$. We have $\sigma_1(0) = \sigma_2(0) = 0$ and by $(g_0) - (g_1)$

$$\sigma_1'(t) = \alpha - 1 - \frac{g'(G^{-1}(t))G^{-1}(t)}{g(G^{-1}(t))} \geq 0 \quad \text{and} \quad \sigma_2'(t) = \frac{g'(G^{-1}(t))G^{-1}(t)}{g(G^{-1}(t))} \geq 0.$$

Thus, $\sigma_1(t) \geq 0$, $\sigma_2(t) \geq 0$ for all $t \geq 0$ and the first part is done. For the second part, just to observe that $G^{-1}(t)t \geq 0$ for all $t \in \mathbb{R}$.

Next, from $(g_0) - (g_2)$ we deduce that $g(s) \geq \beta |s|^{\alpha-1}$ for all $s \in \mathbb{R}$ and taking $s = G^{-1}(t)$ we obtain (5). From item (5) and using integration, the proof of item (6) follows.

Now, let us check (7). By the limit in (g_2), given $\varepsilon > 0$ there exists $R > 0$ such that $g(s) \leq 1 + \beta_\varepsilon s^{\alpha-1}$ for $s \geq R$, where $\beta_\varepsilon = \beta + \varepsilon$. By using (6), (g_0) and the Mean Value Theorem, for $t_0 \geq R$ we get

$$\begin{aligned} G^{-1}(t) - G^{-1}(t_0) &= \int_{t_0}^{t} \frac{1}{g(G^{-1}(s))} \mathrm{d}s \geq \int_{t_0}^{t} \frac{1}{g\left(\left(\frac{\alpha}{\beta}\right)^{1/\alpha} s^{1/\alpha}\right)} \mathrm{d}s \\ &\geq \int_{t_0}^{t} \frac{1}{1 + \beta_\varepsilon \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha-1}{\alpha}} s^{\frac{\alpha-1}{\alpha}}} \mathrm{d}s \\ &\geq \int_{t_0}^{t} \frac{1}{\beta_\varepsilon \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha-1}{\alpha}} s^{\frac{\alpha-1}{\alpha}}} \mathrm{d}s - \int_{t_0}^{t} \frac{1}{\beta_\varepsilon^2 \left(\frac{\alpha}{\beta}\right)^{\frac{2(\alpha-1)}{\alpha}} s^{\frac{2(\alpha-1)}{\alpha}}} \mathrm{d}s. \end{aligned}$$

If $\alpha > 2$ and by calculating the last two integrals, there exists a positive constant C_1 such that

$$\begin{aligned} G^{-1}(t) &\geq G^{-1}(t_0) - \frac{\alpha}{\beta_\varepsilon \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha-1}{\alpha}}} t_0^{1/\alpha} + \frac{\alpha}{\beta_\varepsilon^2 (\alpha - 2) \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha-1}{\alpha}}} \left(t_0^{\frac{2-\alpha}{\alpha}} - t^{\frac{2-\alpha}{\alpha}}\right) \\ &\quad + \frac{\alpha}{\beta_\varepsilon \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha-1}{\alpha}}} t^{1/\alpha} \\ &\geq -C_1 + \frac{\alpha}{\beta_\varepsilon \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha-1}{\alpha}}} t^{1/\alpha}. \end{aligned}$$

As $\beta_\varepsilon \to \beta$ when $\varepsilon \to 0^+$, we conclude that

$$\liminf_{t \to +\infty} \frac{G^{-1}(t)}{t^{1/\alpha}} \geq \left(\frac{\alpha}{\beta}\right)^{1/\alpha}.$$

Using again (6) we establish the desired limit for $\alpha > 2$. If $\alpha = 2$, for all $t > t_0 + 1 \geq R + 1$ there exists a positive constant C_2 satisfying

$$\begin{aligned} G^{-1}(t) &\geq G^{-1}(t_0) + \frac{2}{\beta_\varepsilon \left(\frac{2}{\beta}\right)^{1/2}}(t^{1/2} - t_0^{1/2}) - \frac{1}{\beta_\varepsilon^2 \left(\frac{2}{\beta}\right)} \int_{t_0}^{t} \frac{1}{s} \mathrm{d}s \\ &\geq -C_2 \log t + \frac{2}{\beta_\varepsilon \left(\frac{2}{\beta}\right)^{1/2}} t^{1/2}, \end{aligned}$$

from where we reach

$$\liminf_{t \to +\infty} \frac{G^{-1}(t)}{t^{1/2}} \geq \left(\frac{2}{\beta}\right)^{1/2}.$$

which is the desired limit. Finally, for $1 < \alpha < 2$ we have the estimate

$$G^{-1}(t) \geq -\left(\frac{\alpha}{\beta}\right)^{1/\alpha} t_0^{1/\alpha} - \frac{\alpha^{\frac{2-\alpha}{\alpha}}}{\beta^{2/\alpha}(2-\alpha)} t^{\frac{2-\alpha}{\alpha}} + \frac{\alpha}{\beta_\varepsilon \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha-1}{\alpha}}} t^{1/\alpha}$$

and similarly we get the result. To conclude, item (8) follows directly from (7). □

The next proposition presents an important compactness result.

Proposition 2.3. *Suppose that (V) is satisfied. Then, the map $v \to G^{-1}(v)$ from X into $L^p(\mathbb{R}^2, |x|^{-a}\mathrm{d}x)$ is compact for $2 \leq p < \infty$.*

Proof. Let $(v_n) \subset X$ be a bounded sequence in X. By Lemma 2.2-(2),(3) we have $\|G^{-1}(v_n)\| \leq \|v_n\|$. Thus, $(G^{-1}(v_n))$ is bounded in X and since the embedding $X \hookrightarrow L^p(\mathbb{R}^2, |x|^{-a}\mathrm{d}x)$ is compact for $2 \leq p < \infty$, up to a subsequence, there exists $w \in L^p(\mathbb{R}^2, |x|^{-a}\mathrm{d}x)$ such that $G^{-1}(v_n) \to w$ in $L^p(\mathbb{R}^2, |x|^{-a}\mathrm{d}x)$ and the proof is done. □

It is standard to see that under the assumptions on V, g and f, the functional I is of class C^1 on X with

$$I'(v)\varphi = \int_{\mathbb{R}^2} \left(\nabla v \nabla \varphi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi\right) \mathrm{d}x - \int_{\mathbb{R}^2} \frac{f(G^{-1}(v))}{g(G^{-1}(v))|x|^a}\varphi \mathrm{d}x, \tag{2.4}$$

for $v, \varphi \in X$ and therefore critical points of I turn out to be weak solutions of the semilinear equation

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{f(G^{-1}(v))}{g(G^{-1}(v))|x|^a} \quad \text{in} \quad \mathbb{R}^2. \tag{2.5}$$

We also observe that given $\varepsilon > 0$, $q \geq 1$ and $\varsigma > \varsigma_0$, by (f_1) and (1.8) there exists a constant $C_\varepsilon > 0$ satisfying

$$|f(s)| \leq \varepsilon|s| + C_\varepsilon |s|^{q-1}(e^{\varsigma s^{2\alpha}} - 1) \quad \text{for all} \quad s \in \mathbb{R}. \tag{2.6}$$

We will see in Proposition 2.6 that if $v \in H^1(\mathbb{R}^2)$ is a critical point of the functional I, then $u = G^{-1}(v)$ is a weak solution of (1.1). Therefore, to obtain weak solutions of (1.1), it will be sufficient to look for critical points of I.

