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Weighted Transplantation for Laguerre Coefficients

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Abstract. We present a transplantation theorem for Laguerre coefficients in weighted spaces by means of a discrete local Calderón–Zygmund theory.

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1. Introduction

The Laguerre polynomials L_n^α are defined by means of its corresponding Rodrigues' type formula (see [19, Eq. 5.1.5])

$$x^\alpha e^{-x} L_n^\alpha(x) = \frac{1}{\Gamma(n+1)} \frac{d^n}{dx^n} \left(x^{\alpha+n} e^{-x} \right), \qquad n \in \mathbb{N} = \{0,1,2,\dots\},$$

where the order α is restricted to $\alpha > -1$ for integrability purposes. They are orthogonal on $(0,\infty)$ with respect to the measure

$$d\mu_\alpha(x) = x^\alpha e^{-x}\, dx.$$

Let us consider the family of functions $\{\mathcal{L}_n^\alpha(x)\}_{n\geq 0}$ defined by

$$\mathcal{L}_n^\alpha(x) = \omega_n^\alpha x^{\alpha/2} e^{-x/2} L_n^\alpha(x), \qquad x \in (0,\infty), \tag{1}$$

with ω_n^α the normalization factor

$$\omega_n^\alpha = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right)^{1/2}.$$

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This family, sometimes known as Laguerre functions, is a complete orthonormal system in the space $L^2(0,\infty)$, the set of all measurable and square integrable functions on $(0,\infty)$ with respect to the Lebesgue measure.

Let us define the discrete Fourier–Laguerre transform

$$\mathcal{F}_\alpha f(x) = \sum_{n=0}^{\infty} f(n)\mathcal{L}_n^\alpha(x)$$

for each f in the space of square summable sequences $\ell^2(\mathbb{N})$. It turns out that under this assumption $\mathcal{F}_\alpha f$ is a function in $L^2(0,\infty)$ and the identity $f(n) = c_n^\alpha(\mathcal{F}_\alpha f)$ holds, where

$$c_n^\alpha(F) = \int_0^\infty F(x)\mathcal{L}_n^\alpha(x)\,dx, \qquad F \in L^2(0,\infty),$$

is the usual n-th Fourier–Laguerre coefficient. Furthermore, the Parseval's type identity

$$\int_0^\infty |\mathcal{F}_\alpha f(x)|^2\,dx = \sum_{n=0}^{\infty} |f(n)|^2$$

holds, as well as

$$\int_0^\infty \mathcal{F}_\alpha f_1(x)\mathcal{F}_\alpha f_2(x)\,dx = \sum_{n=0}^{\infty} f_1(n)f_2(n), \qquad f_1, f_2 \in \ell^2(\mathbb{N}). \tag{2}$$

Put in other words, $\mathcal{F}_\alpha$ is an isometric bijection from $\ell^2(\mathbb{N})$ onto $L^2(0,\infty)$ whose inverse is given by

$$\mathcal{F}_\alpha^{-1}F(n) = c_n^\alpha(F),$$

which implies that it is possible to recover the original sequence by means of it, that is, $f = \mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha f)$.

In view of the above, we define the transplantation operator

$$T_\alpha^\beta f = \mathcal{F}_\beta^{-1}(\mathcal{F}_\alpha f), \qquad f \in \ell^2(\mathbb{N}),$$

for any $\alpha, \beta > -1$, which of course becomes the identity operator when $\alpha = \beta$. The mapping properties of this operator in $\ell^p(\mathbb{N})$ have been already studied in the special case $\beta = \alpha + 2$ by R. Askey in [3, Theorem 3]. To be precise, he stated that the size of the coefficients c_n^α and $c_n^{\alpha+2}$, measured in the power $\ell^p(\mathbb{N})$ norm, remain equivalent.

In the aforementioned paper [3], Askey proved the same result for Jacobi coefficients, extending a previous one regarding ultraspherical polynomials in [4] by himself and S. Wainger. The latter was generalized in [5] by J. J. Betancor et al. considering general weights with some additional restrictions. Recently, that work, as well as Askey's work on Jacobi coefficients, has been improved in [2] by the authors, where fairly general weights were considered. Finally, it is worth to mention the study [16] by K. Stempak on Fourier–Bessel coefficients.

It turns out that the dual problem in the continuous setting has considerably much more fruitful results, since the celebrated paper [11] by D. L. Guy regarding the Hankel transform on the positive half-line. That is why we only give here a brief state of the art regarding Laguerre expansions, but the interested reader in transplantation theorems for other expansions is urged to consult the excellent survey [17] and the references therein. Historically, the first transplantation result considering Laguerre expansions in terms of the functions given in (1) is due to Y. Kanjin in [12], which was enhanced in a weighted setting by K. Stempak and W. Trebels in [18]. The latter was refined in a power weight setting by G. Garrigós et al. in [9], given a sharp result for that weights. Other transplantation theorems for Laguerre expansions defined in terms of different functions that (1), such as the so-called Laguerre functions of Hermite type, could be looked up in the monograph [20] by S. Thangavelu.

