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On the Connected Power Graphs of Semigroups of Homogeneous Elements of Graded Rings

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Abstract. In this paper, by the power graph $\mathcal{G}(S)$ of a semigroup S we mean an undirected graph whose vertices are elements of S and where two vertices are adjacent if and only if they are distinct and one of them is a power of the other. Let $R = \bigoplus_{s\in S} R_s$ be a ring graded by a groupoid S. Inspired by the problems raised in Abawajy et al. (Electron J Graph Theory Appl 1(2):125–147, 2013) we investigate the question of connectedness of the power graph of the multiplicative semigroup $H_R = \bigcup_{s\in S} R_s$ of homogeneous elements of R. We establish that $\mathcal{G}(H_R)$ is connected if and only if all of the homogeneous elements of R are nilpotent. If $\mathcal{G}(H_R)$ is connected, then the power graphs $\mathcal{G}(R_e)$ of the multiplicative semigroups R_e, where e runs through the set of all idempotent elements of S, are also connected. The converse, however, does not hold in general, but we prove that it does hold under some additional assumptions. If R has no nontrivial homogeneous right or left zero divisors, then $H_R^* = H_R\setminus\{0\}$ is a semigroup under the multiplication of R, and S is a semigroup. If, moreover, R is with unity and S is cancellative, we prove that $\mathcal{G}(H_R^*)$ is connected if and only if S is a monoid with unity e, and the power graphs $\mathcal{G}(R_e\setminus\{0\})$ and $\mathcal{G}(S)$ are connected.

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1. Introduction

Let S be a semigroup. The *directed power graph* $\overrightarrow{\mathcal{G}}(S)$ of S is a directed graph with S as the set of vertices and (u, v) is an arc if and only if $u \neq v$ and v is a power of u. The *undirected power graph* $\mathcal{G}(S)$ of S is the underlying undirected graph of $\overrightarrow{\mathcal{G}}(S)$, that is, S is the set of vertices and two distinct vertices are adjacent if and only if one of them is a power of the other. The directed power graphs of groups are defined in [27], and the directed power graphs of semigroups are first introduced and studied in [28–30]. In these papers, as explained in [1], the term 'power graph' is used for the directed power graph, which covers the notion of the undirected power graph as the underlying undirected graph of the directed power graph. The undirected power graphs are the main object of study in [6], as well as in [3,4], and where they are also briefly referred to as the power graphs. In this paper, we use the brief term 'power graph' for an 'undirected power graph' since we focus on undirected power graphs only.

Studying not only power graphs, but also enhanced power graphs and Cayley graphs of algebraic structures is important since such graphs have many useful applications, see for instance [2,18,20,22–24,26,31,32,36–38] and references therein.

Power graphs (both directed and undirected) of both groups and semigroups are widely present in the recent literature. The reader is referred to the survey [1] and references therein, and for the more recent results, one may consult for instance [5,34,35] and their references. As observed in [1], it is interesting to investigate the power graphs of ring constructions. Graded rings form one of the most important classes of ring constructions (cf. [22]).

Let R be a ring, and S a partial groupoid, that is, a set with a partial binary operation. Also, let $\{R_s\}_{s\in S}$ be a family of additive subgroups of R, called *components*. We say that $R=\bigoplus_{s\in S} R_s$ is S-*graded* and R *induces* S (or R is an S-*graded ring inducing* S) [21,22,25] if the following two conditions hold:

(i) $R_sR_t\subseteq R_{st}$ whenever st is defined;
(ii) $R_sR_t\neq 0$ implies that the product st is defined.

The set $H_R=\bigcup_{s\in S} R_s$ is called the *homogeneous part of* R, and it is obviously a semigroup with respect to multiplication of R. Elements of H_R are called *homogeneous elements of* R.

The definition of an S-graded ring inducing S applies to both associative and nonassociative rings but throughout the paper, all rings are assumed to be associative. Examples of S-graded rings inducing S can be found in [11,12,16,17,22,25]. Note that the notion of an S-graded ring inducing S covers all the other notions of graded rings, including group rings and crossed products.

Problem 1.1. (*Problem* 11 *in* [1]) Let G be a group, R a ring, and let $R[G]=\bigoplus_{g\in G} Rg$ be a group ring. Reduce various parameters of the graphs $\overrightarrow{\mathcal{G}}(H_{R[G]})$ and $\mathcal{G}(H_{R[G]})$ to the corresponding properties of the coefficient ring R and the group G.

Problem 1.2. (*Problem* 12 *in* [1]) Let G be a group with unity e, and let $R=\bigoplus_{g\in G} R_g$ be a crossed product or a group-graded ring. Reduce the various parameters of the graphs $\overrightarrow{\mathcal{G}}(H_R)$ and $\mathcal{G}(H_R)$ to the corresponding properties of the subring R_e and the group G.

In this paper, we are interested in the connectedness of the power graph $\mathcal{G}(H_R)$, where $R=\bigoplus_{s\in S} R_s$ is generally an S-graded ring inducing S. The aim is to investigate how the connectedness of $\mathcal{G}(H_R)$, and, in some cases, of $\mathcal{G}(H_R\backslash\{0\})$, depends on the connectedness of the power graphs $\mathcal{G}(R_e)$ of the multiplicative semigroups R_e, where e runs through the set of all idempotent elements of S, and, where appropriate, on the connectedness of the power graph of S.

2. Preliminaries

To formulate and prove the results, we recall some notions and facts regarding semigroups and S-graded rings inducing S. For more details, the reader is referred to [8,22].

Let S be a semigroup. A subsemigroup I of S is said to be an *ideal* if $SI \cup IS \subseteq I$. Let I and J be ideals of S such that $J \subseteq I$. The *Rees quotient semigroup* I/J is defined as the semigroup with zero 0 obtained from I by identifying all elements of the ideal J with 0. If I is with zero and $J = 0$, then $I/J = I$. In case $J = \emptyset$, we put $I/J = I$. The quotient I/J is called a *factor* of S.