At first, let us recall the following Trudinger-Moser inequality due to [16]:

Lemma 2.4. *If $\varsigma > 0$, $a \in (0,2)$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} \frac{(e^{\varsigma u^2} - 1)}{|x|^a} \mathrm{d}x < \infty. \tag{2.7}$$

Moreover, if $0 < \varsigma < 2\pi(2-a)$ and $\|u\|_2 \le M$, then there exists a positive constant $C = C(\varsigma, a, M)$, which depends only on M, a and ς, such that

$$\sup_{\|\nabla u\|_2 \le 1} \int_{\mathbb{R}^2} \frac{(e^{\varsigma u^2} - 1)}{|x|^a} \mathrm{d}x \le C. \tag{2.8}$$

In many arguments, we will need of the following lemma:

Lemma 2.5. *Let $\varsigma > 0$ and $r \ge 1$. Then*

$$(e^{\varsigma s^2} - 1)^r \le e^{r\varsigma s^2} - 1, \quad \text{for all } s \in \mathbb{R}.$$

Proof. Just analyze the limits of the function $\xi(s) = (e^{\varsigma s^2} - 1)^r/(e^{r\varsigma s^2} - 1)$ at the origin and at infinity applying the L'Hôpital rule. □

Proposition 2.6 (Critical points of I and solutions of (1.1)). *Every critical point v of I belongs to $C^{0,\vartheta}_{loc}(\mathbb{R}^2)$ for some $\vartheta \in (0,1)$ and $u = G^{-1}(v)$ is a weak solution of (1.1).*

Proof. Every critical point v of I satisfies the equation $-\Delta v = w$ in $\mathbb{R}^2$ in weak sense, where

$$w(x) = \frac{1}{g(G^{-1}(v))}\left[\frac{f(G^{-1}(v))}{|x|^a} - V(x)G^{-1}(v)\right].$$

From this, for $t > 1$, according to (2.6), (5) and (10) of Lemma 2.2, Lemma 2.5, for almost everywhere $x \in B_R \equiv B_R(0)$, we obtain

$$\begin{aligned} |w(x)|^t &\le \left[\frac{|G^{-1}(v)|}{g(G^{-1}(v))}\right]^t \left[\frac{C_1}{|x|^a} + \frac{C_2}{|x|^a}(e^{\varsigma[G^{-1}(v)]^{2\alpha}} - 1) + V(x)\right]^t \\ &\le C_3\left[\frac{1}{|x|^{at}} + \frac{1}{|x|^{at}}\left(e^{t\left(\frac{\alpha}{\beta}\right)^2\varsigma v^2} - 1\right) + M_R^t\right] \end{aligned}$$

where $M_R := \sup\{V(x) : x \in \overline{B_R}\}$. Now, considering $t > 1$ such that $0 < at < 2$ and using Lemma 2.4 we conclude that $w \in L^t(B_R)$. So, applying Schauder regularity theory, it follows that $v \in C^{0,\vartheta}_{loc}(\mathbb{R}^2)$ to some $\vartheta \in (0,1)$. In particular, $v \in L^\infty_{loc}(\mathbb{R}^2)$. The rest of the argument follows in a similar way to the proof of Proposition 2.9 in [14]. □

To conclude this section, we present a version of the Mountain-Pass Theorem, which is a consequence of the Ekeland Variational Principle as developed in [2]. We will also need to establish a local version of the same theorem.

Theorem 2.7. (Mountain-Pass Theorem) *Let X be a Banach space and $\Phi \in C^1(X;\mathbb{R})$ with $\Phi(0)=0$. Let $\mathcal{S}$ be a closed subset of X which disconnects (archwise) X. Let $v_0=0$ and $v_1 \in X$ be points belonging to distinct connected components of $X\backslash\mathcal{S}$. Suppose that*

$$\inf_{\mathcal{S}} \Phi \geq \sigma > 0 \quad \textit{and} \quad \Phi(v_1) \leq 0 \tag{2.9}$$

and let

$$\Gamma = \{\gamma \in C([0,1];X) : \gamma(0)=0 \quad \textit{and} \quad \gamma(1)=v_1\}. \tag{2.10}$$

Then

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)) \geq \sigma$$

and there exists a Cerami sequence[1] *for Φ at the level c. The number c is called the mountain-pass level of Φ.*

3. Geometric Properties

In this section, we are going to show that the functional I satisfies the geometric conditions (2.9). For this, we need to obtain some technical lemmas.

Lemma 3.1. *Assume that (V) and $(g_0)-(g_2)$ hold. If $v \in X$, $\varsigma > 0$, $t > 0$ and $\|v\|_2 \leq M$ with $\left(\frac{\alpha}{\beta}\right)^2 \varsigma\|\nabla v\|_2^2 < 2\pi(2-a)$, then there exists $C = C(a,\alpha,\varsigma,M,t) > 0$ such that*

$$\int_{\mathbb{R}^2} \frac{e^{\varsigma|G^{-1}(v)|^{2\alpha}}-1}{|x|^a}|G^{-1}(v)|^t \mathrm{d}x \leq C\|G^{-1}(v)\|^t.$$

Proof. Consider $r > 1$ close to 1 such that $\left(\frac{\alpha}{\beta}\right)^2 r\varsigma\|\nabla v\|_2^2 < 2\pi(2-ar)$, $ar < 2$ and $ts \geq 2$, where $s = r/(r-1)$. Using (5) of Lemma 2.2 and Holder's inequality, we have

$$\int_{\mathbb{R}^2} \frac{e^{\varsigma|G^{-1}(v)|^{2\alpha}}-1}{|x|^a}|G^{-1}(v)|^t \mathrm{d}x \leq \left[\int_{\mathbb{R}^2} \frac{(e^{\left(\frac{\alpha}{\beta}\right)^2\varsigma v^2}-1)^r}{|x|^{ar}}\mathrm{d}x\right]^{1/r} \|G^{-1}(v)\|_{ts}^t$$

and by Lemmas 2.4, 2.5 and the continuous embedding $H^1(\mathbb{R}^2) \hookrightarrow L^{ts}(\mathbb{R}^2)$, we conclude

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{e^{\varsigma|G^{-1}(v)|^{2\alpha}}-1}{|x|^a}|G^{-1}(v)|^t \mathrm{d}x &\leq \left[\int_{\mathbb{R}^2} \frac{e^{\left(\frac{\alpha}{\beta}\right)^2 r\varsigma\|\nabla v\|_2^2\left(\frac{v}{\|\nabla v\|_2}\right)^2}-1}{|x|^{ar}}\mathrm{d}x\right]^{\frac{1}{r}} \|G^{-1}(v)\|_{ts}^t \\ &\leq C_1\|G^{-1}(v)\|_{ts}^t \leq C\|G^{-1}(v)\|^t,\end{aligned}$$

which proves the lemma. □

[1] (v_n) such that $\Phi(v_n) \to c$ and $\|\Phi'(v_n)\|(1+\|v_n\|)$.

Lemma 3.2. *Assume that (V) holds. If* $v \in H^1(\mathbb{R}^2)$ *and* $t \geq 2$*, then there exists* $C = C(t) > 0$ *such that*

$$\int_{\mathbb{R}^2} \frac{|G^{-1}(v)|^t}{|x|^a} \mathrm{d}x \leq C\|G^{-1}(v)\|^t.$$

Proof. Let $r > 1$ be close to 1 such that $ar < 2$ and $s = r/(r-1)$. Using Hölder's inequality and the continuous embedding $X \hookrightarrow L^q(\mathbb{R}^2)$ for all $2 \leq q < \infty$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|G^{-1}(v)|^t}{|x|^a} \mathrm{d}x &\leq \int_{|x|>1} |G^{-1}(v)|^t \mathrm{d}x + \left(\int_{|x|\leq 1} \frac{1}{|x|^{ar}} \mathrm{d}x\right)^{1/r} \left(\int_{|x|\leq 1} |G^{-1}(v)|^{ts} \mathrm{d}x\right)^{1/s} \\ &\leq \|G^{-1}(v)\|_t^t + C_1\|G^{-1}(v)\|_{ts}^t \\ &\leq C\|G^{-1}(v)\|^t\end{aligned}$$

and the proof follows. □

In view of the last estimates, we can prove that the functional I has the mountain-pass geometry. For this purpose, for $\rho > 0$, we define

$$S_\rho = \left\{ v \in X : \int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}x + \int_{\mathbb{R}^2} V(x)[G^{-1}(v)]^2 \mathrm{d}x = \rho^2 \right\}.$$

Since $Q : X \to \mathbb{R}$, defined by

$$Q(v) = \int_{\mathbb{R}^2} \{|\nabla v|^2 + V(x)[G^{-1}(v)]^2\} \mathrm{d}x,$$

is a continuous function, it follows that S_ρ is a closed subset that disconnects the space X.