Our aim in the present work is to prove the boundedness of the transplantation operator T_α^β with some weights and, as a corollary, improve Askey's result for a natural range of the parameters.

Before formulating our results, we need some previous definitions. A weight sequence in $\mathbb{N}$ will be a strictly positive sequence $w = \{w(n)\}_{n\geq 0}$. We consider the weighted space of p-summable sequences

$$\ell^p(\mathbb{N}, w) = \left\{ f = \{f(n)\}_{n\geq 0} : \|f\|_{\ell^p(\mathbb{N},w)} := \left(\sum_{n=0}^{\infty} |f(n)|^p w(n) \right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \leq p < \infty$. We simply write $\ell^p(\mathbb{N})$ when $w(n) = 1$ for all $n \in \mathbb{N}$.

Given $\alpha, \beta > -1$ fixed and power weights $w_a(m) = (m+1)^a$, with $a \in \mathbb{R}$, for a weight sequence w, we consider the following set of conditions when $1 < p < \infty$ and $1/p + 1/q = 1$:

(A1)

$$[w]_{H_p}^\alpha := \sup_{M\geq 0} \left(\sum_{m=M}^{\infty} w(m) w_{-p(\alpha/2+1)}(m) \right)^{1/p} \left(\sum_{m=0}^{M} w(m)^{-q/p} w_{q\alpha/2}(m) \right)^{1/q} < \infty,$$

(A2)

$$[w]_{H_p^*}^\beta := \sup_{M\geq 0} \left(\sum_{m=M}^{\infty} w(m)^{-q/p} w_{-q(\beta/2+1)}(m) \right)^{1/q} \left(\sum_{m=0}^{M} w(m) w_{p\beta/2}(m) \right)^{1/p} < \infty,$$

(A3)

$$[w]_{A_p^{\mathrm{loc}}} := \sup_{0\leq m\leq n\leq 2(m+1)} \frac{1}{n-m+1} \left(\sum_{k=m}^{n} w(k) \right)^{1/p} \left(\sum_{k=m}^{n} w(k)^{-q/p} \right)^{1/q} < \infty.$$

The usual maximum interpretation could be considered in the case $p = 1$, but we will skip it in this paper.

The values $[w]_{H_p}^\alpha$, $[w]_{H_p^*}^\beta$, and $[w]_{A_p^{\mathrm{loc}}}$ are called the constants of the weight w. First two conditions are adjoint in the sense that $[w]_{H_p}^\alpha < \infty$ if and only if $[w^{-q/p}]_{H_q^*}^\alpha < \infty$. Moreover, note that for any non-negative value $\delta \geq 0$, the inequalities $[w]_{H_p}^\alpha \geq [w]_{H_p}^{\alpha+\delta}$ and $[w]_{H_p^*}^\beta \geq [w]_{H_p^*}^{\beta+\delta}$ hold. Weights satisfying (A3) are known as local $A_p(\mathbb{N})$ weights and, as usual, $[w]_{A_p^{\mathrm{loc}}} < \infty$ if and only if $[w^{-q/p}]_{A_q^{\mathrm{loc}}} < \infty$. Finally, we remark that

$$[w_a]_{H_p}^{\alpha} < \infty \Longleftrightarrow \frac{a+1}{p} < \frac{\alpha}{2} + 1 \tag{3}$$

and

$$[w_a]_{H_p^*}^{\beta} < \infty \Longleftrightarrow -\frac{\beta}{2} < \frac{a+1}{p}. \tag{4}$$

Moreover, $[w_a]_{A_p^{\mathrm{loc}}} < \infty$ for any $a \in \mathbb{R}$.

Throughout the paper, we will use $a \lesssim b$ to denote that two positive quantities a and b fulfil the relation $a \leq Cb$ for a constant C independent of significative quantities. On its behalf, we will use $a \simeq b$ if there are two constants C_1 and C_2 independent of significative quantities such that $C_1 b \leq a \leq C_2 b$.

The main theorem of this paper is the following one.

Theorem 1.1. *Let $\alpha, \beta > -1$ with $\alpha \neq \beta$, $1 < p < \infty$, and w be a weight sequence that satisfies: $w(m) \simeq w(m+1)$ and condition (A1) if $\beta = \alpha + 2k$ for some $k \in \mathbb{N}$; $w(m) \simeq w(m+1)$ and condition (A2) if $\alpha = \beta + 2k$ for some $k \in \mathbb{N}$; and conditions (A1), (A2), and (A3) if $|\beta - \alpha| \neq 2k$ for every $k \in \mathbb{N}$. Then,*

$$\|T_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w). \tag{5}$$

Consequently, the operator T_α^β extends uniquely to a bounded linear operator from $\ell^p(\mathbb{N}, w)$ into itself.

The reason to split in three different cases the hypotheses of the theorem according to $\alpha = \beta + 2k$, $\beta = \alpha + 2k$ and $|\beta - \alpha| \neq 2k$ is because in the first and second cases the transplantation operator is essentially equivalent to the discrete Hardy operator and its adjoint. This phenomenon is not strange and, for example, it is the same as the one occurring in [13] for the Hankel transform. On its behalf, in the last case $|\beta - \alpha| \neq 2k$, the transplantation operator is bounded again by the discrete Hardy operator and its adjoint in the global part, whereas it is bounded by a Calderón–Zygmund operator in the local part (see next section for details).