Let S be a semigroup with zero 0. An element $x \in S$ is said to be *nilpotent* if there exists a positive integer n such that $x^n = 0$. We say that S is *nil* if it entirely consists of nilpotent elements.

Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. *The degree* $\delta(a)$ of a nonzero homogeneous element a of R is defined as the unique $s \in S$ such that $a \in R_s$. Let us define $\delta(0) = 0$, $R_0 = 0$, and $S^0 = S \cup \{0\}$. By putting $st = 0$ for pairs (s,t) for which st was not originally defined, we make S^0 a groupoid throughout the article (see for instance [19]). Obviously, $R = \bigoplus_{s\in S} R_s = \bigoplus_{s\in S^0} R_s$. Moreover, without loss of generality, we assume throughout the article, unless otherwise stated, that $0 \in S$, since the zero element of R may be regarded as a component of R, in which case, of course, $S^0 = S$. However, in case S is a groupoid with or without a zero, we put $S^0 = S \cup \{0\}$ (here if S has a zero, 0 is a new adjoined zero). Moreover, if S is a groupoid with zero 0, then R is said to be a *contracted S-graded ring* if it is S-graded with $R_0 = 0$. If S is without a zero, then an S-graded ring R is a contracted S^0-graded ring.

A subring A of R is said to be *homogeneous* if $A = \bigoplus_{s\in S} A \cap R_s$. Moreover, the largest homogeneous subring of R contained in a subring A of R is equal to $\bigoplus_{s\in S} A \cap R_s$. In particular, if A is a homogeneous ideal of R, then $R/A = \bigoplus_{s\in S} R_s/A \cap R_s$ is a graded ring too. If $T \subseteq S$, then $R_T := \bigoplus_{t\in T} R_t$. Of course, if T is a subgroupoid (ideal) of S, we have that R_T is a homogeneous subring (ideal) of R. The ring R is said to be *graded-nil* if all of its homogeneous elements are nilpotent.

We also recall that R is said to be graded by a *cancellative* S if each of the equalities $su = tu \neq 0$ or $us = ut \neq 0$ implies $s = t$ for s, t, $u \in S$.

Next, we recall some facts on the structure of semigroups and its relation to graded rings, to facilitate the proof of Theorem 3.14.

By a 0-*simple semigroup* we mean a semigroup S with zero 0 such that there are no other ideals of S except for $\{0\}$ and S itself.

If $E(S)$ denotes the set of all idempotent elements of a semigroup S, then $E(S)$ can be partially ordered by the relation $\leq$ defined by $e \leq f$ if and only if $e = ef = fe$. A 0-simple semigroup S is said to be *completely* 0*-simple* if it contains a primitive idempotent element, that is, an idempotent which is minimal among nonzero idempotent elements of S under $\leq$.

Let G be a group, I and Λ nonempty sets, and let $P = (p_{\lambda i})$ be a $\Lambda \times I$-matrix with entries in G^0. The *Rees matrix semigroup* $M^0(G^0; I, \Lambda; P)$ over G^0 with *sandwich matrix* P consists of all triples $(g; i, \lambda)$, for $i \in I$, $\lambda \in \Lambda$, and $g \in G^0$, where all triples of the form $(0; i, \lambda)$ are identified with 0, and multiplication is defined by the rule $(g; i, \lambda)(h; j, \mu) = (gp_{\lambda j}h; i, \mu)$. In case G is trivial, the Rees matrix semigroup $M^0(G^0; I, \Lambda; P)$, denoted by $M^0(e^0; I, \Lambda; P)$, is called *elementary Rees matrix semigroup*, which is actually the rectangular 0-band $I \times \Lambda$.

By the Rees Theorem, a completely 0-simple semigroup is isomorphic to a Rees matrix semigroup in which every row and every column of the sandwich matrix contains a nonzero entry, and conversely, every such semigroup is completely 0-simple.

Let $S = M^0(G^0; I, \Lambda; P)$ be a Rees matrix semigroup, and let e be the unity of G. Define $\sim \subseteq S \times S$ by $(g; i, \lambda) \sim (h; j, \mu)$ if and only if $i = j$ and $\lambda = \mu$. Clearly, $\sim$ is a congruence on S, and $S' = S/\sim$ is isomorphic to the elementary Rees matrix semigroup $M^0(e^0; I, \Lambda; P')$ with $P' = (p'_{\lambda i})$, where $p'_{\lambda i} = 0$ if $p_{\lambda i} = 0$ and $p'_{\lambda i} = e$ otherwise. If $R = \sum_{(g;i,\lambda) \in S} R_{(g;i,\lambda)}$ is a contracted S-graded ring, then we may regard R as a contracted S'-graded ring with the components $R'_{(e;i,\lambda)} = \sum_{g \in G} R_{(g;i,\lambda)}$ (see Remark 1 in [7]). Obviously, each $R'_{(e;i,\lambda)}$ is a G-graded ring.

3. Results

The following result from [6] is frequently used throughout the article.

Proposition 3.1. (Proposition 2.7 in [6]) *Let S be a semigroup. Then, if the power graph $\mathcal{G}(S)$ is connected, S contains at most one idempotent element.*

The following lemma is clear but we include its proof for the sake of completeness.