Lemma 3.3. *Suppose that* (V), (g_0) *and* (f_1) *are satisfied. Then, there exist* $\rho > 0$ *and* $\sigma > 0$ *satisfying*

$$I(v) \geq \sigma, \quad \textit{for all} \quad v \in S_\rho.$$

Proof. From the estimate (2.6), given $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|F(s)| \leq \frac{\varepsilon}{2} s^2 + C_\varepsilon |s|^t (e^{\varsigma s^{2\alpha}} - 1), \quad \text{for all} \quad s \in \mathbb{R},\ t > 2. \tag{3.1}$$

Now, if $\left(\frac{\alpha}{\beta}\right)^2 \varsigma\rho^2 < 2\pi(2-a)$, by using (3.1), Lemma 3.1, Lemma 3.2, Lemma 2.2-(2) and the continuous embedding $H^1(\mathbb{R}^2) \hookrightarrow L^t(\mathbb{R}^2)$, we obtain

$$\begin{aligned} I(v) &\geq \frac{1}{2}Q(v) - \frac{\varepsilon}{2}C\|G^{-1}(v)\|^2 - C_1\|G^{-1}(v)\|^t \\ &\geq \left(\frac{1}{2} - \frac{\varepsilon}{2}C\right) Q(v) - C_1 Q(v)^{t/2}. \end{aligned}$$

Taking $0 < \varepsilon < 1/C$ and since $t > 2$, we may choose $0 < \rho < \frac{\beta}{\alpha}\left(\frac{2\pi(2-a)}{\varsigma}\right)^{1/2}$ such that $\left(\frac{1}{2} - \frac{\varepsilon}{2}C\right)\rho^2 - C_1\rho^t > 0$. Thus, considering $\sigma = \left(\frac{1}{2} - \frac{\varepsilon}{2}C\right)\rho^2 - C_1\rho^t > 0$ we conclude $I(v) \geq \sigma$ for all $v \in S_\rho$. □

Lemma 3.4. *Suppose that* (V), $(g_0)-(g_2)$ *and* (f_2) *are satisfied. Then, there exists* $e \in X$ *such that* $Q(e) > \rho^2$ *and*

$$I(e) < 0 < \sigma \leq \inf_{v \in S_\rho} I(v).$$

Proof. First, consider $\varphi \in C_0^\infty(\mathbb{R}^2, [0,1])\backslash\{0\}$ such that $\operatorname{supp}(\varphi) = \overline{B_1}$. From (f_2), there are positive constants C_1 and C_2 such that $F(s) \geq C_1|s|^{2\theta} - C_2$ for all $s \in \mathbb{R}$. Thus, for $t > 0$ we have

$$\begin{aligned} I(t\varphi) &= \frac{1}{2}\int_{\overline{B_1}}(|\nabla(t\varphi)|^2 + V(x)[G^{-1}(t\varphi)]^2)\mathrm{d}x - \int_{\overline{B_1}} \frac{F(G^{-1}(t\varphi))}{|x|^a}\mathrm{d}x \\ &\leq \frac{t^2}{2}\int_{\overline{B_1}}(|\nabla\varphi|^2 + V(x)\varphi^2)\mathrm{d}x - C_1\int_{\overline{B_1}} \frac{|G^{-1}(t\varphi)|^{2\theta}}{|x|^a}\mathrm{d}x + C_2\int_{\overline{B_1}}\frac{1}{|x|^a}\mathrm{d}x \\ &\leq t^2\left[\frac{\|\varphi\|^2}{2} - C_1\int_{\overline{B_1}}\frac{|G^{-1}(t\varphi)|^{2\theta}}{t^2|x|^a}\mathrm{d}x + \frac{C_2}{t^2}\int_{\overline{B_1}}\frac{1}{|x|^a}\mathrm{d}x\right]. \end{aligned}$$

Since $2\theta - 2\alpha > 0$, for $x \in \overline{B_1}$, by using Lemma 2.2-(7), it follows that

$$\frac{|G^{-1}(t\varphi(x))|^{2\theta}}{t^2} = \left(\frac{G^{-1}(t\varphi(x))}{\sqrt[\alpha]{t\varphi(x)}}\right)^{2\alpha}|G^{-1}(t\varphi(x))|^{2\theta-2\alpha}\varphi(x)^2 \to +\infty \quad \text{as} \quad t \to +\infty.$$

Thus, according to Fatou's Lemma, we obtain

$$\int_{\overline{B_1}}\frac{|G^{-1}(t\varphi)|^{2\theta}}{t^2|x|^a}\mathrm{d}x \to +\infty \quad \text{as} \quad t \to +\infty.$$

and therefore $I(t\varphi) \to -\infty$. Setting $e := t\varphi$ with t large enough, the proof is finished. □

4. On Cerami Sequences for I

The purpose of this section is to prove some results about the Cerami sequences for the functional I. The first one is the following:

Lemma 4.1. *Suppose that* (V), $(g_0)-(g_1)$ *and* (f_2) *are satisfied. Let* (v_n) *be in* X *such that* $I(v_n) \to c \in \mathbb{R}$ *and* $I'(v_n)v_n \to 0$ *as* $n \to +\infty$. *Then,* $Q(v_n)$ *is bounded and* (v_n) *is bounded in* $H^1(\mathbb{R}^2)$.

Proof. Using Lemma 2.2-(4) and (f_2), we obtain

$$\begin{aligned} I(v_n) - \frac{\alpha}{2\theta}I'(v_n)v_n &= \left(\frac{1}{2} - \frac{\alpha}{2\theta}\right)\int_{\mathbb{R}^2}|\nabla v_n|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2}V(x)[G^{-1}(v_n)]^2\mathrm{d}x \\ &\quad - \frac{\alpha}{2\theta}\int_{\mathbb{R}^2}V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\mathrm{d}x - \int_{\mathbb{R}^2}\frac{F(G^{-1}(v_n))}{|x|^a}\mathrm{d}x \\ &\quad + \frac{\alpha}{2\theta}\int_{\mathbb{R}^2}\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))|x|^a}v_n\mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{\alpha}{2\theta}\right)Q(v_n) \\ &\quad + \frac{1}{2\theta}\int_{\{G^{-1}(v_n)>0\}}\frac{f(G^{-1}(v_n))G^{-1}(v_n) - 2\theta F(G^{-1}(v_n))}{|x|^a}\mathrm{d}x \end{aligned}$$

$$\geq \left(\frac{1}{2} - \frac{\alpha}{2\theta}\right) Q(v_n).$$

Since $I(v_n) = c + o_n(1)$ and $I'(v_n)v_n = o_n(1)$, as $n \to +\infty$, it follows that

$$\left(\frac{1}{2} - \frac{\alpha}{2\theta}\right) Q(v_n) \leq c + o_n(1). \tag{4.1}$$

Now, since $\theta > \alpha$, for some constant $C > 0$ we have

$$Q(v_n) = \int_{\mathbb{R}^2} \{|\nabla v_n|^2 + V(x)[G^{-1}(v_n)]^2\}\mathrm{d}x \leq C. \tag{4.2}$$

In view of (4.1), it remains to show that $\int_{\mathbb{R}^2} v_n^2 \mathrm{d}x$ is bounded. By condition (V) and Lemma 2.2-(8) there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^2} v_n^2 \mathrm{d}x &= \int_{\{|v_n|\leq 1\}} v_n^2 \mathrm{d}x + \int_{\{|v_n|>1\}} v_n^2 \mathrm{d}x \\ &\leq \frac{1}{C_1^2 V_0} \int_{\mathbb{R}^2} V(x)[G^{-1}(v_n)]^2 \mathrm{d}x + \frac{1}{C_1^{2\alpha}} \int_{\mathbb{R}^2} [G^{-1}(v_n)]^{2\alpha} \mathrm{d}x. \end{aligned} \tag{4.3}$$

Next, we will use the Gagliardo-Nirenberg inequality (see [22], p. 31), which asserts

$$\|u\|_q \leq C(\vartheta)\|u\|_r^{1-\vartheta}\|\nabla u\|_2^{\vartheta} \tag{4.4}$$

for all $u \in H^1(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)$, where $1 \leq r < \infty$, $0 < \vartheta \leq 1$ and $\frac{1}{q} = \frac{1-\vartheta}{r}$. Setting $u = G^{-1}(v_n)$, $\vartheta = 1 - \frac{1}{\alpha}$ and $r = 2$, we have $q = 2\alpha$. Hence, by using (V) and (4.4), we get

$$\int_{\mathbb{R}^2} |G^{-1}(v_n)|^{2\alpha} \mathrm{d}x \leq \frac{C(\vartheta)^{2\alpha}}{V_0} \left(\int_{\mathbb{R}^2} V(x)[G^{-1}(v_n)]^2 \mathrm{d}x\right) \left(\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}x\right)^{\alpha-1}. \tag{4.5}$$

From (4.2), (4.3) and (4.5), it follows that $\int_{\mathbb{R}^2} v_n^2 \mathrm{d}x$ is bounded and the lemma is proved. □