We have to observe that the condition $w(m) \simeq w(m+1)$, that we consider in the cases in which $|\beta - \alpha| = 2k$ only, is required to have the boundedness in Theorem 1.1 because an extra factor appears when we write the transplantation operator in terms of the Hardy operator and its adjoint. When $|\beta - \alpha| \neq 2k$ the condition (A3) is required for the weight w to deduce (5) and, proceeding as in [2, Lemma 2.2], it is possible to prove that (A3) implies $w(m) \simeq w(m+1)$, then this condition does not appear explicitly in this case.

An immediate consequence of Theorem 1.1 is the following result.

Corollary 1.2. *Let $\alpha, \beta > -1$ with $\alpha \neq \beta$, $1 < p < \infty$, and w be a weight sequence that satisfies conditions (A1), (A2), and (A3). Then, there exists a positive constant C independent of f such that*

$$\frac{1}{C}\|f\|_{\ell^p(\mathbb{N},w)} \leq \|T_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w).$$

Next theorem also follows from the main theorem when power weights w_a are considered.

Theorem 1.3. *Let $\alpha, \beta > -1$ with $\alpha \neq \beta$, $\gamma = \min\{\alpha, \beta\}$, $1 < p < \infty$, and w_a be a power weight sequence with $a \in \mathbb{R}$. Then,*

$$\|T_\alpha^\beta f\|_{\ell^p(\mathbb{N}, w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N}, w_a)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w_a),$$

provided

$$-\frac{\gamma}{2} < \frac{a+1}{p} < \frac{\gamma}{2} + 1.$$

Previous theorem is the discrete counterpart of the sufficiency part of [9, Theorem 1.4]. The necessity of the condition $-\gamma/2 < (a+1)/p < \gamma/2 + 1$ is conjectured to be true for all possible values of the parameters α and β, but unfortunately we are not in position to prove it. However, there are two special situations in which the characterization is obtained. Indeed, for $\alpha, \beta > -1$ and $k \in \mathbb{N} \setminus \{0\}$,

$$\|T_\alpha^{\alpha+2k} f\|_{\ell^p(\mathbb{N}, w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N}, w_a)} \Longleftrightarrow \frac{a+1}{p} < \frac{\alpha}{2} + 1 \tag{6}$$

and

$$\|T_{\beta+2k}^{\beta} f\|_{\ell^p(\mathbb{N}, w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N}, w_a)} \Longleftrightarrow -\frac{\beta}{2} < \frac{a+1}{p}. \tag{7}$$

Theorem 1.3 is, in fact, a transplantation theorem with powers weights and it extends [3, Theorem 3] for functions $F \in L_\delta^1(0, \infty)$, which is defined as the set of measurable functions on $(0, \infty)$ such that

$$\|F\|_{L_\delta^1(0,\infty)} := \int_0^\infty |F(x)| x^\delta \, dx$$

is finite. It is known (see [6]) that

$$W_t^\alpha F(x) = \sum_{n=0}^\infty e^{-t\left(n + \frac{\alpha+1}{2}\right)} c_n^\alpha(F) \mathcal{L}_n^\alpha(x) \xrightarrow[t \to 0^+]{} F(x) \quad \text{a.e.}$$

for functions in $L_\delta^1(0, \infty)$ when $-\frac{\alpha}{2} \leq \delta + 1 \leq \frac{\alpha}{2} + 1$ for $\alpha \neq 0$ and $0 < \delta + 1 \leq 1$ for $\alpha = 0$ (the upper bounds ensure that W_t^α is finite for each function in $L_\delta^1(0, \infty)$ and the lower ones are necessary and sufficient to deduce the boundedness of the corresponding maximal operator from $L_\delta^1(0, \infty)$ into $L_\delta^{1,\infty}(0, \infty)$). Then, using the procedure given in [4] in the case of the ultraspherical expansions, from that convergence we deduce that

$$c_m^\beta(F) = \lim_{t \to 0^+} \int_0^\infty W_t^\alpha F(x) \mathcal{L}_m^\beta(x) \, dx.$$

In this way, using (8) as follows:

$$c_m^\beta(F) = \lim_{t \to 0^+} \sum_{n=0}^\infty e^{-t\left(n + \frac{\alpha+1}{2}\right)} c_n^\alpha(F) \int_0^\infty \mathcal{L}_n^\alpha(x) \mathcal{L}_m^\beta(x) \, dx = T_\alpha^\beta f(m),$$

with $f(n) = c_n^\alpha(F)$. The previous argument proves the following corollary, for which we will need to define the sets U and V given by

$$U = \{(\alpha,0) : \alpha \geq 0\} \cup \{(0,\beta) : \beta \geq 0\} \quad \text{and} \quad V = \{(\alpha,\beta) : \alpha,\beta > -1\} \setminus U.$$

They are motivated by the aforementioned conditions for the convergence of the operator W_t^α.

