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Generating the Mapping Class Group by Two Torsion Elements

Oğuz Yıldız

Abstract. We prove that the mapping class group of a closed connected orientable surface of genus $g \geq 6$ is generated by two elements of order g. Moreover, for $g \geq 7$, we obtain a generating set of two elements, of order g and g', where g' is the least divisor of g greater than 2. We also prove that the mapping class group is generated by two elements of order $g/\gcd(g,k)$ for $g \geq 3k^2 + 4k + 1$ and any positive integer k.

Mathematics Subject Classification. 20F65.

1. Introduction

The mapping class group $\mathrm{Mod}(\Sigma_g)$ of a closed, connected orientable surface Σ_g is the group of orientation-preserving diffeomorphisms of $\Sigma_g \to \Sigma_g$ up to isotopy. Dehn [3] showed that $\mathrm{Mod}(\Sigma_g)$ is generated by $2g(g-1)$ many Dehn twists. Afterwards, Lickorish [12] decreased this number to $3g-1$. Humphries [6] introduced a generating set consisting of $2g+1$ many Dehn twists and proved that this is the least such number.

Note that, the above-generating sets contain only elements of infinite order. Maclachlan [15] proved that $\mathrm{Mod}(\Sigma_g)$ can also be generated by only using torsions. Wajnryb [20] proved that $\mathrm{Mod}(\Sigma_g)$ can be generated by two elements; one of order $4g+2$ and the other a product of opposite Dehn twists. In this paper, we study the problem of generating $\mathrm{Mod}(\Sigma_g)$ by two torsion elements of small orders. Korkmaz [8] found a generating set for $\mathrm{Mod}(\Sigma_g)$ consisting of two torsion elements of order $4g+2$. He also posed the following problem [10]: for which $k < 4g+2$, $\mathrm{Mod}(\Sigma_g)$ can be generated by two elements of order k (A similar question is also asked by Margalit [16])? In particular, what is the smallest such k?

We first prove that $\mathrm{Mod}(\Sigma_g)$ is generated by two elements of order g if $g \geq 6$.

Theorem 1. *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by two elements of order* g *for* $g \geq 6$.

We also obtain generating sets consisting of the elements of smaller orders.

Theorem 2. *For $g \geq 7$ the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by two elements of order g and order g' where g' is the least divisor of g such that $g' > 2$.*

Theorem 3. *For $g \geq 3k^2 + 4k + 1$ and any positive integer k, the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by two elements of order $g/\gcd(g,k)$.*

Since there is a surjective homomorphism from $\mathrm{Mod}(\Sigma_g)$ onto the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$, we have the following immediate result:

Corollary 4. *The symplectic group* $\mathrm{Sp}(2g,\mathbb{Z})$ *is generated by two elements of order g for $g \geq 6$.*

See [2,7,15,17] or [14] for generating sets consisting of involutions, [11, 13,18] or [4] for generating sets consisting of torsions and [19] or [1] for other generating sets for the mapping class groups.

2. Preliminaries

Throughout the paper, we always consider Σ_g, where all genera are depicted as in Fig. 1. Note that the rotation by $2\pi/g$ degrees about z-axis, denoted by R, is a well-defined self-diffeomorphism of Σ_g. Following the notation in [21], we denote simple closed curves by lowercase letters a_i, b_i, c_i and corresponding positive Dehn twists by uppercase letters A_i, B_i, C_i or with the usual notation t_{a_i},t_{b_i},t_{c_i}, respectively. All indices should be considered modulo g. For the composition of diffeomorphisms, $f_1 f_2$ means that f_2 is first and then f_1 comes second as usual.

Commutativity, braid relation and the following basic facts on the mapping class group are used throughout the paper for many times: For any simple closed curves c_1 and c_2 on Σ_g and diffeomorphism $f : \Sigma_g \to \Sigma_g$, $f t_{c_1} f^{-1} = t_{f(c_1)}$; c_1 is isotopic to c_2 if and only if $t_{c_1} = t_{c_2}$ in $\mathrm{Mod}(\Sigma_g)$; and if c_1 and c_2 are disjoint, then $t_{c_1}(c_2) = c_2$. We always refer to [5] for all the remaining properties of the mapping class groups.

Now, let us present Humphries minimal generating set for $\mathrm{Mod}(\Sigma_g)$:

Theorem 5. (Dehn–Lickorish–Humphries) *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the set* $\{A_1, A_2, B_1, B_2, \ldots, B_g, C_1, C_2, \ldots, C_{g-1}\}$.

It is easy to see that the rotation R satisfies that $R(a_k) = a_{k+1}$, $R(b_k) = b_{k+1}$ and $R(c_k) = c_{k+1}$. Deducing from Theorem 5, Korkmaz [9] showed that the mapping class group is generated by four elements. Note that his first element is the rotation R and others are products of one positive and one negative Dehn twists.

Theorem 6. *If $g \geq 3$, then the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the four elements* $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}$.

The next result easily follows from Theorem 6.

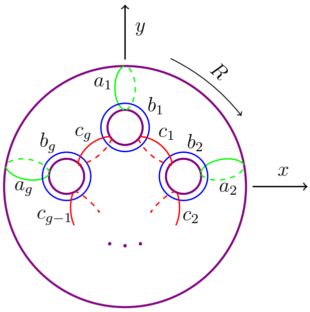

Figure 1. The curves a_i, b_i, c_i and the rotation R on the surface Σ_g

Corollary 7. *If* $g \geq 3$*, then the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the four elements* $R, A_1B_1^{-1}, B_1C_1^{-1}, C_1B_2^{-1}$.

Proof. Let H be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, A_1B_1^{-1}, B_1C_1^{-1}, C_1B_2^{-1}\}$.

It is enough to show that H contains the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$ by Theorem 6.

It is easy to see that $B_2A_2^{-1} \in H$ since $B_2A_2^{-1} = R(B_1A_1^{-1})R^{-1} \in H$ and $B_2C_2^{-1} = R(B_1C_1^{-1})R^{-1} \in H$.

One can also show that $B_1B_2^{-1} = (B_1C_1^{-1})(C_1B_2^{-1}) \in H$. Similarly, we have that $C_1C_2^{-1} = (C_1B_2^{-1})(B_2C_2^{-1}) \in H$ and we also have that $A_1A_2^{-1} = (A_1B_1^{-1})(B_1B_2^{-1})(B_2A_2^{-1}) \in H$.

It follows from Theorem 6 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. □

3. Twelve New Generating Sets for $\mathrm{Mod}(\Sigma_g)$

In this section, we introduce twelve new generating sets consisting of two elements of small orders for the mapping class group. Following the ideas in [9], we construct generating sets consisting of R, an element of order g, and another element which can be expressed as a product of Dehn twists.

The corollaries in this section are mainly the corollaries of Theorem 6. We use the first four corollaries to create generating sets of elements of order g. We use Corollaries 12, 13, 14, 15, 16 and 20 to create generating sets of elements of order g and g', where g' is the least divisor of g greater than 2. In the following, we give four new generating sets to prove Theorem 1.

Corollary 8. *If* $g = 6$*, then the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the two elements* R *and* $C_1B_4A_6A_1^{-1}B_5^{-1}C_2^{-1}$.

Proof. Let $F_1 = C_1B_4A_6A_1^{-1}B_5^{-1}C_2^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_6)$ generated by the set $\{R, F_1\}$.

If H contains the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$, then we are done by Theorem 6 (Fig. 2).