Corollary 4.2. *Suppose that (V), $(g_0) - (g_1)$ and (f_2) are satisfied. Let (v_n) be a Cerami sequence for I in X. Then, there exists $C > 0$ such that*

$$\int_{\mathbb{R}^2} \frac{|f(G^{-1}(v_n))v_n|}{g(G^{-1}(v_n))|x|^a} \mathrm{d}x \leq C.$$

Proof. By Lemma 2.2-(4) and since $I'(v_n)v_n \to 0$ as $n \to +\infty$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{f(G^{-1}(v_n))v_n}{g(G^{-1}(v_n))|x|^a} \mathrm{d}x &\leq \int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}x + \int_{\mathbb{R}^2} V(x)[G^{-1}(v_n)]^2 \mathrm{d}x + o_n(1) \\ &\leq Q(v_n) + o_n(1). \end{aligned}$$

By the previous lemma, $Q(v_n)$ is bounded and the above estimate shows the result. □

Lemma 4.3. *Suppose that (V), $(g_0) - (g_1)$ and $(f_1) - (f_2)$ are satisfied. Let (v_n) be a Cerami sequence for I. Then, (v_n) has a subsequence, still denoted* *by (v_n), such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} V(x)|G^{-1}(v)|^2 \mathrm{d}x < \infty$ and*

$$\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))|x|^a} \to \frac{f(G^{-1}(v))}{g(G^{-1}(v))|x|^a} \quad \textit{in} \quad L^1_{loc}(\mathbb{R}^2), \quad \textit{as } n \to +\infty.$$

Proof. According to Lemma 4.1, (v_n) is bounded in $H^1(\mathbb{R}^2)$. Thus, up to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$. Furthermore, the function v satisfies $\int_{\mathbb{R}^2} V(x)|G^{-1}(v)|^2 \mathrm{d}x < \infty$, because $Q(v_n)$ is bounded and by Fatou's Lemma

$$\int_{\mathbb{R}^2} V(x)|G^{-1}(v)|^2 \mathrm{d}x \le \liminf_{n\to+\infty} \int_{\mathbb{R}^2} V(x)|G^{-1}(v_n)|^2 \mathrm{d}x \le C.$$

Now, it is sufficient to prove that

$$\int_{B_R} \frac{|f(G^{-1}(v_n))|}{g(G^{-1}(v_n))|x|^a} \mathrm{d}x \to \int_{B_R} \frac{|f(G^{-1}(v))|}{g(G^{-1}(v))|x|^a} \mathrm{d}x, \quad \text{as } n \to +\infty.$$

By using Lemma 4.1, Lemma 2.2-(3) and since the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^t_{loc}(\mathbb{R}^2)$, for all $t \ge 1$, is compact, we can assume that $G^{-1}(v_n) \to G^{-1}(v)$ strongly in $L^t(B_R)$ for any $t \in [1, +\infty)$. Moreover, by using items (2) and (3) of Lemma 2.2, Lemma 2.4, Corollary 4.2, estimate (2.6) and Holder's inequality, we obtain

$$|G^{-1}(v)| \in L^1(B_R), \quad \frac{f(G^{-1}(v))}{g(G^{-1}(v))|x|^a} \in L^1(B_R) \;\text{ and }\; \int_{\mathbb{R}^2} \frac{|f(G^{-1}(v_n))v_n|}{g(G^{-1}(v_n))|x|^a} \le C.$$

The rest of the argument follows the same steps as in the proof of Lemma 4.3 in [14]. □

Lemma 4.4. *Suppose that (V), $(g_0)-(g_1)$ and $(f_1)-(f_3)$ are satisfied. Let (v_n) be a Cerami sequence for I in X. Then, (v_n) has a subsequence, still denoted by (v_n), such that*

$$\frac{F(G^{-1}(v_n))}{|x|^a} \to \frac{F(G^{-1}(v))}{|x|^a} \quad \textit{in} \quad L^1(\mathbb{R}^2), \quad \textit{as} \quad n \to +\infty,$$

where v is the weak limit of (v_n) in $H^1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} V(x)|G^{-1}(v)|^2 \mathrm{d}x < \infty$.

Proof. From Lemma 2.2-(4) and Corollary 4.2 we have

$$\frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{|f(G^{-1}(v_n))G^{-1}(v_n)|}{|x|^a} \mathrm{d}x \le \int_{\mathbb{R}^2} \frac{|f(G^{-1}(v_n))v_n|}{g(G^{-1}(v_n))|x|^a} \mathrm{d}x \le C.$$

Thus, similarly to Lemma 4.3, we get

$$\frac{f(G^{-1}(v_n))}{|x|^a} \to \frac{f(G^{-1}(v))}{|x|^a} \quad \text{in} \quad L^1_{loc}(\mathbb{R}^2), \quad \text{as} \quad n \to +\infty. \tag{4.6}$$

Next, by using (f_2) and (f_3), for each $R > 0$, there exists $C > 0$ such that $F(G^{-1}(v_n)) \le C[f(G^{-1}(v_n))]$ in $\overline{B_R}$. This together with (4.6) and the generalized Lebesgue dominated convergence theorem, up to a subsequence, implies that

$$\frac{F(G^{-1}(v_n))}{|x|^a} \to \frac{F(G^{-1}(v))}{|x|^a} \quad \text{in} \quad L^1(B_R), \quad \text{for all } R > 0.$$

To conclude the convergence of the lemma, it is sufficient to prove that given $\delta > 0$, there exists $R > 0$ such that

$$\int_{B_R^c} \frac{F(G^{-1}(v_n))}{|x|^a} \mathrm{d}x \leq \delta \quad \text{and} \quad \int_{B_R^c} \frac{F(G^{-1}(v))}{|x|^a} \mathrm{d}x \leq \delta.$$

For this, we also note that by (f_2) and (f_3), there exists $C_1 > 0$ satisfying

$$|F(x,s)| \leq C_1 |f(x,s)|, \quad \text{for all} \quad (x,s) \in \mathbb{R}^2 \times \mathbb{R}.$$

Thus, for each $A > 0$, we obtain

$$\begin{aligned} \int_{\substack{|x|>R \\ |G^{-1}(v_n)|>A}} \frac{F(G^{-1}(v_n))}{|x|^a} \mathrm{d}x &\leq C_1 \int_{\substack{|x|>R \\ |G^{-1}(v_n)|>A}} \frac{|f(G^{-1}(v_n))|}{|x|^a} \mathrm{d}x \\ &\leq \frac{C_1}{A} \int_{\mathbb{R}^2} \frac{|f(G^{-1}(v_n))G^{-1}(v_n)|}{|x|^a} \mathrm{d}x. \end{aligned}$$

Since

$$\int_{\mathbb{R}^2} \frac{|f(G^{-1}(v_n))G^{-1}(v_n)|}{|x|^a} \mathrm{d}x \leq C,$$

given $\delta > 0$, we may choose $A > 0$ such that

$$\frac{C_1}{A} \int_{\mathbb{R}^2} \frac{|f(G^{-1}(v_n))G^{-1}(v_n)|}{|x|^a} \mathrm{d}x < \frac{\delta}{2}.$$

Thus,

$$\int_{\substack{|x|>R \\ |G^{-1}(v_n)|>A}} \frac{F(G^{-1}(v_n))}{|x|^a} \mathrm{d}x \leq \frac{\delta}{2}. \tag{4.7}$$

Moreover, since f has critical exponential growth and satisfies (f_1) and (f_2), there exists $C(A) > 0$ such that

$$F(x, G^{-1}(s)) \leq C(A)|G^{-1}(s)|^2, \quad \text{for all} \quad (x, G^{-1}(s)) \in \mathbb{R}^2 \times [-A, A].$$

Therefore,

$$\begin{aligned} \int_{\substack{|x|>R \\ |G^{-1}(v_n)|\leq A}} \frac{F(G^{-1}(v_n))}{|x|^a} \mathrm{d}x &\leq C(A) \int_{\substack{|x|>R \\ |G^{-1}(v_n)|\leq A}} \frac{|G^{-1}(v_n)|^2}{|x|^a} \mathrm{d}x \\ &\leq 2C(A) \int_{\substack{|x|>R \\ |G^{-1}(v_n)|\leq A}} \frac{|G^{-1}(v_n) - G^{-1}(v)|^2}{|x|^a} \mathrm{d}x \\ &\quad + 2C(A) \int_{\substack{|x|>R \\ |G^{-1}(v_n)|\leq A}} \frac{|G^{-1}(v)|^2}{|x|^a} \mathrm{d}x. \end{aligned}$$