Corollary 1.4. *Let $n \in \mathbb{N}$, $\alpha,\beta > -1$ with $\alpha \neq \beta$, $\gamma = \min\{\alpha,\beta\}$, $1 < p < \infty$, w_a be a power weight sequence with $a \in \mathbb{R}$, and $\delta \in \mathbb{R}$ such that $-\frac{\gamma}{2} \leq \delta+1 \leq \frac{\gamma}{2}+1$ if $(\alpha,\beta) \in V$ and $0 < \delta+1 \leq 1$ if $(\alpha,\beta) \in U$. Then, there exists a positive constant C independent of F such that*

$$\frac{1}{C}\|c_n^\alpha(F)\|_{\ell^p(\mathbb{N},w_a)} \leq \|c_n^\beta(F)\|_{\ell^p(\mathbb{N},w_a)} \leq C\|c_n^\alpha(F)\|_{\ell^p(\mathbb{N},w_a)}, \qquad F \in L^1_\delta(0,\infty),$$

provided

$$-\frac{\gamma}{2} < \frac{a+1}{p} < \frac{\gamma}{2} + 1.$$

On the other hand, it is possible to repeat the showed procedure for functions in $L^p_\delta(0,\infty)$ having in mind the corresponding modifications in the convergence of the operator W_t^α, which are also treated in [6].

The structure of the paper is the following: in Section 2 some preliminary results related to Hardy operators and basic aspects of a discrete local Calderón–Zygmund theory are showed. Section 3 is devoted to the proofs of Theorems 1.1 and 1.3. In last two sections, the proofs of an auxiliary proposition and several lemmas are given.

2. Preliminary Results

First, we note that the transplantation operator T_α^β can be expressed for sequences $f \in \ell^2(\mathbb{N})$ by the series

$$T_\alpha^\beta f(m) = \sum_{n=0}^\infty f(n) K_\alpha^\beta(n,m), \tag{8}$$

where

$$K_\alpha^\beta(n,m) = \int_0^\infty \mathcal{L}_n^\alpha(x)\mathcal{L}_m^\beta(x)\,dx$$

is the kernel of the operator. Note that the trivial identity $K_\alpha^\beta(n,m) = K_\beta^\alpha(m,n)$ holds. In addition, the kernel satisfies the Markovian property

$$K_\alpha^\beta(n,m) = \sum_{k=0}^\infty K_\alpha^\delta(n,k)K_\delta^\beta(k,m),$$

which is a consequence of the decomposition $T_\alpha^\beta f(m) = T_\delta^\beta \circ T_\alpha^\delta f(m)$, obtained directly from the definition of the operator T_α^β.

To prove Theorem 1.1, we will study separately the transplantation operator T_α^β according to the three different regions $0 \leq n < m_0 := 2m/3$, $m_0 \leq n \leq m_0^* := 3m/2$, and $m_0^* < n < \infty$. The operator restricted to the second region is usually known as the local part, whereas when it is restricted to the union of the remaining ones is denominated as global part. From now on, we will use this denomination.

In the global part, a fundamental tool to prove our results is the boundedness with weights of the discrete Hardy operator and its adjoint, which are given by

$$Hf(m) = \frac{1}{m+1}\sum_{n=0}^{m} f(n) \quad \text{and} \quad H^*f(m) = \sum_{n=m}^{\infty} \frac{f(n)}{n+1},$$

respectively. It is well-known (cf. [14] for instance) that, for a weight sequence w and $1 < p < \infty$, condition (A1) is necessary and sufficient for the weighted inequality

$$\|w_{-\alpha/2}H(w_{\alpha/2}f)\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \tag{9}$$

whereas (A2) is necessary and sufficient for

$$\|w_{\beta/2}H^*(w_{-\beta/2}f)\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}. \tag{10}$$

Therefore, for $1 < p < \infty$, using (3) and (4), we have the following characterization:

$$\|w_{-\alpha/2}H(w_{\alpha/2}f)\|_{\ell^p(\mathbb{N},w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N},w_a)} \Longleftrightarrow \frac{a+1}{p} < \frac{\alpha}{2} + 1 \tag{11}$$

and

$$\|w_{\beta/2}H^*(w_{-\beta/2}f)\|_{\ell^p(\mathbb{N},w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N},w_a)} \Longleftrightarrow -\frac{\beta}{2} < \frac{a+1}{p}. \tag{12}$$

On its behalf, in the local part the proof relies on a discrete local version of the Calderón–Zygmund theory analogue of the one developed by A. Nowak and K. Stempak in [13].