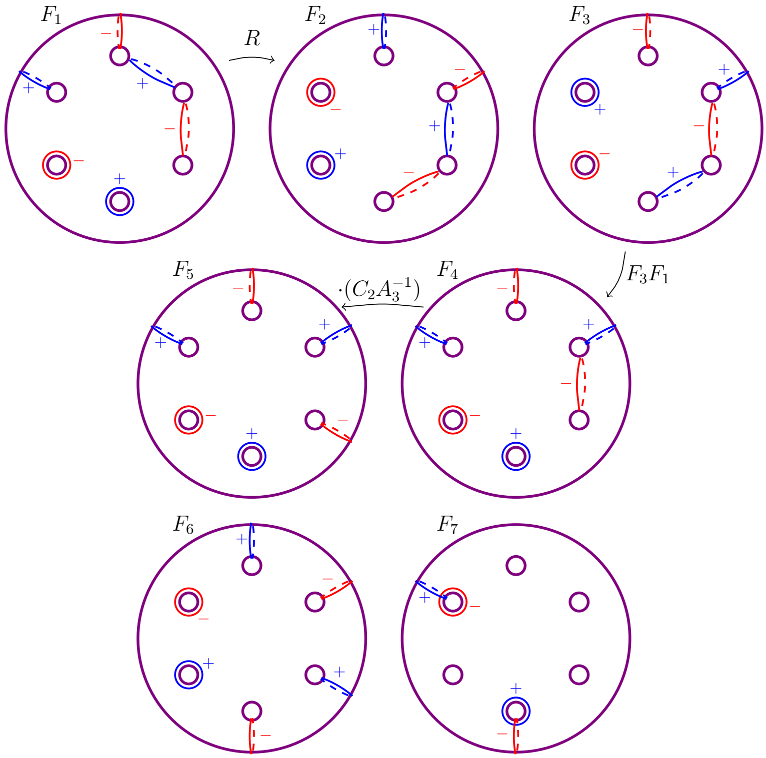

Figure 2. Proof of Corollary 8

Let

$$
\begin{aligned}
F_2 &= RF_1R^{-1}\\
&= R(C_1B_4A_6A_1^{-1}B_5^{-1}C_2^{-1})R^{-1}\\
&= RC_1R^{-1}RB_4R^{-1}RA_6R^{-1}RA_1^{-1}R^{-1}RB_5^{-1}R^{-1}RC_2^{-1}R^{-1}\\
&= Rt_{c_1}R^{-1}Rt_{b_4}R^{-1}Rt_{a_6}R^{-1}Rt_{a_1}^{-1}R^{-1}Rt_{b_5}^{-1}R^{-1}Rt_{c_2}^{-1}R^{-1}\\
&= t_{R(c_1)}t_{R(b_4)}t_{R(a_6)}t_{R(a_1)}^{-1}t_{R(b_5)}^{-1}t_{R(c_2)}^{-1}\\
&= t_{c_2}t_{b_5}t_{a_1}t_{a_2}^{-1}t_{b_6}^{-1}t_{c_3}^{-1}\\
&= C_2B_5A_1A_2^{-1}B_6^{-1}C_3^{-1}
\end{aligned}
$$

and

$$F_3 = F_2^{-1} = C_3B_6A_2A_1^{-1}B_5^{-1}C_2^{-1}.$$

We have $F_3F_1(c_3, b_6, a_2, a_1, b_5, c_2) = (b_4, a_6, a_2, a_1, b_5, c_2)$ so that $F_4 = B_4A_6A_2A_1^{-1}B_5^{-1}C_2^{-1} \in H$. Note that $F_3F_1(c_3) = b_4$ since

$$\begin{aligned} t_{F_3F_1(c_3)} &= (F_3F_1)t_{c_3}(F_3F_1)^{-1} \\ &= F_3F_1C_3F_1^{-1}F_3^{-1} \\ &= C_3B_4C_3B_4^{-1}C_3^{-1} \\ &= (t_{c_3}t_{b_4})t_{c_3}(t_{c_3}t_{b_4})^{-1} \\ &= t_{t_{c_3}t_{b_4}(c_3)} \\ &= t_{b_4}. \end{aligned}$$

We get $F_1F_4^{-1} = C_1A_2^{-1} \in H$. Hence, by conjugating $C_1A_2^{-1}$ with R iteratively, we get $C_iA_{i+1}^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_5 &= F_4(C_2A_3^{-1}) = B_4A_6A_2A_1^{-1}B_5^{-1}A_3^{-1}, \\ F_6 &= RF_5R^{-1} = B_5A_1A_3A_2^{-1}B_6^{-1}A_4^{-1} \end{aligned}$$

and

$$F_7 = F_5F_6 = B_4A_6B_6^{-1}A_4^{-1}.$$

Hence, $(C_4A_5^{-1})F_7(c_4, a_5) = (b_4, a_5)$ so that $B_4A_5^{-1} \in H$. We then get $B_iA_{i+1}^{-1} \in H$ for all i and $B_iC_i^{-1} = (B_iA_{i+1}^{-1})(A_{i+1}C_i^{-1}) \in H$ for all i.

Similarly, we see that $(A_4B_3^{-1})F_7(a_4, b_3) = (b_4, b_3)$ so that $B_4B_3^{-1} \in H$ implying that $B_{i+1}B_i^{-1} \in H$ for all i. In particular, we get $B_1B_2^{-1} \in H$.

Finally, we have $C_1C_2^{-1} = (C_1B_1^{-1})(B_1B_2^{-1})(B_2C_2^{-1}) \in H$ and $A_1A_2^{-1} = (A_1B_6^{-1})(B_6B_1^{-1})(B_1A_2^{-1}) \in H$.

It follows from Theorem 6 that $H = \mathrm{Mod}(\Sigma_6)$, completing the proof of the corollary. □

Corollary 9. *If $g = 7$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $C_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}$ Fig. 3.*

Proof. Let $F_1 = C_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}$. Let H denote the subgroup of $\mathrm{Mod}(\Sigma_7)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = C_2B_5A_7A_1^{-1}B_6^{-1}C_3^{-1}$$

and

$$F_3 = F_2^{-1} = C_3B_6A_1A_7^{-1}B_5^{-1}C_2^{-1}.$$

We have $F_3F_1(c_3, b_6, a_1, a_7, b_5, c_2) = (b_4, a_6, a_1, a_7, b_5, c_2)$ so that $F_4 = B_4A_6A_1A_7^{-1}B_5^{-1}C_2^{-1} \in H$.

Let

$$F_5 = RF_4R^{-1} = B_5A_7A_2A_1^{-1}B_6^{-1}C_3^{-1}$$

and

$$F_6 = F_5^{-1} = C_3B_6A_1A_2^{-1}A_7^{-1}B_5^{-1}.$$

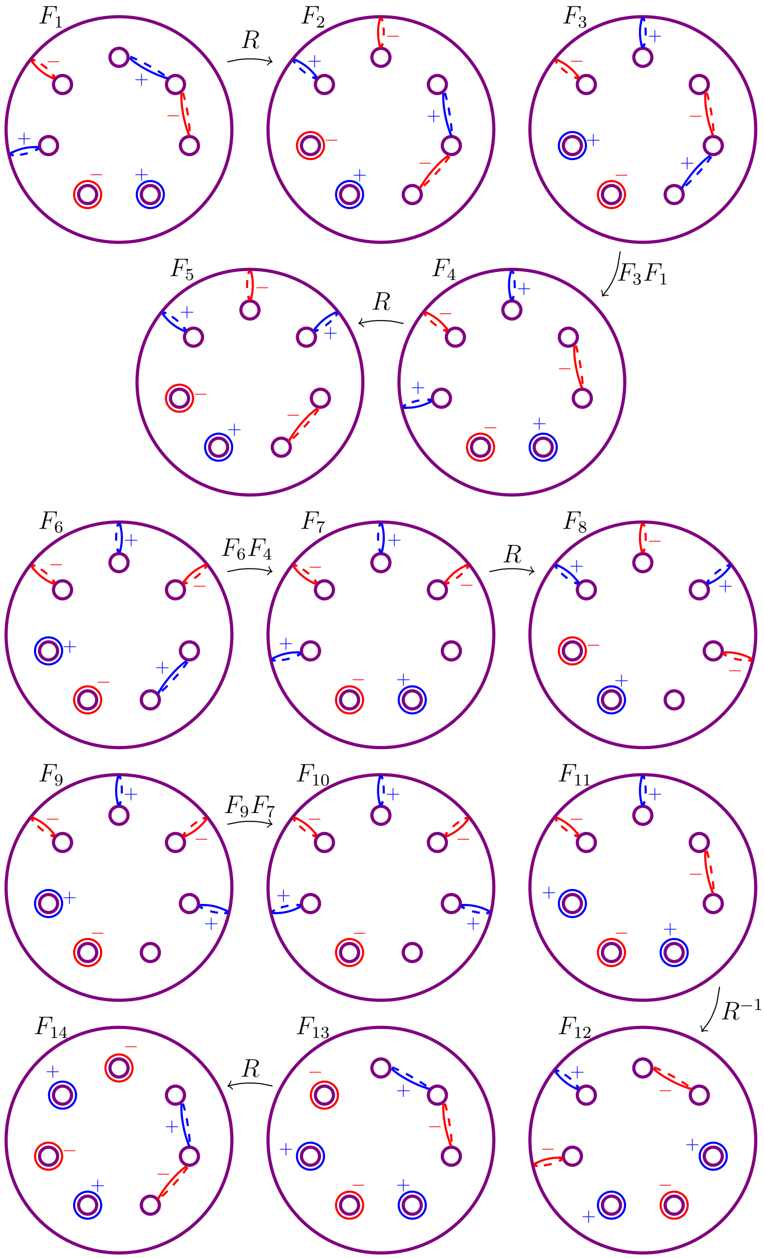

Figure 3. Proof of Corollary 9

We get $F_6F_4(c_3, b_6, a_1, a_2, a_7, b_5) = (b_4, a_6, a_1, a_2, a_7, b_5)$ so that $F_7 = B_4A_6A_1A_2^{-1}A_7^{-1}B_5^{-1} \in H$.

Let

$$F_8 = RF_7R^{-1} = B_5A_7A_2A_3^{-1}A_1^{-1}B_6^{-1}$$

and

$$F_9 = F_8^{-1} = B_6 A_1 A_3 A_2^{-1} A_7^{-1} B_5^{-1}.$$

Hence, we have $F_9 F_7(b_6, a_1, a_3, a_2, a_7, b_5) = (a_6, a_1, a_3, a_2, a_7, b_5)$ so that $F_{10} = A_6 A_1 A_3 A_2^{-1} A_7^{-1} B_5^{-1} \in H$.

We then see that $F_{10} F_8 = A_6 B_6^{-1} \in H$ and by conjugating $A_6 B_6^{-1}$ with R iteratively, we get $A_i B_i^{-1} \in H$ for all i.