Hence, by using Proposition (2.3), given $\delta > 0$, we may choose $R > 0$ satisfying

$$\int_{\substack{|x|>R \\ |G^{-1}(v_n)|\leq A}} \frac{F(G^{-1}(v_n))}{|x|^a} \mathrm{d}x \leq \frac{\delta}{2}. \tag{4.8}$$

From (4.7) and (4.8), given $\delta > 0$, there exists $R > 0$ such that

$$\int_{|x|>R} \frac{F(G^{-1}(v_n))}{|x|^a} \mathrm{d}x \leq \delta.$$

Similarly, we obtain

$$\int_{|x|>R} \frac{F(G^{-1}(v))}{|x|^a} \mathrm{d}x \leq \delta.$$

Combining all the above estimates and since $\delta > 0$ is arbitrary, it follows that

$$\int_{\mathbb{R}^2} \frac{F(G^{-1}(v_n))}{|x|^a} \mathrm{d}x \to \int_{\mathbb{R}^2} \frac{F(G^{-1}(v))}{|x|^a} \mathrm{d}x, \quad \text{as } n \to +\infty,$$

and this completes the proof. □

Lemma 4.5. *Suppose that* (V), $(g_0) - (g_1)$ *and* $(f_1) - (f_2)$ *are satisfied. If* $(v_n) \subset X$ *is a Cerami sequence for* I *such that* $v_n \rightharpoonup v$ *weakly in* $H^1(\mathbb{R}^2)$ *with* $\int_{\mathbb{R}^2} V(x)|G^{-1}(v)|^2 \mathrm{d}x < \infty$, *then*

$$\begin{aligned} &\int_{\mathbb{R}^2} \nabla v \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^2} \frac{V(x) G^{-1}(v)}{g(G^{-1}(v))} \varphi \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^2} \frac{f(G^{-1}(v))}{g(G^{-1}(v)|x|^a} \varphi \mathrm{d}x, \quad \textit{for all} \ \ \varphi \in C_0^\infty(\mathbb{R}^2). \end{aligned}$$

Proof. First, we have that $I'(v)\varphi$ is well defined for $\varphi \in C_0^\infty(\mathbb{R}^2)$ and therefore just prove that $I'(v)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Note that

$$\begin{aligned} &I'(v_n)\varphi - I'(v)\varphi - \int_{\mathbb{R}^2} (\nabla v_n - \nabla v) \nabla \varphi \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^2} \left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))} \right] V(x) \varphi \mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^2} \left[\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))|x|^a} - \frac{f(G^{-1}(v))}{g(G^{-1}(v))|x|^a} \right] \varphi \mathrm{d}x. \end{aligned} \tag{4.9}$$

In view of $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^2)$, we have $v_n \to v$ in $L^p_{loc}(\mathbb{R}^2)$, with $p \geq 1$. Then, up to a subsequence,

$$\begin{aligned} &v_n(x) \to v(x) \ \text{ a.e. in } \mathcal{K} := \text{supp } \varphi, \ \text{ as } n \to +\infty, \\ &|v_n(x)| \leq |w_p(x)| \ \text{ for every } n \in \mathbb{N} \text{ and a.e. in } \mathcal{K}, \ \text{ with } w_p \in L^p(\mathcal{K}). \end{aligned}$$

Consequently,

$$\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \to \frac{G^{-1}(v)}{g(G^{-1}(v))} \quad \text{a.e. in } \ \mathcal{K}, \ \text{as } n \to +\infty.$$

Furthermore, by the continuity of V and Lemma 2.2-(2) and (3), there exists a constant $C > 0$ such that

$$\frac{|V(x) G^{-1}(v_n) \varphi|}{g(G^{-1}(v_n))} \leq |V(x) v_n \varphi| \leq C |w_2| |\varphi| \in L^1(\mathcal{K}).$$

Using these estimates, Lebesgue Dominated Convergence Theorem and the weak convergence $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$, we obtain

$$\int_{\mathbb{R}^2} (\nabla v_n - \nabla v) \nabla \varphi \mathrm{d}x \to 0 \ \text{ and } \ \int_{\mathbb{R}^2} \left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))} \right] V(x) \varphi \mathrm{d}x \to 0,$$

as $n \to +\infty$. In addition, by Lemma 4.3, we have

$$\int_{\mathbb{R}^2} \frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))|x|^a}\varphi \mathrm{d}x \to \int_{\mathbb{R}^2} \frac{f(G^{-1}(v))}{g(G^{-1}(v))|x|^a}\varphi \mathrm{d}x.$$

Hence, taking the limit in (4.9), we get $I'(v_n)\varphi - I'(v)\varphi \to 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ and once $I'(v_n) \to 0$, we conclude $I'(v)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. This finalizes the proof. □

Lemma 4.6. *Suppose that* (V), $(g_0)-(g_1)$ *and* $(f_1)-(f_2)$ *are satisfied. Let* (v_n) *be a Cerami sequence for* I *in* X *such that* $\left(\frac{\alpha}{\beta}\right)^2 \varsigma_0\|\nabla v_n\|_2^2 < 2\pi(2-a)$. *Then,* (v_n) *has a subsequence, still denoted by* (v_n), *such that*

$$\int_{\mathbb{R}^2} \frac{f(G^{-1}(v_n))(v-v_n)}{g(G^{-1}(v_n))|x|^a}\mathrm{d}x \to 0,$$

as $n \to +\infty$, *where* v *is the weak limit of* (v_n) *in* $H^1(\mathbb{R}^2)$ *with* $\int_{\mathbb{R}^2} V(x)|G^{-1}(v)|^2 \mathrm{d}x < \infty$.

Proof. By (2.6), given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|\frac{f(G^{-1}(v_n))(v-v_n)}{g(G^{-1}(v_n))}\right| \le \varepsilon|G^{-1}(v_n)||v-v_n| + C_\varepsilon[e^{(\varsigma_0+\varepsilon)|G^{-1}(v_n)|^{2\alpha}} - 1]|v-v_n|.$$

Hence, by Lemma 2.2-(5), one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \frac{f(G^{-1}(v_n))(v-v_n)}{g(G^{-1}(v_n))|x|^a}\mathrm{d}x\right| \\
&\quad\le \varepsilon C_1 \int_{\mathbb{R}^2} \frac{|G^{-1}(v_n)||G^{-1}(v-v_n)|}{|x|^a}\mathrm{d}x \\
&\qquad + \varepsilon C_1 \int_{\mathbb{R}^2} \frac{|G^{-1}(v_n)||G^{-1}(v-v_n)|^\alpha}{|x|^a}\mathrm{d}x \\
&\qquad + C_\varepsilon \int_{\mathbb{R}^2} \frac{[e^{(\varsigma_0+\varepsilon)|G^{-1}(v_n)|^{2\alpha}} - 1]|G^{-1}(v-v_n)|}{|x|^a}\mathrm{d}x \\
&\qquad + C_\varepsilon \int_{\mathbb{R}^2} \frac{[e^{(\varsigma_0+\varepsilon)|G^{-1}(v_n)|^{2\alpha}} - 1]|G^{-1}(v-v_n)|^\alpha}{|x|^a}\mathrm{d}x.
\end{aligned}$$

By Hölder's inequality and choosing $t > 1$ such that $t' = t/(t-1) \geq 2$, we get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \frac{f(G^{-1}(v_n))(v - v_n)}{g(G^{-1}(v_n))|x|^a} \mathrm{d}x \right| \\
&\quad \leq \varepsilon C_1 \left(\int_{\mathbb{R}^2} \frac{|G^{-1}(v_n)|^2}{|x|^a} \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{|G^{-1}(v - v_n)|^2}{|x|^a} \mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad + \varepsilon C_1 \left(\int_{\mathbb{R}^2} \frac{|G^{-1}(v_n)|^2}{|x|^a} \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{|G^{-1}(v - v_n)|^{2\alpha}}{|x|^a} \mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad + C_\varepsilon \left\{ \int_{\mathbb{R}^2} \frac{[e^{t(\varsigma_0 + \varepsilon)|G^{-1}(v_n)|^{2\alpha}} - 1]}{|x|^a} \mathrm{d}x \right\}^{\frac{1}{t}} \left\{ \int_{\mathbb{R}^2} \frac{|G^{-1}(v - v_n)|^{t'}}{|x|^a} \mathrm{d}x \right\}^{\frac{1}{t'}} \\
&\qquad + C_\varepsilon \left\{ \int_{\mathbb{R}^2} \frac{[e^{t(\varsigma_0 + \varepsilon)|G^{-1}(v_n)|^{2\alpha}} - 1]}{|x|^a} \mathrm{d}x \right\}^{\frac{1}{t}} \left\{ \int_{\mathbb{R}^2} \frac{|G^{-1}(v - v_n)|^{\alpha t'}}{|x|^a} \mathrm{d}x \right\}^{\frac{1}{t'}}.
\end{aligned} \tag{4.10}$$