Let us suppose that

$$K : (\mathbb{N}\times\mathbb{N}) \setminus \Delta \longrightarrow \mathbb{R},$$

where $\Delta = \{(n,n) : m \in \mathbb{N}\}$, is supported in the set

$$D = \{(n,m) : m_0 \le n \le m_0^*\}.$$

Moreover, let us suppose that the following conditions hold:

(a) the size condition

$$|K(n,m)| \lesssim \frac{1}{|n-m|},$$

(b) the regularity properties

$$|K(n,m) - K(n,l)| \lesssim \frac{|m-l|}{|m-n|^2}, \quad |n-m| > 2|m-l|, \tag{b1}$$

$$|K(n,m) - K(l,m)| \lesssim \frac{|n-l|}{|m-n|^2}, \quad |n-m| > 2|n-l|. \tag{b2}$$

A kernel $K(n,m)$ satisfying conditions (a) and (b) is called a discrete local standard kernel. By a discrete local Calderón–Zygmund operator, we mean a linear and bounded operator T from $\ell^r(\mathbb{N})$ into $\ell^r(\mathbb{N})$ for some $1 < r < \infty$, and such that there exists a discrete local standard kernel K so that, for every sequence $f \in c_{00}$, the space of sequences having a finite number of non-null terms,

$$Tf(m) = \sum_{\substack{n\in\mathbb{N} \\ m_0 \le n \le m_0^*}} f(n)K(n,m),$$

for every $m \in \mathbb{N}$ such that $f(m) = 0$.

Theorem 2.1. *Assume that T is a discrete local Calderón–Zygmund operator. Let $1 < p < \infty$ and w be a weight sequence that satisfies condition (A3). Then, the operator T can be extended from $\ell^r(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$ to $\ell^p(\mathbb{N}, w)$ as a bounded operator from $\ell^p(\mathbb{N}, w)$ into itself.*

As usual, previous theorem can be extended to the case $p = 1$ with a weak type inequality and weights satisfying a proper version of condition (A3), but as it has already been mentioned, we do not focus on this question in this paper.

Some comments about the proof of Theorem 2.1 are in order. Following the ideas in [13, Proposition 4.1] (see [5] for the details in the discrete case in a more general setting), from the conditions (a) and (b) for the kernel K, it is possible to prove some Hörmander type estimates. Indeed, if $I = [a,b] \cap \mathbb{N}$, $2I = [a-(b-a)/2, b+(b-a)/2] \cap \mathbb{N}$, and $W_m = \{j \in \mathbb{N} : m_0 \le j \le m_0^*\}$, it is verified that

$$\sum_{n\in\mathbb{N}\setminus 2I} \chi_{W_m}(n)|K(n,m) - K(n,l)||f(n)| \le C\mathcal{M}(|f|)(m), \quad m,l \in I, \tag{13}$$

and

$$\sum_{m\in\mathbb{N}\setminus 2I} \chi_{W_n}(m)|K(n,m) - K(s,m)||f(n)| \le C\mathcal{M}(|f|)(n), \quad n,s \in I, \tag{14}$$

where $\mathcal{M}$ denotes the non-centered discrete Hardy-Littlewood maximal function. Then, using some general results for operators satisfying estimates as (13) and (14) with $f(n) = 1$ in homogeneous spaces (see [10]), the result without weights is deduced. The extension including weights in the discrete $A_p(\mathbb{N})$ follows the standard procedure. It is described in [7, Chapter 7] and involve (13) and (14). Finally, following [13, Theorem 4.3] with appropriate adjustments, it is possible to pass from weights in $A_p(\mathbb{N})$ to local $A_p(\mathbb{N})$ weights satisfying (A3). This last step uses ideas coming from [1].

3. Proofs of Theorems 1.1 and 1.3

The proof of Theorem 1.1 will be based on particular cases of the transplantation operator. First one is Askey's case $\beta = \alpha + 2$, $\alpha > -1$, for which the expression of the kernel is closed.

Lemma 3.1. *Let $n, m \in \mathbb{N}$ and $\alpha > -1$. Then,*

$$K_\alpha^{\alpha+2}(n,m) = (\alpha+1)\frac{\omega_m^{\alpha+2}}{\omega_n^\alpha}, \qquad \text{for } 0 \le n \le m,$$

$$K_\alpha^{\alpha+2}(m+1,m) = -\left(\frac{m+1}{m+\alpha+2}\right)^{1/2},$$

and $K_\alpha^{\alpha+2}(n,m) = 0$ for $0 \le m < n-1$.

Previous lemma, whose proof will be given in the last section, actually shows how the transplantation operator $T_\alpha^{\alpha+2}$ is decomposed in the difference

$$T_\alpha^{\alpha+2} f(m) = (\alpha+1)\omega_m^{\alpha+2}\sum_{n=0}^{m}\frac{f(n)}{\omega_n^\alpha} - \left(\frac{m+1}{m+\alpha+2}\right)^{1/2} f(m+1). \quad (15)$$

Last identity, in conjunction with the characterization involved in (9), is the crucial point to prove the following proposition.

Proposition 3.2. *Let $\alpha > -1$, $k \in \mathbb{N}$, and $1 < p < \infty$. Let w be a weight sequence that satisfies $w(m) \simeq w(m+1)$. Then, the weighted inequality*

$$\|T_\alpha^{\alpha+2k} f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N},w), \quad (16)$$

holds for $k \ge 1$ if and only if w satisfies (A1).