Let

$$F_{11} = (B_6 A_6^{-1}) F_4 = B_4 B_6 A_1 A_7^{-1} B_5^{-1} C_2^{-1}$$

and

$$F_{12} = R^{-1} F_{11} R = B_3 B_5 A_7 A_6^{-1} B_4^{-1} C_1^{-1}.$$

We also have $F_{12} F_1 = B_3 C_2^{-1} \in H$ and then $B_{i+1} C_i^{-1} \in H$ for all i.

Let

$$F_{13} = (B_6 A_6^{-1}) F_1 (A_7 B_7^{-1}) = C_1 B_4 B_6 B_7^{-1} B_5^{-1} C_2^{-1}$$

and

$$F_{14} = R F_{13} R^{-1} = C_2 B_5 B_7 B_1^{-1} B_6^{-1} C_3^{-1}.$$

Finally, $F_{13} F_{14} (C_3 B_4^{-1}) = C_1 B_1^{-1} \in H$ which gives $C_i B_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_7)$, which finishes the proof. □

Corollary 10. *If $g = 8$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $B_1 C_4 A_7 A_8^{-1} C_5^{-1} B_2^{-1}$ Fig. 4.*

Proof. Let $F_1 = B_1 C_4 A_7 A_8^{-1} C_5^{-1} B_2^{-1}$ and let H be the subgroup of $\mathrm{Mod}(\Sigma_8)$ generated by the set $\{R, F_1\}$.

Let us consider the elements

$$F_2 = R F_1 R^{-1} = B_2 C_5 A_8 A_1^{-1} C_6^{-1} B_3^{-1}$$

and

$$F_3 = F_2^{-1} = B_3 C_6 A_1 A_8^{-1} C_5^{-1} B_2^{-1}.$$

We have $F_3 F_1(b_3, c_6, a_1, a_8, c_5, b_2) = (b_3, c_6, b_1, a_8, c_5, b_2)$ so that $F_4 = B_3 C_6 B_1 A_8^{-1} C_5^{-1} B_2^{-1} \in H$.

We get that $F_4 F_3^{-1} = B_1 A_1^{-1} \in H$ and then by conjugating $B_1 A_1^{-1}$ with R iteratively, we get $B_i A_i^{-1} \in H$ for all i.

Let

$$F_5 = R^2 F_1 R^{-2} = B_3 C_6 A_1 A_2^{-1} C_7^{-1} B_4^{-1},$$
$$F_6 = F_5^{-1} = B_4 C_7 A_2 A_1^{-1} C_6^{-1} B_3^{-1}$$

and

$$F_7 = (B_2 A_2^{-1}) F_6 (A_1 B_1^{-1}) = B_4 C_7 B_2 B_1^{-1} C_6^{-1} B_3^{-1}.$$

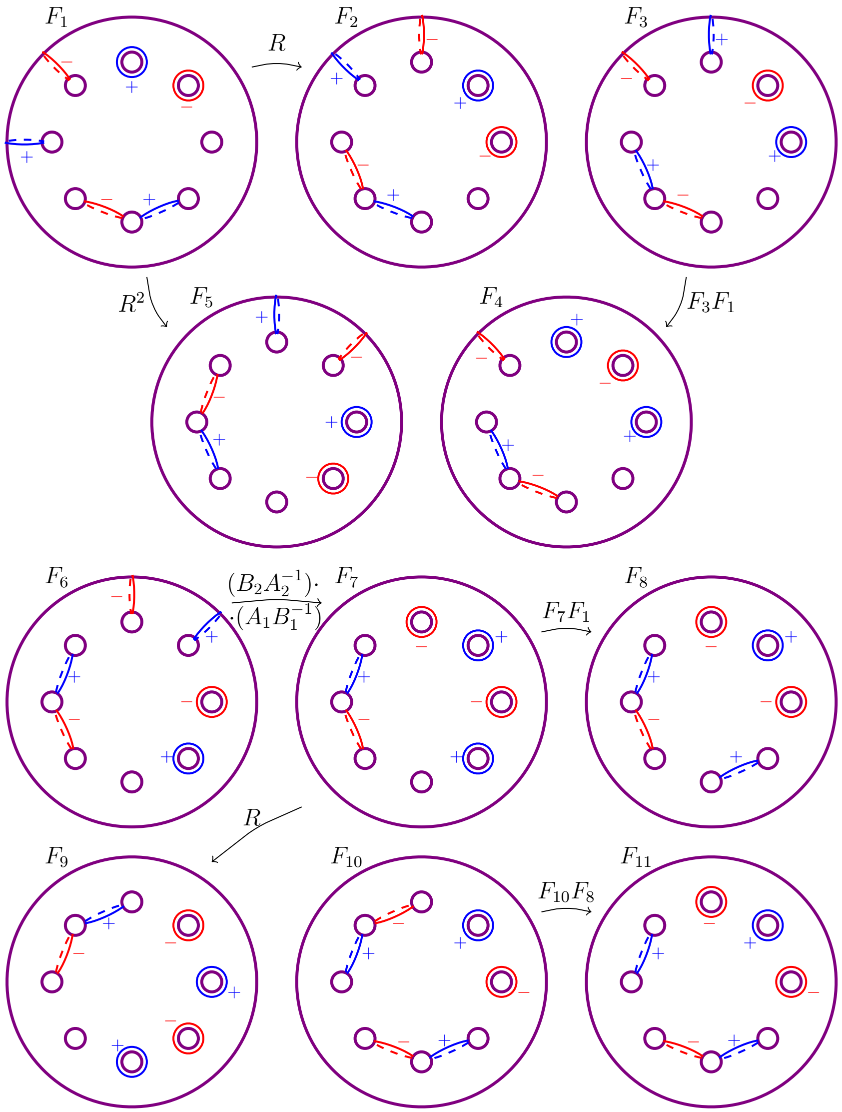

Figure 4. Proof of Corollary 10

We also have $F_7F_1(b_4, c_7, b_2, b_1, c_6, b_3) = (c_4, c_7, b_2, b_1, c_6, b_3)$ so that $F_8 = C_4C_7B_2B_1^{-1}C_6^{-1}B_3^{-1} \in H$. It is easy to check that $F_8F_7^{-1} = C_4B_4^{-1} \in H$ and then we get $C_iB_i^{-1} \in H$ for all i.

Let

$$F_9 = RF_7R^{-1} = B_5C_8B_3B_2^{-1}C_7^{-1}B_4^{-1}$$

and

$$F_{10} = (C_4B_4^{-1})F_9^{-1}(B_5C_5^{-1}) = C_4C_7B_2B_3^{-1}C_8^{-1}C_5^{-1}.$$

Similarly, we see that $F_{10}F_8(c_4, c_7, b_2, b_3, c_8, c_5) = (c_4, c_7, b_2, b_3, b_1, c_5)$ so that $F_{11} = C_4C_7B_2B_3^{-1}B_1^{-1}C_5^{-1} \in H$. Thus, $F_{10}^{-1}F_{11} = C_8B_1^{-1} \in H$ and then we get $C_iB_{i+1}^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_8)$, completing the proof of the corollary. □

Corollary 11. *If $g \geq 9$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $C_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}$ (Fig. 5).*

Proof. Let $F_1 = C_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = C_2B_5A_8A_9^{-1}B_6^{-1}C_3^{-1}$$

and

$$F_3 = F_2^{-1} = C_3B_6A_9A_8^{-1}B_5^{-1}C_2^{-1}.$$

We have $F_3F_1(c_3, b_6, a_9, a_8, b_5, c_2) = (b_4, b_6, a_9, a_8, b_5, c_2)$ so that $F_4 = B_4B_6A_9A_8^{-1}B_5^{-1}C_2^{-1} \in H$.

Hence, we see that $F_4F_3^{-1} = B_4C_3^{-1} \in H$ and then by conjugating $B_4C_3^{-1}$ with R iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all i.

Let

$$F_5 = F_4(C_2B_3^{-1}) = B_4B_6A_9A_8^{-1}B_5^{-1}B_3^{-1},$$
$$F_6 = R^{-2}F_5R^2 = B_2B_4A_7A_6^{-1}B_3^{-1}B_1^{-1}$$

and

$$F_7 = F_6^{-1} = B_1B_3A_6A_7^{-1}B_4^{-1}B_2^{-1}.$$

We get $F_7F_5(b_1, b_3, a_6, a_7, b_4, b_2) = (b_1, b_3, b_6, a_7, b_4, b_2)$ so that $F_8 = B_1B_3B_6A_7^{-1}B_4^{-1}B_2^{-1} \in H$.

We also have $F_8F_7^{-1} = B_6A_6^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

Let

$$F_9 = F_5(A_8B_8^{-1})(B_8C_7^{-1}) = B_4B_6A_9C_7^{-1}B_5^{-1}B_3^{-1},$$
$$F_{10} = R^{-1}F_9R = B_3B_5A_8C_6^{-1}B_4^{-1}B_2^{-1}$$

and

$$F_{11} = F_{10}^{-1} = B_2B_4C_6A_8^{-1}B_5^{-1}B_3^{-1}.$$

Hence, we have $F_{11}F_9(b_2, b_4, c_6, a_8, b_5, b_3) = (b_2, b_4, b_6, a_8, b_5, b_3)$ so that $F_{12} = B_2B_4B_6A_8^{-1}B_5^{-1}B_3^{-1} \in H$.