Next, note that there exists $t > 1$ sufficiently close to 1, $\varepsilon > 0$ sufficiently small and $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{e^{t(\varsigma_0 + \varepsilon)|G^{-1}(v_n)|^{2\alpha}} - 1}{|x|^a} \mathrm{d}x \leq C. \tag{4.11}$$

Indeed, we can infer that for n sufficiently large, there exists $t > 1$, sufficiently close to 1, and $\varepsilon > 0$ sufficiently small so that $\left(\frac{\alpha}{\beta}\right)^2 t(\varsigma_0 + \varepsilon)\|\nabla v_n\|_2^2 < 2\pi(2-a)$. Hence, by Lemma 2.2-(7) and Lemma 2.4, we get

$$\int_{\mathbb{R}^2} \frac{e^{t(\varsigma_0 + \varepsilon)|G^{-1}(v_n)|^{2\alpha}} - 1}{|x|^a} \mathrm{d}x \leq \int_{\mathbb{R}^2} \frac{e^{t(\varsigma_0 + \varepsilon)\left(\frac{\alpha}{\beta}\right)^2 \|\nabla v_n\|_2^2 \left(\frac{|v_n|}{\|\nabla v_n\|_2}\right)^2} - 1}{|x|^a} \mathrm{d}x \leq C,$$

which proves (4.11). Since $G^{-1}(v_n - v)$ is a bounded sequence in X and for $p \in [2, +\infty)$ the embedding $X \hookrightarrow L^p(\mathbb{R}^2; |x|^{-a}\mathrm{d}x)$ is compact, up to a subsequence, we have

$$\int_{\mathbb{R}^2} \frac{|G^{-1}(v - v_n)|^{t'}}{|x|^a} \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{|G^{-1}(v - v_n)|^{\alpha t'}}{|x|^a} \mathrm{d}x \to 0.$$

Therefore, from (4.10) and (4.11) we conclude the proof of the theorem. □

We recall that the minimax level of I is given by

$$0 < c_m = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{4.12}$$

where $\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$ and e was given in Lemma 3.4.

As a consequence of Lemma 4.6, we have the following result, which is essential for the proof of Theorem 1.1.

Corollary 4.7. *Suppose that (V), $(g_0) - (g_2)$ and $(f_1) - (f_2)$ are satisfied. Let (v_n) be a Cerami sequence for I in X at the level c_m satisfying $(\alpha/\beta)^2 \varsigma_0 \|\nabla$*

$v_n\|_2^2 < 2\pi(2-a)$ and $v_n \rightharpoonup 0$ weakly in X. Then $c_m = 0$, where c_m is given in (4.12).

Proof. Indeed, since $I'(v_n)v_n \to 0$,

$$\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}x + \int_{\mathbb{R}^2} \frac{V(x)G^{-1}(v_n)}{g(G^{-1}(v_n)}v_n = \int_{\mathbb{R}^2} \frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n)|x|^a}v_n + o_n(1).$$

Hence, by Lemma 2.2-(4) we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}x + \frac{1}{\alpha}\int_{\mathbb{R}^2} V(x)[G^{-1}(v_n)]^2 \mathrm{d}x \\
&\quad \leq \int_{\mathbb{R}^2} \frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n)|x|^a}v_n + o_n(1) \leq \int_{\mathbb{R}^2} \frac{f(G^{-1}(v_n))G^{-1}(v_n)}{|x|^a}\mathrm{d}x + o_n(1).
\end{aligned} \tag{4.13}$$

Moreover, as $I(v_n) \to c_m$ we get

$$\begin{aligned}
c_m = &\frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} V(x)[G^{-1}(v_n)]^2 \mathrm{d}x \\
&- \int_{\mathbb{R}^2} \frac{F(G^{-1}(v_n))}{|x|^a}\mathrm{d}x + o_n(1).
\end{aligned} \tag{4.14}$$

Then, by (4.13), (4.14) and Lemma 4.6-(2),(3), we conclude that $c_m = 0$ as we desired. □

5. Minimax Level Estimate

In this section, we obtain an estimate for the mountain pass level of I, which will be crucial to study the behavior of Cerami sequences for I. For this, let $r > 0$ and consider the Moser's sequence defined by

$$M_n(x,r) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & \text{if } |x| \leq \frac{r}{n}, \\ \dfrac{\log(r/|x|)}{\sqrt{\log n}}, & \text{if } \frac{r}{n} \leq |x| \leq r, \\ 0, & \text{if } |x| > r, \end{cases}$$

which satisfies $M_n \in H_0^1(B_r)$, $\|\nabla M_n\|_2 = 1$ for all $n \in \mathbb{N}$ and

$$\|M_n\|_2^2 = \frac{r^2}{4\log n} - \frac{r^2}{2n^2} - \frac{r^2}{4n^2\log n}.$$

Proposition 5.1. *Assume that (V), $(g_0)-(g_2)$, (f_1), (f_2) and (f_4) are satisfied. Then, the minimax level c_m satisfies*

$$c_m < \frac{(2-a)\pi}{(\frac{\alpha}{\beta})^2\varsigma_0}. \tag{5.1}$$

Proof. To prove (5.1), it is sufficient to obtain $n \in \mathbb{N}$ such that

$$\max_{t\geq 0} I(t\widetilde{M}_n) < \frac{(2-a)\pi}{(\frac{\alpha}{\beta})^2\varsigma_0},$$

where $\widetilde{M}_n = M_n/\|M_n\|$. Suppose, for the sake of contradiction, that for all $n \in \mathbb{N}$, we have

$$\max_{t \geq 0} I(t\widetilde{M}_n) \geq \frac{(2-a)\pi}{(\frac{\alpha}{\beta})^2 \varsigma_0}. \tag{5.2}$$

In view of Lemma 3.3 and Lemma 3.4, for all $n \in \mathbb{N}$, there exists $t_n > 0$ such that

$$I(t_n\widetilde{M}_n) = \max_{t \geq 0} I(t\widetilde{M}_n). \tag{5.3}$$

By Lemma 2.2-(3), (5.2), (5.3), (f_2) and $\|\widetilde{M}_n\| = 1$, it follows that

$$t_n^2 \geq \frac{2(2-a)\pi}{(\frac{\alpha}{\beta})^2 \varsigma_0}, \tag{5.4}$$

because

$$\begin{aligned} \frac{t_n^2}{2} &= \frac{t_n^2}{2} \int_{\mathbb{R}^2} \left(|\nabla \widetilde{M}_n|^2 + V(x)\widetilde{M}_n^2 \right) \mathrm{d}x \\ &\geq \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla (t_n\widetilde{M}_n)|^2 + V(x)[G^{-1}(t_n\widetilde{M}_n)]^2 \right\} \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} \frac{F(G^{-1}(t_n\widetilde{M}_n))}{|x|^a} \mathrm{d}x \geq \frac{(2-a)\pi}{\left(\frac{\alpha}{\beta}\right)^2 \varsigma_0}. \end{aligned}$$

Next, we will show that the sequence (t_n) is bounded. To achieve this goal, let us remember that $\frac{d}{\mathrm{d}t} I(t\widetilde{M}_n) = 0$ at $t = t_n$, that is, $I'(t_n\widetilde{M}_n) \cdot \widetilde{M}_n = 0$. Thus,

$$\begin{aligned} &t_n^2 \int_{\mathbb{R}^2} \left[|\nabla \widetilde{M}_n|^2 + t_n^{-2} V(x) \frac{G^{-1}(t_n\widetilde{M}_n)}{g(G^{-1}(t_n\widetilde{M}_n))} t_n\widetilde{M}_n \right] \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} \frac{f(G^{-1}(t_n\widetilde{M}_n))}{g(G^{-1}(t_n\widetilde{M}_n))|x|^a} t_n\widetilde{M}_n \mathrm{d}x = 0. \end{aligned}$$