Proof. We start with the case $k = 1$. First, we note that the condition $w(m) \simeq w(m+1)$ implies the equivalence

$$\left(\sum_{m=0}^{\infty} |f(m+1)|^p w(m)\right)^{1/p} \simeq \|f\|_{\ell^p(\mathbb{N},w)}.$$

Then, from (15) and using the equivalence $\omega_n^a \simeq (n+1)^{-a/2}$, it is clear that (16) with $k = 1$ is equivalent to

$$\|w_{-\alpha/2} H(w_{\alpha/2} f)\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}$$

and, in this way, the result follows in this case from (9) because w satisfies (A1).

Now, to finish the proof we only have to prove that (16) holds, for all $k > 1$, when w satisfies (A1). In this situation, the transplantation operator can be written as the composition

$$T_\alpha^{\alpha+2k} = T_{\alpha+2(k-1)}^{\alpha+2k} \circ \cdots \circ T_\alpha^{\alpha+2}.$$

Since condition (A1) holds by hypothesis, it also verified that the constants $[w]_{H_p}^{\alpha+2\ell}$, where $\ell = 1, \ldots, k-1$, are finite (remind that $[w]_{H_p}^{\alpha+\delta} \le [w]_{H_p}^\alpha$ for any $\delta \ge 0$). Then, every operator $T_{\alpha+\ell}^{\alpha+2(\ell+1)}$, with $\ell = 0, 1, \ldots, k-1$, is bounded with the weight sequence w. □

The transplantation operator $T_{\beta+2}^\beta$ is closely related to Askey's case. Since $K_{\beta+2}^\beta(n,m) = K_\beta^{\beta+2}(m,n)$, by Lemma 3.1, we have

$$T_{\beta+2}^\beta f(m) = -\left(\frac{m}{m+\beta+1}\right)^{1/2} f(m-1) + \frac{\beta+1}{\omega_m^\beta}\sum_{n=m}^{\infty} f(n)\omega_n^{\beta+2}.$$

Proposition 3.3. *Let $\beta > -1$, $k \in \mathbb{N}$, and $1 < p < \infty$. Let w be a weight sequence that satisfies $w(m) \simeq w(m+1)$. Then, the weighted inequality*

$$\|T^{\beta}_{\beta+2k} f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N},w),$$

holds for $k \geq 1$ if and only if w satisfies (A2).

The proof is analogue to the one for Proposition 3.2 but using the characterization (10), so we omit it.

Note that (6) and (7) can be deduced from the previous propositions using (3) and (4).

Next particular case in which we will base the proof of Theorem 1.1 corresponds with the restrictions $\alpha, \beta > -1$, $\alpha + \beta > 0$, and $\alpha < \beta < \alpha + 2$.

Proposition 3.4. *Let $\alpha, \beta > -1$ such that $\alpha + \beta > 0$ and $\alpha < \beta < \alpha + 2$, and $1 < p < \infty$. If w is a weight sequence that satisfies (A1), (A2), and (A3), then*

$$\|T^{\beta}_{\alpha} f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N},w).$$

This proposition is the central point to prove Theorem 1.1. Its proof is quite technical and we postpone it to next section. However, from Proposition 3.4 we can deduce the boundedness of the transplantation operator for $\beta < \alpha < \beta + 2$.

Proposition 3.5. *Let $\alpha, \beta > -1$ such that $\alpha + \beta > 0$ and $\beta < \alpha < \beta + 2$, and $1 < p < \infty$. If w is a weight sequence that satisfies (A1), (A2), and (A3), then*

$$\|T^{\beta}_{\alpha} f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N},w).$$

Proof. This result follows from Proposition 3.4 by applying a duality argument. Indeed, by the converse Hölder's inequality (see [8, (6.14)]),

$$\begin{aligned} \|T^{\beta}_{\alpha} f\|_{\ell^p(\mathbb{N},w)} &= \sup_{\substack{g \in c_{00} \\ \|g\|_{\ell^q(\mathbb{N},w)}=1}} \left| \sum_{m=0}^{\infty} g(m) T^{\beta}_{\alpha} f(m) w(m) \right| \\ &= \sup_{\substack{g \in c_{00} \\ \|g\|_{\ell^q(\mathbb{N},w)}=1}} \left| \sum_{m=0}^{\infty} T^{\alpha}_{\beta}(wg)(m) f(m) \right| \\ &\leq \|f\|_{\ell^p(\mathbb{N},w)} \sup_{\substack{g \in c_{00} \\ \|g\|_{\ell^q(\mathbb{N},w)}=1}} \|T^{\alpha}_{\beta}(wg)\|_{\ell^q(\mathbb{N},w^{-q/p})}. \end{aligned}$$

Here, c_{00} denotes the space of all sequences which have only finitely many nonzero elements and, again, q is the conjugate of p. Then, using the identities $[w^{-q/p}]^{\beta}_{H_q} = [w]^{\beta}_{H^*_p}$, $[w^{-q/p}]^{\alpha}_{H^*_q} = [w]^{\alpha}_{H_p}$, and $[w^{-q/p}]_{A^{\text{loc}}_q} = [w]_{A^{\text{loc}}_p}$, and the conditions (A1), (A2), and (A3), by Proposition 3.4, we have

$$\|T^{\alpha}_{\beta}(wg)\|_{\ell^q(\mathbb{N},w^{-q/p})} \lesssim \|wg\|_{\ell^q(\mathbb{N},w^{-q/p})} = \|g\|_{\ell^q(\mathbb{N},w)}$$

and the proof is finished. □

Let us give now the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us prove the cases not covered by Propositions 3.2, 3.3, 3.4, and 3.5.