Finally, we see that $F_{12}F_{11}^{-1} = B_6C_6^{-1} \in H$ and then $B_iC_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. □

We introduce six new generating sets in Corollaries 12, 13, 14, 15, 16 and 20 to prove Theorem 2.

Corollary 12. *If $g = 8$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $B_1A_5C_5C_7^{-1}A_7^{-1}B_3^{-1}$.*

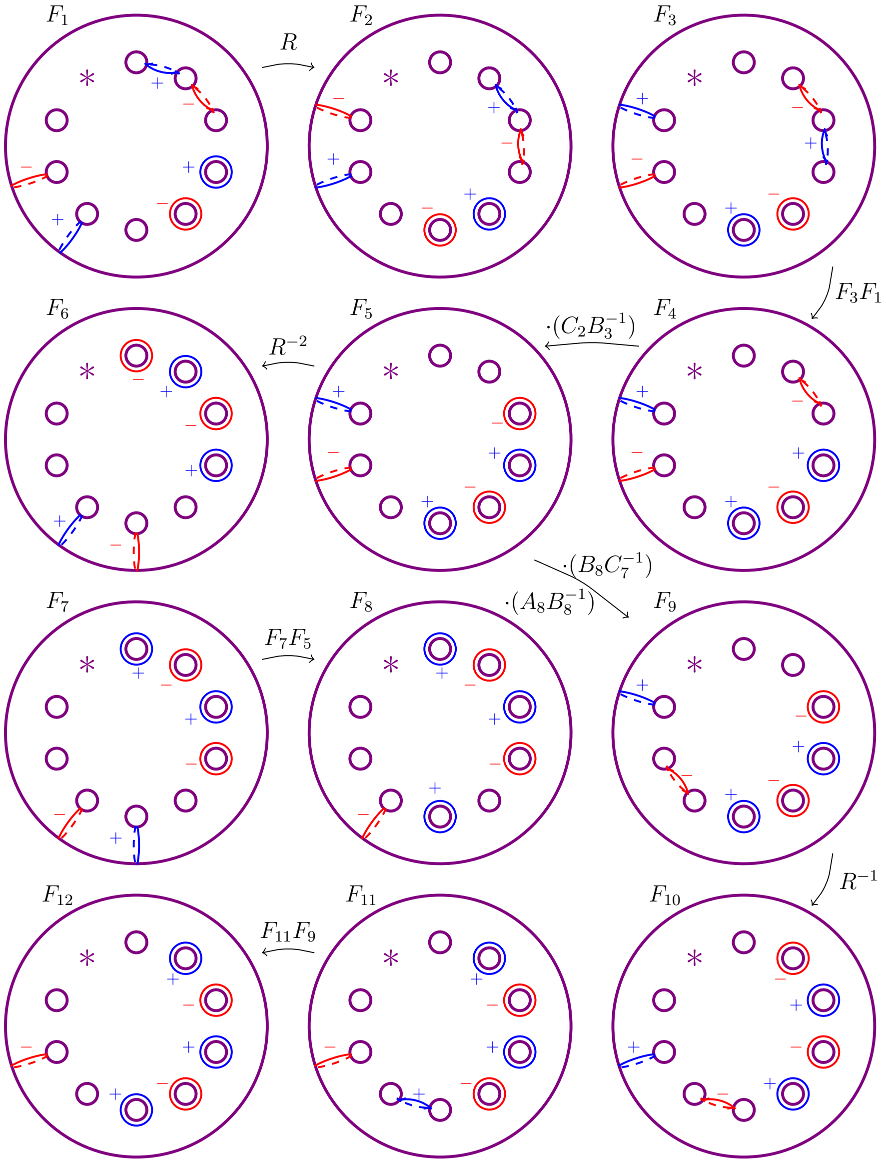

Figure 5. Proof of Corollary 11

Proof. Let $F_1 = B_1A_5C_5C_7^{-1}A_7^{-1}B_3^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = B_2A_6C_6C_8^{-1}A_8^{-1}B_4^{-1}$$

and

$$F_3 = F_2^{-1} = B_4A_8C_8C_6^{-1}A_6^{-1}B_2^{-1}.$$

We have $F_3F_1(b_4, a_8, c_8, c_6, a_6, b_2) = (b_4, a_8, b_1, c_6, a_6, b_2)$ so that $F_4 = B_4A_8B_1C_6^{-1}A_6^{-1}B_2^{-1} \in H$.

We get $F_4F_3^{-1} = B_1C_8^{-1} \in H$ and then by conjugating $B_1C_8^{-1}$ with R iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all i.

Let

$$F_5 = RF_4R^{-1} = B_5A_1B_2C_7^{-1}A_7^{-1}B_3^{-1}.$$

We also have $F_5F_4(b_5, a_1, b_2, c_7, a_7, b_3) = (b_5, b_1, b_2, c_7, a_7, b_3)$ so that $F_6 = B_5B_1B_2C_7^{-1}A_7^{-1}B_3^{-1} \in H$.

Hence, we get $F_6F_5^{-1} = B_1A_1^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

Let

$$F_7 = (C_4B_5^{-1})F_6(C_7B_8^{-1})(A_7B_7^{-1}) = C_4B_1B_2B_3^{-1}B_8^{-1}B_7^{-1},$$
$$F_8 = RF_7R^{-1} = C_5B_2B_3B_4^{-1}B_1^{-1}B_8^{-1}$$

and

$$F_9 = F_8^{-1} = B_8B_1B_4B_3^{-1}B_2^{-1}C_5^{-1}.$$

Similarly, check that $F_9F_7(b_8, b_1, b_4, b_3, b_2, c_5) = (b_8, b_1, c_4, b_3, b_2, c_5)$ so that $F_{10} = B_8B_1C_4B_3^{-1}B_2^{-1}C_5^{-1} \in H$.

Finally, we see that $F_{10}F_9^{-1} = C_4B_4^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_8)$, completing the proof of the corollary. □

Corollary 13. *If $g = 9$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $B_1A_3C_5C_8^{-1}A_6^{-1}B_4^{-1}$.*

Proof. Let $F_1 = B_1A_3C_5C_8^{-1}A_6^{-1}B_4^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_9)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = B_2A_4C_6C_9^{-1}A_7^{-1}B_5^{-1}$$

and

$$F_3 = F_2^{-1} = B_5A_7C_9C_6^{-1}A_4^{-1}B_2^{-1}.$$

We have that $F_3F_1(b_5, a_7, c_9, c_6, a_4, b_2) = (c_5, a_7, b_1, c_6, b_4, b_2)$ so that $F_4 = C_5A_7B_1C_6^{-1}B_4^{-1}B_2^{-1} \in H$.

Let

$$F_5 = RF_4R^{-1} = C_6A_8B_2C_7^{-1}B_5^{-1}B_3^{-1}$$

and

$$F_6 = F_5^{-1} = B_3B_5C_7B_2^{-1}A_8^{-1}C_6^{-1}.$$

We get $F_6F_4(b_3, b_5, c_7, b_2, a_8, c_6) = (b_3, c_5, c_7, b_2, a_8, c_6)$ so that $F_7 = B_3C_5C_7B_2^{-1}A_8^{-1}C_6^{-1} \in H$.

We see that $F_7F_6^{-1} = C_5B_5^{-1} \in H$ and then by conjugating $C_5B_5^{-1}$ with R iteratively, we get $C_iB_i^{-1} \in H$ for all i.

Let

$$F_8 = (B_7C_7^{-1})F_6(C_6B_6^{-1}) = B_3B_5B_7B_2^{-1}A_8^{-1}B_6^{-1},$$
$$F_9 = RF_8R^{-1} = B_4B_6B_8B_3^{-1}A_9^{-1}B_7^{-1}$$

and

$$F_{10} = F_9^{-1} = B_7A_9B_3B_8^{-1}B_6^{-1}B_4^{-1}.$$

We also have $F_{10}F_8(b_7, a_9, b_3, b_8, b_6, b_4) = (b_7, a_9, b_3, a_8, b_6, b_4)$ so that $F_{11} = B_7A_9B_3A_8^{-1}B_6^{-1}B_4^{-1} \in H$.

Finally, we have $F_{11}^{-1}F_{10} = A_8B_8^{-1} \in H$ and then $A_iB_i^{-1} \in H$ for all i. Check $F_4(B_4A_4^{-1})F_2(B_5C_5^{-1}) = B_1C_9^{-1} \in H$ and then $B_{i+1}C_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_9)$, completing the proof of the corollary. □

Corollary 14. *If $g = 10$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $A_1C_1B_3B_7^{-1}C_5^{-1}A_5^{-1}$.*

Proof. Let $F_1 = A_1C_1B_3B_7^{-1}C_5^{-1}A_5^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_{10})$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = A_2C_2B_4B_8^{-1}C_6^{-1}A_6^{-1}.$$

We have $F_2F_1(a_2, c_2, b_4, b_8, c_6, a_6) = (a_2, b_3, b_4, b_8, b_7, a_6)$ so that $F_3 = A_2B_3B_4B_8^{-1}B_7^{-1}A_6^{-1} \in H$.