By Lemma 2.2-(4), (f_2) and $\|\nabla \widetilde{M}_n\|_2 \leq 1$, one has

$$\begin{aligned} t_n^2 &= t_n^2 \int_{\mathbb{R}^2} \left[|\nabla \widetilde{M}_n|^2 + V \frac{t_n^2 \widetilde{M}_n^2}{t_n^2} \right] \mathrm{d}x \geq t_n^2 \int_{\mathbb{R}^2} \left[|\nabla \widetilde{M}_n|^2 + \frac{G^{-1}(t_n\widetilde{M}_n) t_n\widetilde{M}_n}{t_n^2 g(G^{-1}(t_n\widetilde{M}_n))} \right] \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \frac{f(G^{-1}(t_n\widetilde{M}_n))}{g(G^{-1}(t_n\widetilde{M}_n))|x|^a} t_n\widetilde{M}_n \mathrm{d}x \geq \int_{B_{\frac{r}{n}}(0)} \frac{f(G^{-1}(t_n\widetilde{M}_n))}{g(G^{-1}(t_n\widetilde{M}_n))|x|^a} t_n\widetilde{M}_n \mathrm{d}x \\ &\geq \frac{1}{\alpha} \int_{B_{\frac{r}{n}}(0)} \frac{f(G^{-1}(t_n\widetilde{M}_n)) G^{-1}(t_n\widetilde{M}_n)}{|x|^a} \mathrm{d}x. \end{aligned} \tag{5.5}$$

According to (f_4), given $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$sf(s) \geq (\xi_0 - \varepsilon) e^{\varsigma_0 |s|^{2\alpha}}, \quad \text{for all } s \geq R_\varepsilon. \tag{5.6}$$

Since $G^{-1}(t_n\widetilde{M}_n) > R_\varepsilon$ in $B_{\frac{r}{n}}(0)$ for n sufficiently large, using (5.5) and (5.6), we obtain

$$t_n^2 \geq \frac{\xi_0-\varepsilon}{\alpha}\int_{B_{\frac{r}{n}}(0)} \frac{e^{\varsigma_0|G^{-1}(t_n\widetilde{M}_n)|^{2\alpha}}}{|x|^a}\mathrm{d}x. \tag{5.7}$$

In view of Lemma 2.2-(7), given $\eta>0$ there exists $R_\eta>0$ such that

$$|G^{-1}(s)|^{2\alpha} \geq \left[\left(\frac{\alpha}{\beta}\right)^2-\eta\right]s^2, \quad \text{for all } s\geq R_\eta. \tag{5.8}$$

Thus, for n sufficiently large (without loss of generality we can assume $R_\varepsilon > R_\eta$), using (5.7) and (5.8) we get

$$\begin{aligned}
t_n^2 &\geq \frac{\xi_0-\varepsilon}{\alpha}\int_{B_{\frac{r}{n}}(0)} \frac{e^{\varsigma_0\left[\left(\frac{\alpha}{\beta}\right)^2-\eta\right]t_n^2\widetilde{M}_n^2}}{|x|^a}\mathrm{d}x \\
&= \frac{\xi_0-\varepsilon}{\alpha}e^{\varsigma_0\left[\left(\frac{\alpha}{\beta}\right)^2-\eta\right]\frac{1}{2\pi}\frac{\log n}{\|M_n\|^2}t_n^2}\frac{2\pi}{2-a}\left(\frac{r}{n}\right)^{2-a} \\
&= \frac{\xi_0-\varepsilon}{\alpha}e^{\varsigma_0\left[\left(\frac{\alpha}{\beta}\right)^2-\eta\right]\frac{1}{2\pi}\frac{\log n}{\|M_n\|^2}t_n^2-(2-a)\log n}\frac{2\pi}{2-a}r^{2-a}.
\end{aligned} \tag{5.9}$$

Hence,

$$1 \geq \frac{\xi_0-\varepsilon}{\alpha}e^{\varsigma_0\left[\left(\frac{\alpha}{\beta}\right)^2-\eta\right]\frac{1}{2\pi}\frac{\log n}{\|M_n\|^2}t_n^2-(2-a)\log n-2\log t_n}\frac{2\pi}{2-a}r^{2-a}, \tag{5.10}$$

which implies

$$\varsigma_0\left[\left(\frac{\alpha}{\beta}\right)^2-\eta\right]\frac{1}{2\pi}\frac{\log n}{\|M_n\|^2}t_n^2-(2-a)\log n-2\log t_n \leq C.$$

This estimate shows that (t_n) is bounded, otherwise, once $\|M_n\|^2 \leq 1+\|V\|_{L^\infty(B_r)}\|M_n\|_2^2$, we have

$$\begin{aligned}
&\varsigma_0\left[\left(\frac{\alpha}{\beta}\right)^2-\eta\right]\frac{1}{2\pi}\frac{\log n}{\|M_n\|^2}t_n^2-(2-a)\log n-2\log t_n \\
&\geq t_n^2\log n\left\{\frac{\varsigma_0\left[\left(\frac{\alpha}{\beta}\right)^2-\eta\right]}{2\pi\left(1+\|V\|_{L^\infty(B_r)\|M_n\|_2}\right)}-\frac{2-a}{t_n^2}-\frac{2\log t_n}{t_n^2\log n}\right\} \\
&\to +\infty, \quad \text{as } n\to+\infty,
\end{aligned}$$

which is a contradiction with (5.10). Thus, by (5.4), (5.9) and since (t_n) is bounded, there are constants $C_1 = C_1(a,\varsigma_0,\alpha,\beta,\eta) > 0$ and $C_2>0$ such that

$$C_1\frac{\log n}{\|M_n\|^2}-\log n \leq C_2. \tag{5.11}$$

However,

$$
\begin{aligned}
C_1 \frac{\log n}{\|M_n\|^2} - \log n &= \frac{C_1 \log n - \|M_n\|^2 \log n}{\|M_n\|^2} \\
&\geq \frac{C_1 \log n - \left[1 + \|V\|_{L^\infty(B_r)} \left(\frac{r^2}{4\log n} - \frac{r^2}{2n^2} - \frac{r^2}{4n^2 \log n}\right)\right] \log n}{1 + \|V\|_{L^\infty(B_r)} \left(\frac{r^2}{4\log n} - \frac{r^2}{2n^2} - \frac{r^2}{4n^2 \log n}\right)} \\
&= \frac{(C_1 - 1)\log n + \|V\|_{L^\infty(B_r)} \left(\frac{r^2}{4n^2} + \frac{r^2 \log n}{2n^2} - \frac{r^2}{4}\right)}{1 + \|V\|_{L^\infty(B_r)} \left(\frac{r^2}{4\log n} - \frac{r^2}{2n^2} - \frac{r^2}{4n^2 \log n}\right)} \longrightarrow +\infty,
\end{aligned}
$$

as $n \to +\infty$, which contradicts (5.11). The proposition is proved. □

6. Proof of Theorem 1.1

According to Lemma 3.3 and Lemma 3.4, the hypotheses of Theorem 2.7 are satisfied. Thus, the minimax level c_m of I is positive and there is a Cerami sequence (v_n) for I at the level c_m. Applying Lemma 4.1 and 4.3, we may assume, without loss generality, that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^2)$ for some $v \in H^1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} V(x)|G^{-1}(v)|^2 \mathrm{d}x < \infty$. From Lemma 4.5, v is a weak solution of equation (2.5). Now, suppose by contradiction, that v is zero. In view of Lemma 4.4 and since $I(v_n) \to c_m$ as $n \to +\infty$, we reach

$$
\frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla v_n|^2 + V(x) \left[G^{-1}(v_n)\right]^2 \right\} \mathrm{d}x = c_m + o_n(1). \tag{6.1}
$$

From Proposition 5.1, we have

$$
c_m < (2-a)\pi / (\frac{\alpha}{\beta})^2 \varsigma_0. \tag{6.2}
$$

Using condition (V), (6.1) and (6.2), there exists $n_0 \in \mathbb{N}$ such that

$$
\left(\frac{\alpha}{\beta}\right)^2 \varsigma_0 \|\nabla v_n\|_2^2 < 2\pi(2-a), \quad \text{for all } n \geq n_0.
$$

Thus, in view of Corollary 4.7, we get $c_m = 0$, which is a contradiction. Therefore, $v \neq 0$.