Lemma 3.2. *Let S be a semigroup with a nonempty set of idempotent elements $E(S)$. If $e \in E(S)$, and if $\mathcal{G}(S)$ is connected, then s and e are adjacent in $\mathcal{G}(S)$ for every $e \neq s \in S$.*

Proof. Since $\mathcal{G}(S)$ is connected and $E(S)$ is nonempty, Proposition 3.1 tells us that S contains exactly one idempotent element, denote it by e. Let $s \neq e$ be an element of S. Since $\mathcal{G}(S)$ is connected, there exists a path between e and s, say $e, s_1, s_2, \ldots, s_k, s$. Since $s_1 \neq e$, and since s_1 and e are adjacent in $\mathcal{G}(S)$, there exists a positive integer n_1 such that $s_1^{n_1} = e$. Now, s_1 and s_2 are adjacent in $\mathcal{G}(S)$, so $s_1^{n_2'} = s_2$, for some positive integer n_2', or $s_2^{n_2''} = s_1$, for some positive integer n_2''. In any case, $s_2^{n_2} = e$, for some positive integer n_2. So, inductively, like in the proof of Proposition 2.7 in [6], we obtain that there exists a positive integer n such that $s^n = e$. Hence, s and e are adjacent in $\mathcal{G}(S)$. □

Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. Our interest in relating the connectedness of the power graphs of the semigroups of the ring components of R with the connectedness of the power graph of H_R stems out the following observation.

Theorem 3.3. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. Then the power graph $\mathcal{G}(H_R)$ is connected if and only if R is a graded-nil ring.*

Proof. Let the power graph $\mathcal{G}(H_R)$ be connected. By Proposition 3.1, the connectedness of $\mathcal{G}(H_R)$ implies that 0 is the only idempotent element of the semigroup H_R. Let $0 \neq x \in H_R$. According to Lemma 3.2, we have that x and 0 are adjacent in $\mathcal{G}(H_R)$. Hence, $x^n = 0$, for some positive integer n, that is, x is nilpotent. It follows that H_R is a nil semigroup. Hence, R is a graded-nil ring.

Conversely, if R is a graded-nil ring, then H_R is a nil semigroup. Hence, every vertex of $\mathcal{G}(H_R)$ is adjacent to 0. Therefore, the power graph $\mathcal{G}(H_R)$ is connected. □

The previous theorem holds in greater generality. Namely, we note that Lemma 3.2, together with Proposition 3.1, tells us in particular that the power graph of an arbitrary semigroup S with zero is connected if and only if S is a nil semigroup. We record this in the form of the following corollary.

Corollary 3.4. *Let S be a semigroup with zero. Then, $\mathcal{G}(S)$ is connected if and only if S is nil.*

There are many graded rings which are in particular graded-nil, provided that some conditions on their homogeneous subrings or on their grading sets are satisfied. For instance, results from Section 6.3 in [22], together with Theorem 3.3, give many examples of graded rings such that the power graphs of their homogeneous parts are connected.

Likewise, there are many examples of rings which are not graded-nil. For instance, group rings over fields, observed as group graded rings. However, according to Theorem 2.9 in [6], the power graph of a group G is connected if and only if G is periodic. Let F be a finite field, G a finite group, and $F[G]$ the group ring. Observe $F[G]$ as a G-graded ring $\bigoplus_{g\in G} Fg$, and put $H_{F[G]} = \bigcup_{g\in G} Fg$. Clearly, $H_{F[G]}$ is not a nil semigroup, and so, by Theorem 3.3, the power graph of $H_{F[G]}$ is not connected. On the other hand, $H^*_{F[G]} = H_{F[G]}\backslash\{0\}$ forms a group. Since $H^*_{F[G]}$ is finite, the power graph of $H^*_{F[G]}$ is connected. Let us take a look at this situation more closely in the general setting.

Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S, which has no nontrivial homogeneous right or left zero divisors. Then, if R_s and R_t are nonzero, we have that $R_s R_t \neq 0$, and so st exists. Therefore, in this case, there is no need to assume that zero is a component of R, and therefore, no need of introducing the zero element 0 into S. We simply index by S only the nonzero additive subgroups of R. Then, it follows that S is an associative groupoid, that is, a semigroup. Namely, $R_sR_tR_u \neq 0$ implies $(st)u = s(tu)$. Also, the set $H_R^* = H_R\backslash\{0\}$ is a semigroup under the multiplication of R. Since there are no nontrivial homogeneous right or left zero divisors, 0 is an isolated vertex in $\mathcal{G}(H_R)$. Hence, the power graph $\mathcal{G}(H_R)$ is not connected. However, as we have seen, the connectedness of the power graph $\mathcal{G}(H_R^*)$ is not seldom. Note that, in this case, $\mathcal{G}(H_R^*)$ coincides with the graph obtained from $\mathcal{G}(H_R)$ by removing the vertex 0. Now, let us assume that R is moreover a ring with unity 1, and that S is cancellative. Then we know that the set of all idempotent elements $E(S)$ of S is finite, the ring components R_e of R are rings with unities 1_e, and $1 = \sum_{e\in E(S)} 1_e$ (see for instance [19]).

Theorem 3.5. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S, which has no nontrivial homogeneous right or left zero divisors. Moreover, let R be with unity 1, and let S be cancellative. Then the power graph $\mathcal{G}(H_R^*)$ is connected if and only if the following conditions are satisfied*:

(i) *S is a monoid with unity e*;

(ii) *The power graph $\mathcal{G}(R_e^*)$ of the multiplicative semigroup $R_e^* = R_e\backslash\{0\}$ is connected*;

(iii) *The power graph $\mathcal{G}(S)$ is connected.*

Proof. Let the power graph $\mathcal{G}(H_R^*)$ be connected. Then H_R^* contains at most one idempotent element by Proposition 3.1. Hence, by the discussion preceding this theorem, since S is cancellative, and R is with unity 1, we have that S contains exactly one idempotent element e, and $1_e = 1$ is the only idempotent element in H_R^*. Therefore, e is the unity of S. We have already established that S is a semigroup in case R has no nontrivial homogeneous right or left zero divisors. Hence (i) holds.

Let x_e and y_e be distinct elements from R_e^*.

Case a. $1 \in \{x_e, y_e\}$. Without loss of generality, let $x_e = 1$. Since $\mathcal{G}(H_R^*)$ is connected, and 1 is the only idempotent element of H_R^*, by Lemma 3.2, we get that 1 and y_e are adjacent in $\mathcal{G}(R_e^*)$.