Let

$$F_4 = R^4F_3R^{-4} = A_6B_7B_8B_2^{-1}B_1^{-1}A_{10}^{-1}$$

and

$$F_5 = F_4^{-1} = A_{10}B_1B_2B_8^{-1}B_7^{-1}A_6^{-1}.$$

We get $F_5F_3(a_{10}, b_1, b_2, b_8, b_7, a_6) = (a_{10}, b_1, a_2, b_8, b_7, a_6)$ so that $F_6 = A_{10}B_1A_2B_8^{-1}B_7^{-1}A_6^{-1} \in H$.

We see that $F_6F_5^{-1} = A_2B_2^{-1} \in H$ and then by conjugating $A_2B_2^{-1}$ with R iteratively, we get $A_iB_i^{-1} \in H$ for all i.

Let

$$F_7 = (B_2A_2^{-1})(A_3B_3^{-1})F_3(B_7A_7^{-1})(A_6B_6^{-1}) = B_2A_3B_4B_8^{-1}A_7^{-1}B_6^{-1},$$
$$F_8 = RF_2F_3^{-1}R^{-1}F_7 = B_2A_3C_3C_7^{-1}A_7^{-1}B_6^{-1},$$
$$F_9 = F_8^{-1} = B_6A_7C_7C_3^{-1}A_3^{-1}B_2^{-1}$$

and

$$F_{10} = R^4F_9R^{-4} = B_{10}A_1C_1C_7^{-1}A_7^{-1}B_6^{-1}.$$

We also have $F_{10}F_8(b_{10}, a_1, c_1, c_7, a_7, b_6) = (b_{10}, a_1, b_2, c_7, a_7, b_6)$ so that $F_{11} = B_{10}A_1B_2C_7^{-1}A_7^{-1}B_6^{-1} \in H$.

We then get $F_{11}F_{10}^{-1} = B_2C_1^{-1} \in H$ and then $B_{i+1}C_i^{-1} \in H$ for all i.

Let
$F_{12} = (B_2A_2^{-1})F_3(A_6B_6^{-1})(B_6C_5^{-1})(B_7A_7^{-1})(B_8A_8^{-1}) = B_2B_3B_4A_8^{-1}A_7^{-1}C_5^{-1}$,
$F_{13} = F_{12}^{-1} = C_5A_7A_8B_4^{-1}B_3^{-1}B_2^{-1}$
and

$$F_{14} = RF_{13}R^{-1} = C_6A_8A_9B_5^{-1}B_4^{-1}B_3^{-1}.$$

Hence, we have $F_{14}F_{12}(c_6, a_8, a_9, b_5, b_4, b_3) = (c_6, a_8, a_9, c_5, b_4, b_3)$ so that $F_{15} = C_6A_8A_9C_5^{-1}B_4^{-1}B_3^{-1} \in H$.

Finally, we see that $F_{15}^{-1}F_{14} = C_5B_5^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_{10})$, completing the proof of the corollary. □

Corollary 15. *If $g \geq 13$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $A_1B_4C_8C_{10}^{-1}B_6^{-1}A_3^{-1}$.*

Proof. Let $F_1 = A_1B_4C_8C_{10}^{-1}B_6^{-1}A_3^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = A_2B_5C_9C_{11}^{-1}B_7^{-1}A_4^{-1}$$

and

$$F_3 = F_2^{-1} = A_4B_7C_{11}C_9^{-1}B_5^{-1}A_2^{-1}.$$

We have $F_3F_1(a_4, b_7, c_{11}, c_9, b_5, a_2) = (b_4, b_7, c_{11}, c_9, b_5, a_2)$ so that $F_4 = B_4B_7C_{11}C_9^{-1}B_5^{-1}A_2^{-1} \in H$.

We see that $F_4F_3^{-1} = B_4A_4^{-1} \in H$ and then by conjugating $B_4A_4^{-1}$ with R iteratively, we get $B_iA_i^{-1} \in H$ for all i.

Let

$$F_5 = R^2F_1R^{-2} = A_3B_6C_{10}C_{12}^{-1}B_8^{-1}A_5^{-1}$$

and

$$F_6 = F_5^{-1} = A_5B_8C_{12}C_{10}^{-1}B_6^{-1}A_3^{-1}.$$

We also have $F_6F_1(a_5, b_8, c_{12}, c_{10}, b_6, a_3) = (a_5, c_8, c_{12}, c_{10}, b_6, a_3)$ so that $F_7 = A_5C_8C_{12}C_{10}^{-1}B_6^{-1}A_3^{-1} \in H$.

We get $F_7F_6^{-1} = C_8B_8^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

Let

$$F_8 = (A_4B_4^{-1})F_1(A_3B_3^{-1}) = A_1A_4C_8C_{10}^{-1}B_6^{-1}B_3^{-1},$$
$$F_9 = R^3F_8R^{-3} = A_4A_7C_{11}C_{13}^{-1}B_9^{-1}B_6^{-1}$$

and

$$F_{10} = F_9^{-1} = B_6B_9C_{13}C_{11}^{-1}A_7^{-1}A_4^{-1}.$$

Hence, check that $F_{10}F_8(b_6, b_9, c_{13}, c_{11}, a_7, a_4) = (b_6, c_8, c_{13}, c_{11}, a_7, a_4)$ so that $F_{11} = B_6C_8C_{13}C_{11}^{-1}A_7^{-1}A_4^{-1} \in H$.

Finally, we have $F_{11}F_{10}^{-1} = C_8B_9^{-1} \in H$ and then $C_iB_{i+1}^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. □

Corollary 16. *If $g \geq 12$, then the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the two elements R and $B_1A_3C_6C_{10}^{-1}A_7^{-1}B_5^{-1}$.*

Proof. Let $F_1 = B_1A_3C_6C_{10}^{-1}A_7^{-1}B_5^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = B_2A_4C_7C_{11}^{-1}A_8^{-1}B_6^{-1}$$

and

$$F_3 = F_2^{-1} = B_6A_8C_{11}C_7^{-1}A_4^{-1}B_2^{-1}.$$

We have $F_3F_1(b_6, a_8, c_{11}, c_7, a_4, b_2) = (c_6, a_8, c_{11}, c_7, a_4, b_2)$ so that $F_4 = C_6A_8C_{11}C_7^{-1}A_4^{-1}B_2^{-1} \in H$.

We get $F_4F_3^{-1} = C_6B_6^{-1} \in H$ and then by conjugating $C_6B_6^{-1}$ with R iteratively, we get $C_iB_i^{-1} \in H$ for all i.

Let

$$F_5 = F_1(C_{10}B_{10}^{-1})(B_5C_5^{-1}) = B_1A_3C_6B_{10}^{-1}A_7^{-1}C_5^{-1}$$

and

$$F_6 = R^2F_5R^{-2} = B_3A_5C_8B_{12}^{-1}A_9^{-1}C_7^{-1}.$$

We also have $F_6F_5(b_3, a_5, c_8, b_{12}, a_9, c_7) = (a_3, a_5, c_8, b_{12}, a_9, c_7)$ so that $F_7 = A_3A_5C_8B_{12}^{-1}A_9^{-1}C_7^{-1} \in H$.

We get $F_7F_6^{-1} = A_3B_3^{-1} \in H$ and then $A_iB_i^{-1} \in H$ for all i.

Let

$$F_8 = (C_1B_1^{-1})(B_3A_3^{-1})F_1(B_5C_5^{-1}) = C_1B_3C_6C_{10}^{-1}A_7^{-1}C_5^{-1}$$

and

$$F_9 = RF_8R^{-1} = C_2B_4C_7C_{11}^{-1}A_8^{-1}C_6^{-1}.$$

Then check that $F_9F_8(c_2, b_4, c_7, c_{11}, a_8, c_6) = (b_3, b_4, c_7, c_{11}, a_8, c_6)$ so that $F_{10} = B_3B_4C_7C_{11}^{-1}A_8^{-1}C_6^{-1} \in H$.

Finally, we have $F_{10}F_9^{-1} = B_3C_2^{-1} \in H$ and then $B_{i+1}C_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. □

Lemma 17. *If $g \geq 11$, then the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the two elements R and $A_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1}$.*

Proof. Let $F_1 = A_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = A_2B_3C_5C_g^{-1}B_{g-2}^{-1}A_{g-3}^{-1}.$$

We have $F_2F_1(a_2, b_3, c_5, c_g, b_{g-2}, a_{g-3}) = (b_2, b_3, c_5, c_g, b_{g-2}, b_{g-3})$ so that $F_3 = B_2B_3C_5C_g^{-1}B_{g-2}^{-1}B_{g-3}^{-1} \in H$.

Let

$$F_4 = R^{-1}F_3R = B_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}B_{g-4}^{-1}$$

and

$$F_5 = F_3^{-1} = B_{g-3}B_{g-2}C_gC_5^{-1}B_3^{-1}B_2^{-1}.$$

We also have $F_5F_4(b_{g-3}, b_{g-2}, c_g, c_5, b_3, b_2) = (b_{g-3}, b_{g-2}, b_1, c_5, b_3, b_2)$ so that $F_6 = B_{g-3}B_{g-2}B_1C_5^{-1}B_3^{-1}B_2^{-1} \in H$.