Next, we prove that v is nonnegative. Indeed, if $v^- = \max\{-v, 0\}$ then $v^- \in H^1(\mathbb{R}^2)$ and by density we get

$$
\int_{\mathbb{R}^2} |\nabla v^-|^2 \mathrm{d}x + \int_{\mathbb{R}^2} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} (-v^-) \mathrm{d}x = \int_{\mathbb{R}^2} \frac{f(G^{-1}(v))}{g(G^{-1}(v))|x|^a} (-v^-) \mathrm{d}x \leq 0.
$$

On the other hand, we know that $\frac{G^{-1}(v)}{g(G^{-1}(v))}(-v^-) \geq 0$ and this implies that $\int_{\mathbb{R}^2} |\nabla v^-|^2 \mathrm{d}x = 0$. Thus, $v^- = 0$ almost everywhere in $\mathbb{R}^2$ and therefore $v \geq 0$. In order to prove that $v > 0$ in $\mathbb{R}^2$, we suppose, otherwise, that there exists $x_0 \in \mathbb{R}^2$ such that $v(x_0) = 0$. Notice that 2.5 can be written in the form

$$
-\Delta v + c(x)v = V(x) \frac{v - G^{-1}(v)}{g(G^{-1}(v))} + \frac{f(G^{-1}(v))}{g(G^{-1}(v))|x|^a} \geq 0
$$

where $c(x) = V(x)\frac{v}{g(G^{-1}(v))} > 0$ for all $x \in \mathbb{R}^2$. Recalling that $v \in C^{0,\vartheta}_{loc}(\mathbb{R}^2)$, using Strong Maximum Principle (see [20], Theorem 8.19) in an arbitrary ball centered in x_0, we can conclude that $v \equiv 0$, which is impossible. Therefore, v has to be strictly positive. In view of Proposition 2.6 we reach $u = G^{-1}(v)$ is a positive solution of (1.1) and the proof of Theorem 1.1 is complete.

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References

[1] Adachi, S., Watanabe, T.: Uniqueness of the ground state solutions of quasilinear Schrödinger equations. Nonlinear Anal. **7**(5), 819–833 (2012)

[2] Aubin, J.P., Ekeland, I.: Applied Nonlinear Analysis. Pure and Applied Mathematics. Wiley, New York (1984)

[3] Brull, L., Lange, H.: Solitary waves for quasilinear Schrödinger equations. Expos. Math. **4**, 279–288 (1986)

[4] Cao, D.M.: Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$. Commun. Part. Differ. Equ. **17**, 407–435 (1992)

[5] Chen, J., Tang, X., Cheng, B.: Non-Nehari manifold method for a class of generalized quasilinear Schrödinger equations. Appl. Math. Lett. **74**, 20–26 (2017)

[6] Chu, C., Liu, H.: Existence of positive solutions for a quasilinear Schrödinger equation. Nonlinear Anal. Real World Appl. **44**, 118–127 (2018)

[7] Colin, M., Jeanjean, L.: Solutions for a quasilinear Schrödinger equation: A dual approach. Nonlinear Anal. **56**, 213–226 (2004)

[8] Cuccagna, S.: On instability of excited states of the nonlinear Schrödinger equation. Phys. D **238**, 38–54 (2009)

[9] de Figueiredo, D.G., Miyagaki, O.H., Ruf, B.: Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range. Calc. Var. Part. Differ. Equ. **3**, 139–153 (1995)

[10] Deng, Y.B., Huang, W.T.: Ground state solutions for generalized quasilinear Schrödinger equations without (AR) condition. J. Math. Anal. Appl. **456**, 927–945 (2017)

[11] Deng, Y., Peng, S., Wang, J.: Nodal soliton solutions for generalized quasilinear Schrödinger equations. J. Math. Phys. **55**, 16 (2014)

[12] Deng, Y., Peng, S., Yan, S.: Critical exponents and solitary wave solutions for generalized quasilinear Schrödinger equations. J. Differ. Equ. **260**, 1228–1262 (2016)

[13] Deng, Y., Peng, S., Yan, S.: Positive soliton solutions for generalized quasilinear Schrödinger equations with critical growth. J. Differ. Equ. **258**, 115–147 (2015)

[14] de Souza, M., Severo, U.B., Vieira, G.F.: On a nonhomogeneous and singular quasilinear equation involving critical growth in $\mathbb{R}^2$. Comput. Math. Appl. **74**, 513–531 (2017)

[15] de Souza, M., Severo, U.B., Vieira, G.F.: Solutions for a class of singular quasilinear equations involving critical growth in $\mathbb{R}^2$. Math. Nachr. **2021**, 5 (2021)

[16] DoÓ, J.M., de Souza, M.: On a class of singular Trudinger-Moser type inequalities and its applications. Math. Nachr. **284**, 1754–1776 (2011)

[17] DoÓ, J.M., Miyagaki, O., Soares, S.: Soliton solutions for quasilinear Schrödinger equations: the critical exponential case. Nonlinear Anal. **67**, 3357–3372 (2007)

[18] DoÓ, J.M., Severo, U.: Quasilinear Schrödinger equations involving concave and convex nonlinearities. Commun. Pure Appl. Anal. **8**, 621–644 (2009)

[19] Furtado, M.F., Silva, E.D., Silva, M.L.: Existence of solution for a generalized quasilinear elliptic problem. J. Math. Phys. **58**, 14 (2017)

[20] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order, Reprint of the 1998th edn Classics in Mathematics. Springer, Berlin (2001)

[21] Jeanjean, L.: On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$. Proc. R. Soc. Edinb. Sect. A **129**, 787–809 (1999)

[22] Kavian, O.: Introduction á la Thèorie Des Points Critiques et Applications aux Problèmes Elliptiques. Springer, Paris (1993)

[23] Kurihara, S.: Large-amplitude quasi-solitons in superfluid films. J. Phys. Soc. Jpn. **50**, 3263–3267 (1981)

[24] Laedke, E.W., Spatschek, K.H., Stenflo, L.: Evolution theorem for a class of perturbed envelope soliton solutions. J. Math. Phys. **24**, 2764–2769 (1983)

[25] Landau, L.D., Lifschitz, E.M.: Quantum Mechanics. Non-relativistic Theory. Addison-Wesley, Reading (1968)

[26] Li, Q., Wu, X.: Existence, multiplicity and concentration of solutions for generalized quasilinear Schrödinger equations with critical growth. J. Math. Phys. **58**, 30 (2017)

[27] Litvak, A.G., Sergeev, A.M.: One-dimensional collapse of plasma waves. JETP Lett. **27**, 517–520 (1978)

[28] Liu, J., Wang, Z.Q.: Soliton solutions for quasilinear Schrödinger equations. Proc. Am. Math. Soc. **131**, 441–448 (2003)

[29] Liu, J., Wang, Y., Wang, Z.Q.: Soliton solutions for quasilinear Schrödinger equations II. J. Differ. Equ. **187**, 473–493 (2003)

[30] Moameni, A.: On a class of periodic quasilinear Schrödinger equations involving critical growth in $\mathbb{R}^2$. J. Math. Anal. Appl. **334**, 775–786 (2007)

[31] Nakamura, A.: Damping and modification of exciton solitary waves. J. Phys. Soc. Jpn. **42**, 1824–1835 (1977)

[32] Poppenberg, M., Schmitt, K., Wang, Z.Q.: On the existence of soliton solutions to quasilinear Schrödinger equations. Calc. Var. Part. Differ. Equ. **14**, 329–344 (2002)

[33] Shen, Y., Wang, Y.: Soliton solutions for generalized quasilinear Schrödinger equations. Nonlinear Anal. **80**, 194–201 (2013)

[34] Shen, Y.T., Wang, Y.J.: A class of generalized quasilinear Schrödinger equations. Commun. Pure Appl. Anal. **15**, 853–870 (2016)

[35] Shi, H., Chen, H.: Existence and multiplicity of solutions for a class of generalized quasilinear Schrödinger equations. J. Math. Anal. Appl. **452**, 578–594 (2017)

[36] Shi, H., Chen, H.: Generalized quasilinear asymptotically periodic Schrödinger equations with critical growth. Comput. Math. Appl. **71**, 849–858 (2016)

[37] Wang, Y.: Multiplicity of solutions for singular quasilinear Schrödinger equations with critical exponents. J. Math. Anal. Appl. **458**, 1027–1043 (2018)

[38] Zhang, C., Chenn, L.: Concentration-compactness principle of singular Trudinger-Moser inequalities in $\mathbb{R}^n$ and n-Laplace equations. Adv. Nonlinear Stud. **18**, 567–585 (2018)

Uberlandio B. Severo, Manassés de Souza and Diogo de S. Germano
Departamento de Matemática
Universidade Federal da Paraíba
58051-900 João Pessoa
PB
Brazil
e-mail: uberlandio@mat.ufpb.br

Manassés de Souza
e-mail: manasses@mat.ufpb.br

Diogo de S. Germano
Universidade Federal de Campina Grande, Unidade Acadêmica de Matemática
58109-970 Campina Grande
PB
Brazil
e-mail: diogosg@mat.ufcg.edu.br

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