Case b. $1 \notin \{x_e, y_e\}$. Since $\mathcal{G}(H_R^*)$ is connected, like in the previous case, we obtain that x_e and 1 are adjacent in $\mathcal{G}(R_e^*)$ as well as are y_e and 1. Hence, there exists a path between x_e and y_e in $\mathcal{G}(R_e^*)$.

Therefore, the power graph $\mathcal{G}(R_e^*)$ is connected, that is, (ii) is satisfied.

Now we prove that the power graph $\mathcal{G}(S)$ is connected. Take arbitrary distinct elements s and t from S.

Case a. $e \in \{s, t\}$. Let $s = e$, and let $y \in R_t$ be a nonzero element. Since $\mathcal{G}(H_R^*)$ is connected, Lemma 3.2 implies that y and $1 \in R_e = R_s$ are adjacent in $\mathcal{G}(H_R^*)$. Hence, there exists a positive integer n such that $y^n = 1$. It follows that $\delta(y)^n = t^n = e$. Therefore, t and $e = s$ are adjacent in $\mathcal{G}(S)$.

Case b. $e \notin \{s, t\}$. By the previous case, $s^m = e$ and $t^n = e$, for some positive integers m and n. Therefore, there exists a path between s and t.

It follows that $\mathcal{G}(S)$ is connected, and so, (iii) holds as well.

Conversely, let us assume that conditions (i), (ii) and (iii) hold. Again, by the discussion preceding this theorem, the hypotheses on R and S, together with (i) imply that $1_e = 1$. Now, let x and y be distinct elements of H_R^*.

Case 1. $\delta(x) \neq \delta(y)$ and $1 \in \{x, y\}$. Let $x = 1$. Since $\mathcal{G}(S)$ is connected, Lemma 3.2 implies that there exists a positive integer n such that $\delta(y)^n = e$. Therefore, $y^n \in R_e^*$, since R has no nontrivial homogeneous right or left zero divisors. Since $\mathcal{G}(R_e^*)$ is connected by (ii), it follows, as before, that there exists a positive integer m such that $(y^n)^m = 1 = x$. So, x and y are adjacent in $\mathcal{G}(H_R^*)$.

Case 2. $\delta(x) \neq \delta(y)$ and $1 \notin \{x, y\}$. By the previous case, x is adjacent to 1 and 1 is adjacent to y in $\mathcal{G}(H_R^*)$. Hence, x and y are connected by a path in $\mathcal{G}(H_R^*)$.

Case 3. $\delta(x) = \delta(y)$. Let $\delta(x) = \delta(y) = s$. If $s = e$, we are done, since $\mathcal{G}(R_e^*)$ is connected by assumption. So, let us assume that $s \neq e$. By the connectedness of $\mathcal{G}(S)$, by Lemma 3.2, there exists a positive integer n such that $s^n = e$. Hence, x^n, $y^n \in R_e^*$. So, there exist x_e and $y_e \in R_e^*$ such that $x^n = x_e$ and $y^n = y_e$. However, $\mathcal{G}(R_e^*)$ is connected. Hence, by Lemma 3.2, $x_e^p = 1$ and $y_e^q = 1$ for some positive integers p and q. Therefore, x is adjacent to 1 and 1 is adjacent to y. Thus, x and y are connected by a path in $\mathcal{G}(H_R^*)$.

Hence, the power graph $\mathcal{G}(H_R^*)$ is connected. □

Let S be a semigroup, R a ring, and $R[S]$ a semigroup ring. By observing $R[S] = \bigoplus_{s \in S} Rs$ as an S-graded ring with the components Rs ($s \in S$), we put $H_{R[S]} = \bigcup_{s \in S} Rs$. If e is an idempotent element of S, then the rings Re and R are isomorphic. Therefore, the following corollary is immediate.

Corollary 3.6. *Let $R[S]$ be a semigroup ring over a domain R with unity, where S is a cancellative semigroup. Then the power graph $\mathcal{G}(H_{R[S]}^*)$ is connected if and only if the following conditions hold*:

(i) *S is a monoid*;
(ii) *The power graph $\mathcal{G}(R^*)$ of the multiplicative semigroup $R^* = R \backslash \{0\}$ is connected*;
(iii) *The power graph $\mathcal{G}(S)$ is connected.*

Remark 3.7. Under the assumptions of Theorem 3.5, we note that (iii) implies (i). Namely, if $\mathcal{G}(S)$ is connected, then, by Proposition 3.1, there exists at most one idempotent element in S. Since S is cancellative, by the discussion preceding Theorem 3.5, we have that the set $E(S)$ of all idempotent elements of S is finite, and that $1 = \sum_{e \in E(S)} 1_e$, where 1_e are unities of the ring components R_e. Therefore, S contains exactly one idempotent element $e \in S$, and $1_e = 1$. Hence, since S is a semigroup, it is a monoid with unity e. However, it is known that (i) does not imply (iii). For instance, by Theorem 2.9 in [6], the power graph of the additive group $\mathbb{Z}$ of integers is not connected.

If (iii) holds but (ii) does not, then, since $\mathcal{G}(R_e^*)$ is a subgraph of $\mathcal{G}(H_R^*)$, the graph $\mathcal{G}(H_R^*)$ cannot be connected. Moreover, (ii) without (iii) does not imply the connectedness of $\mathcal{G}(H_R^*)$. Namely, let us observe the group ring $\mathbb{F}_2[\mathbb{Z}]$ as a group graded ring $\bigoplus_{n \in \mathbb{Z}} \mathbb{F}_2 n$, where $\mathbb{F}_2$ is the field with two elements. Now, the graph $\mathcal{G}(\mathbb{F}_2^*)$ is connected, while $\mathcal{G}(\mathbb{Z})$ is not. Hence, (ii) holds but (iii) does not. We may invoke Lemma 3.2 to conclude that the power graph $\mathcal{G}(H_{\mathbb{F}_2[\mathbb{Z}]}^*)$ is not connected.