We see that $F_6F_5^{-1} = B_1C_g^{-1} \in H$ and then by conjugating $B_1C_g^{-1}$ with R iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all i.

Let

$$F_7 = (C_{g-3}B_{g-2}^{-1})(C_{g-4}B_{g-3}^{-1})F_6 = C_{g-3}C_{g-4}B_1C_5^{-1}B_3^{-1}B_2^{-1}$$

and

$$F_8 = R^2F_7R^{-2} = C_{g-1}C_{g-2}B_3C_7^{-1}B_5^{-1}B_4^{-1}.$$

We have $F_8F_7(c_{g-1}, c_{g-2}, b_3, c_7, b_5, b_4) = (c_{g-1}, c_{g-2}, b_3, c_7, c_5, b_4)$ so that $F_9 = C_{g-1}C_{g-2}B_3C_7^{-1}C_5^{-1}B_4^{-1} \in H$.

We then get $F_9F_8^{-1} = C_5B_5^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

Let

$$F_{10} = F_1(B_{g-3}C_{g-3}^{-1}) = A_1B_2C_4C_{g-1}^{-1}C_{g-3}^{-1}A_{g-4}^{-1}$$

and

$$F_{11} = RF_{10}R^{-1} = A_2B_3C_5C_g^{-1}C_{g-2}^{-1}A_{g-3}^{-1}.$$

Hence, we see $F_{11}F_{10}(a_2, b_3, c_5, c_g, c_{g-2}, a_{g-3}) = (b_2, b_3, c_5, c_g, c_{g-2}, a_{g-3})$ so that $F_{12} = B_2B_3C_5C_g^{-1}C_{g-2}^{-1}A_{g-3}^{-1} \in H$.

Finally, we have $F_{12}F_{11}^{-1} = B_2A_2^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the lemma. □

Lemma 18. *If $g \geq 13$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $A_1B_2C_4C_{g-2}^{-1}B_{g-4}^{-1}A_{g-5}^{-1}$.*

Proof. Let $F_1 = A_1B_2C_4C_{g-2}^{-1}B_{g-4}^{-1}A_{g-5}^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = A_2B_3C_5C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1}.$$

We have $F_2F_1(a_2, b_3, c_5, c_{g-1}, b_{g-3}, a_{g-4}) = (b_2, b_3, c_5, c_{g-1}, b_{g-3}, b_{g-4})$ so that $F_3 = B_2B_3C_5C_{g-1}^{-1}B_{g-3}^{-1}B_{g-4}^{-1} \in H$.

Let

$$\begin{aligned} F_4 = F_2F_3^{-1} = A_2B_2^{-1}A_{g-4}^{-1}B_{g-4}, \\ F_5 &= RF_4R^{-1} = A_3B_3^{-1}A_{g-3}^{-1}B_{g-3}, \\ F_6 &= F_5F_3 = B_2A_3C_5C_{g-1}^{-1}A_{g-3}^{-1}B_{g-4}^{-1}, \\ F_7 &= R^{-2}F_6R^2 = B_gA_1C_3C_{g-3}^{-1}A_{g-5}^{-1}B_{g-6}^{-1} \end{aligned}$$

and

$$F_8 = F_7^{-1} = B_{g-6}A_{g-5}C_{g-3}C_3^{-1}A_1^{-1}B_g^{-1}.$$

We get $F_8F_6(b_{g-6}, a_{g-5}, c_{g-3}, c_3, a_1, b_g) = (b_{g-6}, a_{g-5}, c_{g-3}, c_3, a_1, c_{g-1})$ so that $F_9 = B_{g-6}A_{g-5}C_{g-3}C_3^{-1}A_1^{-1}C_{g-1}^{-1} \in H$.

We see that $F_9F_8^{-1} = C_{g-1}B_g^{-1} \in H$ and then by conjugating $C_{g-1}B_g^{-1}$ with R iteratively, we get $C_iB_{i+1}^{-1} \in H$ for all i.

Let

$$F_{10} = F_3(C_{g-1}B_g^{-1}) = B_2B_3C_5B_g^{-1}B_{g-3}^{-1}B_{g-4}^{-1}$$

and

$$F_{11} = R^2F_{10}R^{-2} = B_4B_5C_7B_2^{-1}B_{g-1}^{-1}B_{g-2}^{-1}.$$

We also have $F_{11}F_{10}(b_4, b_5, c_7, b_2, b_{g-1}, b_{g-2}) = (b_4, c_5, c_7, b_2, b_{g-1}, b_{g-2})$ so that $F_{12} = B_4C_5C_7B_2^{-1}B_{g-1}^{-1}B_{g-2}^{-1} \in H$.

We then get $F_{12}F_{11}^{-1} = C_5B_5^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

Let

$$F_{13} = F_1(B_{g-4}C_{g-4}^{-1}) = A_1B_2C_4C_{g-2}^{-1}C_{g-4}^{-1}A_{g-5}^{-1}$$

and

$$F_{14} = RF_{13}R^{-1} = A_2B_3C_5C_{g-1}^{-1}C_{g-3}^{-1}A_{g-4}^{-1}.$$

Hence, $F_{14}F_{13}(a_2, b_3, c_5, c_{g-1}, c_{g-3}, a_{g-4}) = (b_2, b_3, c_5, c_{g-1}, c_{g-3}, a_{g-4})$ so that $F_{15} = B_2B_3C_5C_{g-1}^{-1}C_{g-3}^{-1}A_{g-4}^{-1} \in H$.

Finally, we have $F_{15}F_{14}^{-1} = B_2A_2^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. □

Lemma 19. *If $k \geq 7$ and $g \geq 2k+1$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by elements R and $A_1B_2C_4C_{g-k+4}^{-1}B_{g-k+2}^{-1}A_{g-k+1}^{-1}$.*

Proof. Let $F_1 = A_1B_2C_4C_{g-k+4}^{-1}B_{g-k+2}^{-1}A_{g-k+1}^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = R^{k-3}F_1R^{3-k} = A_{k-2}B_{k-1}C_{k+1}C_1^{-1}B_{g-1}^{-1}A_{g-2}^{-1}$$

and

$$F_3 = F_2^{-1} = A_{g-2}B_{g-1}C_1C_{k+1}^{-1}B_{k-1}^{-1}A_{k-2}^{-1}.$$

$F_3F_1(a_{g-2}, b_{g-1}, c_1, c_{k+1}, b_{k-1}, a_{k-2}) = (a_{g-2}, b_{g-1}, b_2, c_{k+1}, b_{k-1}, a_{k-2})$ so that $F_4 = A_{g-2}B_{g-1}B_2C_{k+1}^{-1}B_{k-1}^{-1}A_{k-2}^{-1} \in H$.

We get $F_4F_3^{-1} = B_2C_1^{-1} \in H$ and then by conjugating $B_2C_1^{-1}$ with R iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all i.

Let

$$F_5 = F_1(B_{g-k+2}C_{g-k+1}^{-1}) = A_1B_2C_4C_{g-k+4}^{-1}C_{g-k+1}^{-1}A_{g-k+1}^{-1}$$

and

$$F_6 = RF_5R^{-1} = A_2B_3C_5C_{g-k+5}^{-1}C_{g-k+2}^{-1}A_{g-k+2}^{-1}.$$

$F_6F_5(a_2, b_3, c_5, c_{g-k+5}, c_{g-k+2}, a_{g-k+2}) = (b_2, b_3, c_5, c_{g-k+5}, c_{g-k+2}, a_{g-k+2})$

so that $F_7 = B_2B_3C_5C^{-1}_{g-k+5}C^{-1}_{g-k+2}A^{-1}_{g-k+2} \in H$.

We then get $F_7F_6^{-1} = B_2A_2^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

Let

$$F_8 = R^{k-2}F_6R^{2-k} = A_kB_{k+1}C_{k+3}C_3^{-1}Cg^{-1}A_g^{-1}$$

and

$$F_9 = F_8^{-1} = A_gC_gC_3C^{-1}_{k-3}B^{-1}_{k+1}A_k^{-1}.$$

We have $F_9F_6(a_g, c_g, c_3, c_{k+3}, b_{k+1}, a_k) = (a_g, c_g, b_3, c_{k+3}, b_{k+1}, a_k)$ so that $F_{10} = A_gC_gB_3C^{-1}_{k-3}B^{-1}_{k+1}A_k^{-1} \in H$.

Finally, we see that $F_{10}F_9^{-1} = B_3C_3^{-1} \in H$ and then $B_iC_i^{-1} \in H$ for all i.

It follows from Corollary 7 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the lemma. □

Corollary 20. *If $k \geq 5$ and $g \geq 2k+1$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by elements R and $A_1B_2C_4C^{-1}_{g-k+4}B^{-1}_{g-k+2}A^{-1}_{g-k+1}$.*

Proof. It directly follows from Lemmas 17, 18 and 19. □

4. Main Results

In this section, we prove the main results of this paper. The following Lemma is useful to decide the order of an element.