(i) Case $\alpha > -1$ and $\alpha + 2k < \beta < \alpha + 2(k+1)$ for $k \in \{1, 2, \ldots\}$. This time the transplantation operator can be written as the composition

$$T_\alpha^\beta = T_{\alpha+2k}^\beta \circ T_\alpha^{\alpha+2k}.$$

Then, in this case, the result follows from Propositions 3.4 and 3.2.

(ii) Case $\beta > -1$ and $\beta + 2k < \alpha < \beta + 2(k+1)$ for $k \in \{1, 2, \ldots\}$. This case can be deduced from (i) by a duality argument as we did to prove Proposition 3.5 from Proposition 3.4, so we omit the details.

(iii) Case $-1 < \alpha \leq 0$, $-1 < \beta \leq 1$, $\alpha + \beta \leq 0$, and $\alpha < \beta$. Here, we put the transplantation operator as the composition

$$T_\alpha^\beta = T_{\alpha+2}^\beta \circ T_\alpha^{\alpha+2}.$$

The operators involved in this case can be controlled applying Propositions 3.5 and 3.2.

(iv) Case $-1 < \alpha \leq 1$, $-1 < \beta \leq 0$, $\alpha + \beta \leq 0$, and $\beta < \alpha$. In this last case, we can use duality and again we omit the details. □

Let us prove now Theorem 1.3 to finish the present section. Before that, we note that clearly $w_a(m) \simeq w_a(m+1)$ for any $a \in \mathbb{R}$.

Proof of Theorem 1.3. Under the hypothesis $-\gamma/2 < (a+1)/p < \gamma/2 + 1$, the weight w_a satisfies the conditions (A1), (A2), and (A3). Then, the boundedness of the transplantation operator with the weight w_a is an immediate consequence of Theorem 1.1. □

4. Proof of Proposition 3.4

The proofs of Lemmas 4.1, 4.2, and 4.4 that we will use in the proof of Proposition 3.4 are quite technical and they are postponed to the last section.

From now on, we will profusely use the notation $\nu_j^a = 4j + 2a + 2$ and

$$\mathcal{K}_\alpha^\beta(n, m) = \int_0^\infty \mathcal{L}_n^\alpha(x) \mathcal{L}_m^\beta(x) \frac{dx}{x}.$$

Note that the last integral is convergent if $\alpha + \beta > 0$ and positive at least under the extra assumption $\alpha < \beta < \alpha + 2$ (see Eq. (26) in next section).

In the following result, we deduce a proper expression for the kernel of the transplantation operator.

Lemma 4.1. *Let $\alpha, \beta > -1$ such that $\alpha + \beta > 0$ and $|\beta - \alpha| \neq 2k$ for every $k \in \mathbb{N}$, and $n, m \in \mathbb{N}$. Then,*

$$K_\alpha^\beta(n, m) = \frac{\beta^2 - \alpha^2}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n, m).$$

From this lemma, it is clear that

$$T_\alpha^\beta f(m) = (\beta^2 - \alpha^2) \sum_{\substack{n=0\\ n\neq m}}^{\infty} \frac{f(n)}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n,m) + f(m) K_\alpha^\beta(m,m).$$

Splitting the transplantation operator into four different operators according to the aforementioned regions, we obtain

$$T_\alpha^\beta f(m) = O_\alpha^\beta f(m) + Q_\alpha^\beta f(m) + (O^*)_\alpha^\beta f(m) + P_\alpha^\beta f(m),$$

where last operators are given explicitly by

$$O_\alpha^\beta f(m) = (\beta^2 - \alpha^2) \sum_{\substack{n\in\mathbb{N}\\ n< m_0}} \frac{f(n)}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n,m),$$

$$(O^*)_\alpha^\beta f(m) = (\beta^2 - \alpha^2) \sum_{\substack{n\in\mathbb{N}\\ n> m_0^*}} \frac{f(n)}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n,m),$$

$$Q_\alpha^\beta f(m) = (\beta^2 - \alpha^2) \sum_{\substack{n\in\mathbb{N}, n\neq m\\ m_0\le n\le m_0^*}} \frac{f(n)}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n,m),$$

and

$$P_\alpha^\beta f(m) = f(m) K_\alpha^\beta(m,m).$$

By applying Cauchy–Schwarz's inequality, it is immediate that $|K_\alpha^\beta(m,m)| \le 1$ and then

$$\|P_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}. \tag{17}$$

To estimate the remaining operators in the decomposition of T_α^β, we will use frequently, without an explicit mention to it, the equivalence

$$|\nu_m^\beta - \nu_n^\alpha| \simeq \begin{cases} m+1, & 0 \le n < m_0, \\ |m-n|, & m_0 \le n \le m_0^*,\ n \neq m, \\ n+1, & m_0^* < n. \end{cases}$$

It holds for $\alpha, \beta > -1$ such that $\alpha < \beta < \alpha + 2$ and its proof is elementary.