A particular case of S-graded rings inducing S with no nontrivial homogeneous right or left zero divisors is a graded division ring, that is a graded ring R for which H_R^* is a group (for instance, when R is a group ring over a field). In that case, S is a group as well (see for instance [33]). Therefore, the following corollary to Theorem 3.5 also holds.

Corollary 3.8. *Let R be an S-graded ring inducing S, which is a graded division ring, and let e be the unity of S. Then the power graph $\mathcal{G}(H_R^*)$ is connected if and only if the following conditions are satisfied*:

- (i) *The power graph $\mathcal{G}(R_e^*)$ of the multiplicative semigroup $R_e^* = R_e \setminus \{0\}$ is connected*;
- (ii) *The power graph $\mathcal{G}(S)$ is connected.*

Let us now return to Theorem 3.3. It turns out we can say more in the case of semigroup rings.

Theorem 3.9. *Let $R[S]$ be a semigroup ring. Then the following statements are equivalent*:

- (i) *The power graph $\mathcal{G}(H_{R[S]})$ is connected*;
- (ii) *The power graph $\mathcal{G}(R)$ of the multiplicative semigroup R is connected.*

Proof. (i) $\Rightarrow$ (ii) Since the power graph $\mathcal{G}(H_{R[S]})$ is connected, by Theorem 3.3, we have that $H_{R[S]}$ is a nil semigroup. In particular, for every idempotent element $e \in S$, it follows that Re is a nil multiplicative semigroup. Hence, the power graph $\mathcal{G}(Re)$ is connected by Corollary 3.4. Since the rings Re and R are isomorphic for every idempotent element $e \in S$, we get that $\mathcal{G}(R)$ is connected too.

(ii) $\Rightarrow$ (i) Corollary 3.4 tells us that the connectedness of the power graph $\mathcal{G}(R)$ implies that R is a nil semigroup as a multiplicative semigroup. It follows that $H_{R[S]}$ is a nil semigroup, that is, $R[S]$ is a graded-nil ring. Hence, according to Theorem 3.3, the power graph $\mathcal{G}(H_{R[S]})$ is connected. □

The situation in the case of S-graded rings inducing S in general is not that nice, even if S is a cancellative semigroup. Namely, let $R = \bigoplus_{s \in S} R_s$ be an S-graded ring inducing S, and let the power graph $\mathcal{G}(H_R)$ be connected. By Theorem 3.3, H_R is then a nil semigroup. In particular, for every idempotent element $e \in S$, the multiplicative semigroup R_e is nil. Therefore, by Corollary 3.4, the power graph $\mathcal{G}(R_e)$ is connected. However, the converse statement does not hold in general. For instance, let A be a ring which is not nil, and let x be an indeterminate. Then the ring $R = A[x]$ of polynomials without nonzero constant terms is a $\mathbb{Z}$-graded ring with the components $R_k = \{0\}$ if $k \leq 0$ and $R_k = Ax^k$ if $k > 0$. The power graph $\mathcal{G}(H_R)$ is not connected, yet $\mathcal{G}(R_0)$, as a graph consisting of a single vertex, is connected.

It is, therefore, natural to search for conditions which would guarantee the connectedness of the power graph $\mathcal{G}(H_R)$, provided that the power graphs $\mathcal{G}(R_e)$ are connected for all idempotent elements $e \in S$. We omit the trivial case of S being finite. So, from now on, we assume that the grading set is infinite.

Before we give desired conditions, we recall a few more notions and facts on S-graded rings inducing S. Throughout, $J(A)$ stands for the Jacobson radical of a ring A, that is, the intersection of all maximal modular right ideals of A.

The notion of an S-graded ring inducing S is equivalent to that of a graded ring studied in [9,10,33]. Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. A homogeneous right ideal I of R is said to be a *graded modular right ideal* [9] if there exists a homogeneous element $u \in R$, called a *left unity modulo* I, such that $ux - x \in I$ for every homogeneous element $x \in R$. If S is cancellative, and I a proper graded modular right ideal of R, then all of the left unities modulo I are of the same degree, which is an idempotent element of S, called the *degree of* I. If S is cancellative, then the *graded Jacobson radical* [9], denoted by $J^g(R)$, is equal to the intersection of all maximal graded modular right ideals of R. For the study of other radicals of S-graded rings inducing S, and related concepts, we refer the reader to [11–19,21,25] and references therein.

If $R = \bigoplus_{s\in S} R_s$ is an S-graded ring inducing S and $R' = \bigoplus_{s'\in S'} R_{s'}$ an S'-graded ring inducing S', then a mapping $\phi : R \to R'$ is called a *homogeneous homomorphism* [9,10,33] if it is a ring homomorphism such that $\phi(H_R) \subseteq H_{R'}$, and if $\delta(\phi(x)) = \delta(\phi(y)) \neq 0'$ implies that $\delta(x) = \delta(y)$ for all $x, y \in H_R$.

Theorem 3.10 [9,10]. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. If S is cancellative, then*:

- (i) *The mapping $I \mapsto I \cap R_e$ defines a one-to-one correspondence between the set of all maximal graded modular right ideals of R of degree e and the set of all maximal modular right ideals of the ring R_e. In particular, $J^g(R) \cap R_e = J(R_e)$ for every idempotent element $e \in S$;*
- (ii) *The largest homogeneous ideal $J_l^g(R)$ of R, contained in $J(R)$, is contained in $J^g(R)$. If $x \in J_l^g(R)$ is a homogeneous element, then there exists an idempotent element $e \in S$ and an integer n such that $x^n \in R_e$.*

Theorem 3.11. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S, and let S be cancellative. Moreover, let us assume that R has a unique maximal right ideal, and that $R_sR_t = 0$ whenever s and t are nonidempotent elements whose product st is an idempotent element of S. Then, the power graph $\mathcal{G}(H_R)$ is connected if and only if the power graph $\mathcal{G}(R_e)$ of the multiplicative semigroup R_e is connected for every idempotent element $e \in S$.*

Proof. If the power graph $\mathcal{G}(H_R)$ is connected, we have already established that the power graphs $\mathcal{G}(R_e)$ are connected, where e runs through the set of all idempotent elements of S.