Lemma 21. *If R is an element of order k in a group G and if x and y are elements in G satisfying $RxR^{-1} = y$, then the order of Rxy^{-1} is also k.*

Proof. $(Rxy^{-1})^k = (yRy^{-1})^k = yR^ky^{-1} = 1$.
On the other hand, if $(Rxy^{-1})^l = 1$ then $(Rxy^{-1})^l = (yRy^{-1})^l = yR^ly^{-1} = 1$ i.e. $R^l = 1$ and hence $k \mid l$. □

Now, we are ready to prove Theorem 2.

Proof. For $g = 10$, we let H_{10} be the subgroup of $\mathrm{Mod}(\Sigma_{10})$ generated by the set $\{R, R^4A_1C_1B_3B_7^{-1}C_5^{-1}A_5^{-1}\}$. We get $H_{10} = \mathrm{Mod}(\Sigma_{10})$ by Corollary 14. Then we are done by Lemma 21 since $R^4(A_1C_1B_3)R^{-4} = A_5C_5B_7$. Note that, order of R^4 is clearly 5 and hence order of the element $R^4(A_1C_1B_3)(A_5C_5B_7)^{-1}$ is also 5 by Lemma 21 since $R^4(a_1) = a_5$, $R^4(c_1) = c_5$ and $R^4(b_3) = b_7$ implies $R^4(A_1C_1B_3)R^{-4} = A_5C_5B_7$.

For $g = 9$, we let H_9 be the subgroup of $\mathrm{Mod}(\Sigma_9)$ generated by the set $\{R, R^3B_1A_3C_5C_8^{-1}A_6^{-1}B_4^{-1}\}$. We have $H_9 = \mathrm{Mod}(\Sigma_9)$ by Corollary 13. Then we are done by Lemma 21 since $R^3(B_1A_3C_5)R^{-3} = B_4A_6C_8$.

For $g = 8$, we let H_8 be the subgroup of $\mathrm{Mod}(\Sigma_8)$ generated by the set $\{R, R^2B_1A_5C_5C_7^{-1}A_7^{-1}B_3^{-1}\}$. Hence, $H_8 = \mathrm{Mod}(\Sigma_8)$ by Corollary 12. Then we are done by Lemma 21 since $R^2(B_1A_5C_5)R^{-2} = B_3A_7C_7$.

For $g = 7$, we let H_7 be the subgroup of $\mathrm{Mod}(\Sigma_7)$ generated by the set $\{R, RC_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}\}$. We have $H_7 = \mathrm{Mod}(\Sigma_7)$ by Corollary 9. Then we are done by Lemma 21 since $R(C_1B_4A_6)R^{-1} = C_2B_5A_7$.

The remaining part of the proof is the case of $g \geq 11$. Let $k = g/g'$ so that k is the greatest divisor of g such that k is strictly less than $g/2$. Clearly, the number k can be any positive integer but three.

If $k = 2$, let K_2 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^2 A_1 B_4 C_8 C_{10}^{-1} B_6^{-1} A_3^{-1}\}$. We get $K_2 = \mathrm{Mod}(\Sigma_g)$ by Corollary 15. Then we are done by Lemma 21 since $R^2(A_1B_4C_8)R^{-2} = A_3B_6C_{10}$.

If $k = 4$, let K_4 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^4 B_1 A_3 C_6 C_{10}^{-1} A_7^{-1} B_5^{-1}\}$. We get $K_4 = \mathrm{Mod}(\Sigma_g)$ by Corollary 16. Then we are done by Lemma 21 since $R^4(B_1A_3C_6)R^{-4} = B_5A_7C_{10}$.

If $k = 1$ or $k = 5$, let K_5 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^{-5} A_1 B_2 C_4 C_{g-1}^{-1} B_{g-3}^{-1} A_{g-4}^{-1}\}$. We get $K_5 = \mathrm{Mod}(\Sigma_g)$ by Corollary 20. Then we are done by Lemma 21 since $R^{-5}(A_1B_2C_4)R^5 = A_{g-4}B_{g-3}C_{g-1}$.

If $k = 6$, let K_6 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^{-6} A_1 B_2 C_4 C_{g-2}^{-1} B_{g-4}^{-1} A_{g-5}^{-1}\}$. We get $K_6 = \mathrm{Mod}(\Sigma_g)$ by Corollary 20. Then we are done by Lemma 21 since $R^{-6}(A_1B_2C_4)R^6 = A_{g-5}B_{g-4}C_{g-2}$.

If $k \geq 7$, let K be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^{-k} A_1 B_2 C_4 C_{g-k+4}^{-1} B_{g-k+2}^{-1} A_{g-k+1}^{-1}\}$. We get $K = \mathrm{Mod}(\Sigma_g)$ by Corollary 20. Then we are done by Lemma 21 since $R^{-k}(A_1B_2C_4)R^k = A_{g-k+1}B_{g-k+2}C_{g-k+4}$. □

Finally, we prove Theorem 1.

Proof. If $g = 6$, let H_6 be the subgroup of $\mathrm{Mod}(\Sigma_6)$ generated by the set $\{R, RC_1B_4A_6A_1^{-1}B_5^{-1}C_2^{-1}\}$. We get $H_6 = \mathrm{Mod}(\Sigma_6)$ by Corollary 8. Then we are done by Lemma 21 since $R(C_1B_4A_6)R^{-1} = C_2B_5A_1$. Note that, since $R(c_1) = c_2$, $R(b_4) = b_5$ and $R(a_6) = a_1$, we have $R(C_1B_4A_6)R^{-1} = C_2B_5A_1$ which implies order of the element $R(C_1B_4A_6)(C_2B_5A_1)^{-1}$ is g.

If $g = 7$, let H_7 be the subgroup of $\mathrm{Mod}(\Sigma_7)$ generated by the set $\{R, RC_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}\}$. We get $H_7 = \mathrm{Mod}(\Sigma_7)$ by Corollary 9. Then we are done by Lemma 21 since $R(C_1B_4A_6)R^{-1} = C_2B_5A_7$.

If $g = 8$, let H_8 be the subgroup of $\mathrm{Mod}(\Sigma_8)$ generated by the set $\{R, RB_1C_4A_7A_8^{-1}C_5^{-1}B_2^{-1}\}$. We get $H_8 = \mathrm{Mod}(\Sigma_8)$ by Corollary 10. Then we are done by Lemma 21 since $R(B_1C_4A_7)R^{-1} = B_2C_5A_8$.

If $g \geq 9$, let H_9 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, RC_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}\}$. We get $H_9 = \mathrm{Mod}(\Sigma_g)$ by Corollary 11. Then we are done by Lemma 21 since $R(C_1B_4A_7)R^{-1} = C_2B_5A_8$. □

5. Further Results

In this section, we prove Theorem 3 which states as: for $g \geq 3k^2 + 4k + 1$ and any positive integer k, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by two elements of order $g/\gcd(g, k)$.

Korkmaz showed the following result in the proof of Theorem 6.

Theorem 22. *If $g \geq 3$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $A_iA_j^{-1}, B_iB_j^{-1}, C_iC_j^{-1}$ for all i, j.*

Sketch of the proof is as follows: $A_1A_2^{-1}B_1B_2^{-1}(a_1,a_3) = (b_1,a_3)$. $B_1A_3^{-1}C_1C_2^{-1}(b_1,a_3) = (c_1,a_3)$. Korkmaz then showed that A_3 can be generated by these elements using lantern relation. Hence, $A_i = (A_iA_3^{-1})A_3$, $B_i = (B_iB_1^{-1})(B_1A_3^{-1})A_3$ and $C_i = (C_iC_1^{-1})(C_1A_3^{-1})A_3$ are generated by given elements. This finishes the proof.

Now, we prove the next statement as a corollary to Theorem 22.

Corollary 23. *If $g \geq 3$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $A_iB_i^{-1}, C_iB_i^{-1}, C_iB_{i+1}^{-1}$ for all i.*

Proof. Let us denote by H the subgroup generated by the elements $A_iB_i^{-1}, C_iB_i^{-1}, C_iB_{i+1}^{-1}$ for all i.

We have $B_iB_j^{-1} = (B_iC_i^{-1})(C_iB_{i+1}^{-1})\cdots(B_{j-1}C_{j-1}^{-1})(C_{j-1}B_j^{-1}) \in H$ for all i,j, we also have $C_iC_j^{-1} = (C_iB_i^{-1})(B_iB_j^{-1})(B_jC_j^{-1}) \in H$ for all i,j and $A_iA_j^{-1} = (A_iB_i^{-1})(B_iB_j^{-1})(B_jA_j^{-1}) \in H$ for all i,j.

It follows from Theorem 22 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the lemma. □

Theorem 24. *If $g \geq 21$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $R^2, B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}$.*

Proof. Let $F_1 = B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R^2, F_1\}$.

Let

$$F_2 = R^2F_1R^{-2} = B_3B_4A_7A_{10}C_{13}C_{16}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}B_5^{-1}$$

and

$$F_3 = F_2^{-1} = B_5B_6A_9A_{12}C_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}.$$

We have $F_3F_1(b_5,b_6,\ldots,b_3) = (a_5,b_6,\ldots,b_3)$ so that $F_4 = A_5B_6A_9A_{12}C_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1} \in H$.