Now, let e be an idempotent element of S, and let the power graph $\mathcal{G}(R_e)$ be connected. Then, it follows from Corollary 3.4 that R_e is nil. We claim that then R is graded-nil under the given hypotheses. Let M be a unique maximal right ideal of R. Since S is cancellative and since R_e is nil for every idempotent element $e \in S$, it is clear that the ring R is without a unity (see for instance [19]).

Case 1. M is not a modular right ideal of R. Then there are no maximal modular right ideals of R. Hence, $R = J(R)$, and by Theorem 3.10, we then have that $J_l^g(R) = J^g(R) = J(R) = R$. Take an arbitrary homogeneous element $x \in R$. By Theorem 3.10, there exists an idempotent element $e \in S$ and an integer n such that $x^n \in R_e$. Since R_e is nil, x is nilpotent. Therefore, R is graded-nil.

Case 2. M is a modular right ideal of R. Then the Jacobson radical $J(R) = M$ is a unique maximal modular right ideal of R. Let I be a maximal graded modular right ideal of R. If it is maximal as a right ideal of R, then $I = J(R)$. Otherwise, the maximal ideal of R which contains I is also a modular right ideal of R, and hence, coincides with $J(R)$. Therefore, in any case, $I \subseteq J(R)$. It follows that all of the maximal graded modular right ideals of R are contained in $J(R)$. Hence, $J^g(R) \subseteq J(R)$. In particular, since the largest homogeneous ideal $J_l^g(R)$ of R, contained in $J(R)$, is contained in $J^g(R)$ by Theorem 3.10(ii), it follows that $J^g(R) = J_l^g(R)$. Therefore, $J^g(R)$ is a maximal homogeneous ideal of R, and a unique maximal graded modular right ideal of R of degree, say $e \in S$.

Now, having in mind that $(R/J^g(R))_e = R_e/J(R_e)$, let $\phi : R/J^g(R) \to R_e/J(R_e)$ be the projection mapping, that is, the mapping defined by $\phi(\overline{x}) = x + J(R_e)$ if $x \in R_e$, and $\phi(\overline{x}) = 0 + J(R_e)$ if $x \notin R_e$ ($\overline{x} \in H_{R/J^g(R)}$). This mapping is well defined. Moreover, ϕ is a surjective homogeneous homomorphism due to our assumption that $R_sR_t = 0$ whenever s and t are nonidempotent elements whose product st is an idempotent element of S (cf. the proof of Theorem 3.2 in [14]). Since $J^g(R)$ is a maximal homogeneous ideal of R, the ring $R/J^g(R)$ has no nontrivial homogeneous ideals. It follows that either $\ker\phi = 0$ or $\ker\phi = R/J^g(R)$. If $\ker\phi = R/J^g(R)$, then $R_e = J(R_e)$. However, this is impossible by Theorem 3.10(i), since $J^g(R) \neq R$. Namely, $R_e = J(R_e)$ implies that a left unity modulo $J^g(R)$ belongs to $J^g(R)$, that is, $J^g(R) = R$. Hence, $\ker\phi = 0$, and so, ϕ is a homogeneous homomorphism which is both injective and surjective. Therefore, every homogeneous element from $R/J^g(R)$ can be identified with a unique element from $R_e/J(R_e)$, and vice-versa. Thus, if $s \in S$ is a nonidempotent element of S, and $x \in R_s$, it follows that $x \in J^g(R)$. However, $J^g(R) = J_l^g(R)$, and so, by Theorem 3.10(ii) there exists an integer n such that $x^n \in R_f$ for some idempotent element $f \in S$. Since R_f is nil, we have that x is a nilpotent element of R. Hence, R is graded-nil.

Thus, in both cases, R is graded-nil, and so, Theorem 3.3 tells us that the power graph $\mathcal{G}(H_R)$ is indeed connected. □

Remark 3.12. Let us note that if R is an S-graded ring inducing S, where S is cancellative, the assumption that $R_sR_t = 0$ whenever s and t are nonidempotent elements whose product st is an idempotent element of S, is equivalent to assuming that R is an S'-graded ring inducing S', where S' is cancellative, and such that the product of nonidempotent elements of S' cannot be a nonzero idempotent element of S'. One example of such a groupoid is the multiplicative semigroup of nonnegative integers.

Also, one cannot remove the assumption of $R = \bigoplus_{s\in S} R_s$ having a unique maximal right ideal as it can be seen from the already given polynomial ring example.

Corollary 3.13. *Let G be a group with unity e, and let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. Moreover, let us assume that R has a maximal right ideal, and that $R_g R_{g^{-1}} = 0$ for every $g \in G\backslash\{e\}$. Then, the power graph $\mathcal{G}(H_R)$ is connected if and only if the power graph $\mathcal{G}(R_e)$ of the multiplicative semigroup R_e is connected.*

Proof. This follows immediately from the previous theorem since R is a contracted G^0-graded ring. □

The final result of this article shows that the cancellativity of the grading set is not necessary to have the equivalence in terms of connectedness between the power graph of the semigroup of homogeneous elements and the power graphs of the semigroups of the ring components.

Recall that an *epigroup* is a semigroup in which some power of any element is contained in a subgroup of the given semigroup (see for instance [22]).

Theorem 3.14. *Let S be an epigroup with a finite number of idempotent elements, and let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring. Moreover, let us assume that for every nontrivial subgroup G of S, a nonzero ring $R_G = \bigoplus_{s\in G} R_s$ has a unique maximal right ideal, and that $R_s R_{s^{-1}} = 0$ whenever $s \in G\backslash\{e\}$, where e is the unity of G. Then the power graph $\mathcal{G}(H_R)$ is connected if and only if the power graph $\mathcal{G}(R_e)$ of the multiplicative semigroup R_e is connected for every idempotent element $e \in S$.*

Proof. If $\mathcal{G}(H_R)$ is connected, then we already know that $\mathcal{G}(R_e)$ is connected for every idempotent element $e \in S$.

Let us assume that $\mathcal{G}(R_e)$ is connected for every idempotent element $e \in S$. Then the ring R_e is nil for every idempotent element $e \in S$. According to Theorem 3.3, it is enough to prove that R is graded-nil, under the given hypotheses. As we know, since S is an epigroup with a finite number of idempotent elements, S^0 has a finite ideal chain

$$S^0 = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_m \supseteq S_{m+1} = \{0\},$$

such that for each i, we have that S_i/S_{i+1} is a completely 0-simple semigroup (with a finite sandwich matrix) or a nil semigroup, see for instance Theorem 1.9 in [22]. Let us observe R as a contracted S^0-graded ring. For every i we know that $R_{S_{i+1}}$ is an ideal of R_{S_i} and that $R_{S_i}/R_{S_{i+1}}$ is a contracted S_i/S_{i+1}-graded ring. Moreover, the ring components of $R_{S_i}/R_{S_{i+1}}$ are the ring components of R corresponding to idempotent elements of $S_i\backslash S_{i+1}$. Generally, a homogeneous subring of $R_{S_i}/R_{S_{i+1}}$ which corresponds to a subgroup G of S_i/S_{i+1} is the homogeneous subring R_G of R. In case S_i/S_{i+1} is a nil semigroup, we have that $R_{S_i}/R_{S_{i+1}}$ is graded-nil as a contracted S_i/S_{i+1}-graded ring. In case S_i/S_{i+1} is a completely 0-simple semigroup, it is isomorphic to a Rees matrix semigroup $M^0(G^0; I, \Lambda; P)$. We claim that $R_{S_i}/R_{S_{i+1}}$ is graded-nil as a contracted S_i/S_{i+1}-graded ring in this case too.

Case I. G^0 is a trivial group with zero, that is, $M^0(G^0; I, \Lambda; P)$ is an elementary Rees matrix semigroup. Then, $R_{S_i}/R_{S_{i+1}}$ may be observed as a ring graded by the rectangular band $I \times \Lambda$. Hence, all of its components are the ring components. Therefore, by the above discussion, and by our assumption on the rings R_e, where e runs through the set of idempotent elements of S, we have that $R_{S_i}/R_{S_{i+1}}$ is a graded-nil ring.

Case II. G^0 is not a trivial group with zero. Then, we may observe $R_{S_i}/R_{S_{i+1}}$ as an elementary Rees matrix semigroup graded ring all of whose components are G-graded rings, as explained in the preliminaries. Denote this elementary Rees matrix semigroup by M_i. Hence, by the hypothesis, each component of an M_i-graded ring $R_{S_i}/R_{S_{i+1}}$, if distinct from the zero ring, has a unique maximal right ideal, and the product of its non-ring components that correspond to mutually inverse elements of G is zero. Now, Corollary 3.13 implies that each nonzero component of an M_i-graded ring $R_{S_i}/R_{S_{i+1}}$ is graded-nil. Of course, each zero ring component of an M_i-graded ring $R_{S_i}/R_{S_{i+1}}$ is graded-nil too. So, it follows that $R_{S_i}/R_{S_{i+1}}$ is graded-nil as a contracted S_i/S_{i+1}-graded ring.

Hence, indeed, if S_i/S_{i+1} is a completely 0-simple semigroup, $R_{S_i}/R_{S_{i+1}}$ is graded-nil as a contracted S_i/S_{i+1}-graded ring.

Now, let $s \in S$ be a nonidempotent element, and let $x \in R_s$. We claim that x is nilpotent. So, we may assume that $x \neq 0$. Now, $x + R_{S_2} \in R_{S_1}/R_{S_2}$. It may be that $x \in R_{S_2}$ (if $s \in S_2$). Suppose $x \notin R_{S_2}$. If S_1/S_2 is a nil semigroup, then $x^{n_1'} \in R_{S_2}$ for some integer n_1'. Otherwise, S_1/S_2 is a Rees matrix semigroup, and, as we concluded above for an arbitrary i, we have that R_{S_1}/R_{S_2} is graded-nil as a contracted S_1/S_2-graded ring. Therefore, $x^{n_1''} \in R_{S_2}$ for some integer n_1''. Hence, in all cases, $x^{n_1} \in R_{S_2}$ for some integer n_1. If $x^{n_1} = 0$, we are done. Otherwise, we may assume that $x^{n_1} + R_{S_3}$ is either a homogeneous element of the ring R_{S_2}/R_{S_3} of a nonidempotent degree or $x^{n_1} \in R_{S_3}$. Namely, if s^{n_1} were an idempotent element of S, then x would be a nilpotent element, and we would be done. Suppose $x^{n_1} \notin R_{S_3}$. Now, S_2/S_3 is either a nil semigroup or a Rees matrix semigroup. By repeating the process, we obtain that $(x^{n_1})^{n_2} \in R_{S_3}$ for some integer n_2, and so on. Eventually we arrive at $x^n \in R_{S_{m+1}}$, for some integer n, that is, $x^n = 0$. Hence, R is graded-nil as an S-graded ring, which completes the proof. $\square$

References

[1] Abawajy, J., Kelarev, A., Chowdhury, M.: Power graphs: a survey. Electron. J. Graph Theory Appl. **1**(2), 125–147 (2013)

[2] Bera, S., Dey, H.K., Mukheryee, S.K.: On the connectivity of enhanced power graphs of finite groups. Graphs Comb. **37**, 591–603 (2021)