Note that $\ldots$ refers to the elements remaining fixed under the given maps.

We also have $F_4F_3^{-1} = A_5B_5^{-1} \in H$ and then by conjugating $A_5B_5^{-1}$ with R^2 iteratively, we get $A_{2i+1}B_{2i+1}^{-1} \in H$ for all i.

Let

$$F_5 = R^4F_1R^{-4} = B_5B_6A_9A_{12}C_{15}C_{18}C_{20}^{-1}C_{17}^{-1}A_{14}^{-1}A_{11}^{-1}B_8^{-1}B_7^{-1}$$

and

$$\begin{aligned} F_6 &= (A_7B_7^{-1})F_5^{-1}(B_5A_5^{-1}) \\ &= A_7B_8A_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1}. \end{aligned}$$

We then have $F_6F_1(a_7,b_8,a_{11},\ldots,b_6,a_5) = (a_7,a_8,a_{11},\ldots,b_6,a_5)$ so that $F_7 = A_7A_8A_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1} \in H$.

$F_7F_6^{-1} = A_8B_8^{-1} \in H$ and then by conjugating $A_8B_8^{-1}$ with R^2 iteratively, we get $A_{2i}B_{2i}^{-1} \in H$ for all i.

Hence, we get $A_iB_i^{-1} \in H$ for all i.

Let

$$F_8 = (B_{12}A_{12}^{-1})F_4 = A_5B_6A_9B_{12}C_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}.$$

We then get $F_8F_1(\ldots, b_{12}, \ldots) = (\ldots, c_{11}, \ldots)$ so that $F_9 = A_5B_6A_9C_{11}C_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1} \in H$.

We have $F_9F_8^{-1} = C_{11}B_{12}^{-1} \in H$ and then by conjugating $C_{11}B_{12}^{-1}$ with R^2 iteratively, we get $C_{2i+1}B_{2i+2}^{-1} \in H$ for all i.

Let

$$F_{10} = (B_{11}A_{11}^{-1})F_7 = A_7A_8B_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1}.$$

Similarly, we have $F_{10}F_1(\ldots, b_{11}, \ldots) = (\ldots, c_{11}, \ldots)$ so that $F_{11} = A_7A_8C_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1} \in H$.

Hence, we get $F_{11}F_{10}^{-1} = C_{11}B_{11}^{-1} \in H$ and we get $C_{2i+1}B_{2i+1}^{-1} \in H$ for all i.

Let

$$F_{12} = (B_{15}C_{15}^{-1})F_4 = A_5B_6A_9A_{12}B_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}.$$

We also have $F_{12}F_1(\ldots, b_{15}, \ldots) = (\ldots, c_{14}, \ldots)$ so that $F_{13} = A_5B_6A_9A_{12}C_{14}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1} \in H$.

Check that $F_{13}F_{12}^{-1} = C_{14}B_{15}^{-1} \in H$ and then we get $C_{2i}B_{2i+1}^{-1} \in H$ for all i. Hence, we have $C_iB_{i+1}^{-1} \in H$ for all i.

Let

$$F_{14} = F_7(C_{15}B_{16}^{-1}) = A_7A_8A_{11}A_{14}C_{17}C_{20}C_{18}^{-1}B_{16}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1}.$$

We then get $F_{14}F_1(\ldots, b_{16}, \ldots) = (\ldots, c_{16}, \ldots)$ so that $F_{15} = A_7A_8A_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{16}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1} \in H$.

Hence, we see that $F_{15}^{-1}F_{14} = C_{16}B_{16}^{-1} \in H$ and then we get $C_{2i}B_{2i}^{-1} \in H$ for all i. Finally, we have $C_iB_i^{-1} \in H$ for all i.

It follows from Corollary 23 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the theorem. □

Corollary 25. *If g is even and $g \geq 22$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by two elements of order $g/2$.*

Proof. Let H be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R^2, R^2B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}\}$.
We get $H = \mathrm{Mod}(\Sigma_g)$ by Theorem 24. Then we are done by Lemma 21 since $R^2(B_1B_2A_5A_8C_{11}C_{14})R^{-2} = B_3B_4A_7A_{10}C_{13}C_{16}$. □

Generalization of Theorem 24 and Corollary 25 is as follows:

Theorem 26. *For $k \geq 2$ and $g \geq 3k^2 + 4k + 1$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $R^k, R^kF(R^kF^{-1}R^{-k})$ where $F = B_1B_2\ldots B_kA_{2k+1}A_{3k+2}\cdots A_{k^2+2k}C_{k^2+3k+1}C_{k^2+4k+2}\ldots C_{2k^2+3k}$ Fig. 6.*

Proof. We define an algorithm to prove the desired result.
Let $F = B_1B_2\cdots B_kA_{2k+1}A_{3k+2}\cdots A_{k^2+2k}C_{k^2+3k+1}C_{k^2+4k+2}\cdots C_{2k^2+3k}$ and $F_1 = F(R^kF^{-1}R^{-k})$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R^k, F_1\}$.

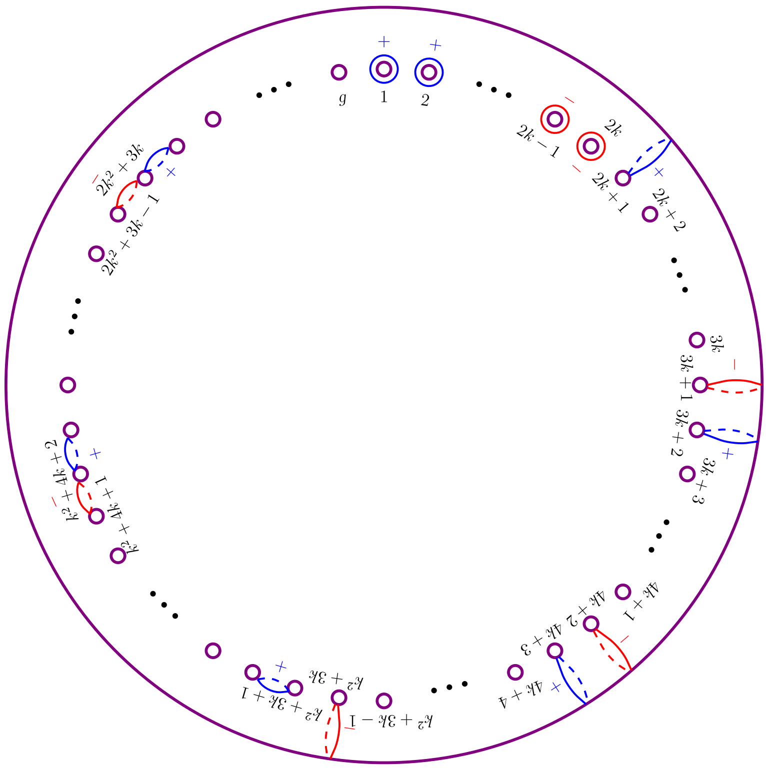

Figure 6. Generator for Theorem 3

A) Use conjugation of F_1 with $R^k, R^{2k}, \ldots, R^{k^2}$ with proper multiplications to get $A_{k+1}B_{k+1}^{-1} \in H$, $A_{k+2}B_{k+2}^{-1} \in H$, ... , $A_{2k-1}B_{2k-1}^{-1} \in H$, $A_{2k}B_{2k}^{-1} \in H$, respectively. Hence, we have $A_iB_i^{-1} \in H$ for all i.

B) Follow the next k steps.

1) Use conjugation of F_1 with R^{kl} for some positive integers l's with proper multiplications to get $C_{ik+1}B_{ik+1}^{-1} \in H$ and $C_{ik+1}B_{ik+2}^{-1} \in H$ for all i.

2) Use conjugation of F_1 with R^{kl} for some positive integers l's with proper multiplications to get $C_{ik+2}B_{ik+2}^{-1} \in H$ and $C_{ik+2}B_{ik+3}^{-1} \in H$ for all i.

...

k) Use conjugation of F_1 with R^{kl} for some positive integers l's with proper multiplications to get $C_{ik}B_{ik}^{-1} \in H$ and $C_{ik}B_{ik+1}^{-1} \in H$ for all i.

Hence, $C_iB_i^{-1} \in H$ and $C_iB_{i+1}^{-1} \in H$ for all i.

It follows from Corollary 23 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the theorem.

See Theorem 24 for an example application of the algorithm. □

Now, we prove Theorem 3.

Proof. For $k \geq 2$ and $g \geq 3k^2 + 4k + 1$, let H be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R^k, R^kF(R^kF^{-1}R^{-k})\}$. Then $H = \mathrm{Mod}(\Sigma_g)$ by Theorem 26. Hence, we are done by Lemma 21 since the orders of R^k and $R^kF(R^kF^{-1}R^{-k})$ are g/d where d is the greatest common divisor of g and k. If $k = 1$, we are done by Theorem 1. □

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Oğuz Yıldız
Department of Mathematics
Middle East Technical University
06800 Ankara
Turkey
e-mail: oguzyildiz16@gmail.com

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