[3] Cameron, P.J.: The power graph of a finite group II. J. Group Theory **13**, 779–783 (2010)

[4] Cameron, P.J., Ghosh, S.: The power graph of a finite group. Discret. Math. **311**, 1220–1222 (2011)

[5] Cameron, P.J., Jafari, S.H.: On the connectivity and independence number of power graphs of groups. Graphs Comb. **36**, 895–904 (2020)

[6] Chakrabarty, I., Ghosh, S., Sen, M.K.: Undirected power graphs of semigroups. Semigroup Forum **78**, 410–426 (2009)

[7] Clase, M.V., Jespers, E.: On the Jacobson radical of semigroup graded rings. J. Algebra **169**, 79–97 (1994)

[8] Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups, vols. I and II. American Mathematical Society, Providence (1961 and 1967)

[9] Halberstadt, E.: Le radical d'un anneide régulier. C. R. Acad. Sci. Paris Sér. A **270**, 361–363 (1970)

[10] Halberstadt, E.: Théorie artinienne homogène des anneaux gradués à grades non commutatifs réguliers. PhD Thesis, University Piere and Marie Curie, Paris, France (1971)

[11] Ilić-Georgijević, E.: On graded Ω-groups. Filomat **29**(10), 2167–2183 (2015)

[12] Ilić-Georgijević, E.: On graded Brown–McCoy radicals of graded rings. J. Algebra Appl. **15**(8), 16501437, 13 (2016). arXiv:1904.00033

[13] Ilić-Georgijević, E.: On graded Thierrin radicals of graded rings. Commun. Algebra **45**(9), 3886–3891 (2017)

[14] Ilić-Georgijević, E.: On graded special radicals of graded rings. J. Algebra Appl. **17**(6), 1850109, 10 (2018)

[15] Ilić-Georgijević, E.: On the Jacobson and simple radicals of semigroups. Filomat **32**(7), 2577–2582 (2018)

[16] Ilić-Georgijević, E.: On radicals of graded ring constructions. In: Leroy, A., et al. (eds.) Rings, Modules and Codes. Contemporary Mathematics, vol. 727, pp. 167–176. American Mathematical Society, Providence (2019)

[17] Ilić-Georgijević, E.: Notes on the graded Jacobson radical: a graded version of the Jacobson stable set. Commun. Algebra **48**(6), 2624–2631 (2020)

[18] Ilić-Georgijević, E.: A description of the Cayley graphs of homogeneous semigroups. Commun. Algebra **48**(12), 5203–5214 (2020)

[19] Ilić-Georgijević, E.: On the Jacobson radical of a groupoid graded ring. J. Algebra **573**, 561–575 (2021)

[20] Ilić-Georgijević, E.: On transitive Cayley graphs of pseudo-unitary homogeneous semigroups. Commun. Algebra **49**(11), 4915–4923 (2021)

[21] Kelarev, A.V.: On groupoid graded rings. J. Algebra **178**, 391–399 (1995)

[22] Kelarev, A.V.: Ring Constructions and Applications. Series in Algebra, vol. 9. World Scientific, London (2002)

[23] Kelarev, A.V.: Graph Algebras and Automata. Pure and Applied Mathematics, vol. 257. Marcel Dekker, New York (2003)

[24] Kelarev, A.V.: Labelled Cayley graphs and minimal automata. Australas. J. Comb. **30**, 95–101 (2004)

[25] Kelarev, A.V., Plant, A.: Bergman's lemma for graded rings. Commun. Algebra **23**(12), 4613–4624 (1995)

[26] Kelarev, A.V., Praeger, C.E.: On transitive Cayley graphs of groups and semigroups. Eur. J. Comb. **24**(1), 59–72 (2003)

[27] Kelarev, A.V., Quinn, S.J.: A combinatorial property and power graphs of groups. In: Dorninger, D., et al. (eds.) Contributions to General Algebra, vol. 12, pp. 229–235. Springer, Berlin (2000)

[28] Kelarev, A.V., Quinn, S.J.: Directed graphs and combinatorial properties of semigroups. J. Algebra **251**, 16–26 (2002)

[29] Kelarev, A.V., Quinn, S.J.: A combinatorial property and power graphs of semigroups. Comment. Math. Univ. Carol. **45**, 1–7 (2004)

[30] Kelarev, A.V., Quinn, S.J., Smolikova, R.: Power graphs and semigroups of matrices. Bull. Aust. Math. Soc. **63**, 341–344 (2001)

[31] Kelarev, A.V., Ryan, J., Yearwood, J.L.: Cayley graphs as classifiers for data mining: the influence of asymmetries. Discret. Math. **309**(17), 5360–5369 (2009)

[32] Khosravi, B., Praeger, C.E.: Normal edge-transitive Cayley graphs and Frattini-like subgroups. J. Algebra (2021). https://doi.org/10.1016/j.jalgebra.2021.03.035

[33] Krasner, M.: Anneaux gradués généraux. Publications mathématiques et informatiques de Rennes, Colloque d'algebre **S3**, 209–308 (1980)

[34] Ma, X.: Perfect codes in proper reduced power graphs of finite groups. Commun. Algebra **48**(9), 3881–3890 (2020)

[35] Panda, R.P., Krishna, K.V.: On the minimum degree, edge-connectivity and connectivity of power graphs of finite groups. Commun. Algebra **46**(7), 3182–3197 (2018)

[36] Panda, R.P., Dalal, S., Kumar, J.: On the enhanced power graph of a finite group. Commun. Algebra **49**(4), 1697–1716 (2021)

[37] Zahirović, S., Bošnjak, I., Madarász, R.: A study of enhanced power graphs of finite groups. J. Algebra Appl. **19**(4), 2050062 (2020)

[38] Zhang, J., Zhou, S.: On subgroup perfect codes in Cayley graphs. Eur. J. Comb. **91**, 103228 (2021)

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