Now, let us focus on O_α^β and $(O^*)_\alpha^\beta$, which are defined in the global part. Following estimation will play a key role.

Lemma 4.2. *Let $n, m \in \mathbb{N}$ and $\alpha, \beta > -1$ such that $\alpha + \beta > 0$ and $\alpha < \beta < \alpha + 2$. Then,*

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim \left(\frac{n+1}{m+1}\right)^{\alpha/2}, \qquad 0 \le n \le m, \tag{18}$$

and

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim \left(\frac{m+1}{n+1}\right)^{\beta/2}, \qquad m \le n < \infty. \tag{19}$$

Moreover,

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim 1, \qquad m_0 \le n \le m_0^*. \tag{20}$$

Now, regarding the operator O_α^β, by (18), it is clear the estimate

$$|O_\alpha^\beta f(m)| \lesssim w_{-(\alpha/2+1)}(m) \sum_{\substack{n\in\mathbb{N}\\ n<m_0}} w_{\alpha/2}(n)|f(n)| \lesssim w_{-\alpha/2}(m) H(w_{\alpha/2}|f|)(m).$$

Therefore, O_α^β is bounded by a Hardy operator and then, for $1<p<\infty$, the weighted norm inequality

$$\|O_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)} \tag{21}$$

holds provided (A1). On its behalf, the treatment of the operator $(O^*)_\alpha^\beta$ is analogous. Indeed, using now (19), we have

$$|(O^*)_\alpha^\beta f(m)| \lesssim w_{\beta/2}(m) H^*(w_{-\beta/2}|f|)(m).$$

Then, $(O^*)_\alpha^\beta$ is bounded by the adjoint of the Hardy operator and, for $1<p<\infty$, the weighted norm inequality

$$\|(O^*)_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)} \tag{22}$$

holds provided condition (A2).

To study the local part Q_α^β, we use Theorem 2.1. First, we have to prove that the kernel $K_\alpha^\beta(n,m)$ satisfies properties (a) and (b), so it is a local standard kernel and we can apply the local Calderón–Zygmund theory. The size condition (a) for the kernel is contained in the next proposition.

Proposition 4.3. *Let $\alpha,\beta>-1$ such that $\alpha+\beta>0$ and $\alpha<\beta<\alpha+2$. Let $n,m\in\mathbb{N}$ such that $n\neq m$ and $m_0\le n\le m_0^*$. Then,*

$$|K_\alpha^\beta(n,m)| \lesssim \frac{1}{|n-m|}.$$

Proof. The proof is immediate by means of Lemma 4.1 and the estimate (20) given in Lemma 4.2. □

To deduce the regularity properties (b), we will need an extra lemma.

Lemma 4.4. *Let $n,m\in\mathbb{N}$, and $\alpha,\beta>-1$ such that $\alpha+\beta>0$ and $\alpha<\beta<\alpha+2$. Then,*

$$\int_0^\infty \mathcal{L}_n^{\alpha+1}(x)\mathcal{L}_m^\beta(x)\frac{dx}{x^{1/2}} \lesssim \frac{(n+1)^{1/2}}{m+1-n}\left(\frac{n+1}{m+1}\right)^{\alpha/2}, \qquad 0\le n\le m,$$

and

$$\int_0^\infty \mathcal{L}_n^{\alpha+1}(x)\mathcal{L}_m^\beta(x)\frac{dx}{x^{1/2}} \lesssim \frac{(n+1)^{1/2}}{n+1-m}\left(\frac{m+1}{n+1}\right)^{\beta/2}, \qquad 0\le m\le n.$$

With the help of this lemma, we can prove the following result.

Proposition 4.5. *Let $\alpha,\beta>-1$ such that $\alpha+\beta>0$ and $\alpha<\beta<\alpha+2$. Let $n,m\in\mathbb{N}$ so that $n\neq m$ and $m_0\le n\le m_0^*$. Then,*

$$|K_\alpha^\beta(n,m)-K_\alpha^\beta(n+1,m)| \lesssim \frac{1}{|m-n|^2} \tag{23}$$

and

$$|K_\alpha^\beta(n,m)-K_\alpha^\beta(n,m+1)| \lesssim \frac{1}{|m-n|^2}. \tag{24}$$

Note that the estimates (23) and (24) ensure the regularity properties (b2) and (b1) for the kernel $K_\alpha^\beta(n,m)$. Let us see for instance that (23) implies (b2) (the proof for (24) implies (b1) is analogous). Let us suppose that $n<l$. By the triangle inequality

$$|K_\alpha^\beta(n,m)-K_\alpha^\beta(l,m)|\lesssim\sum_{j=0}^{l-n-1}|K_\alpha^\beta(n+j,m)-K_\alpha^\beta(n+1+j,m)|.$$

If $n>m$, we apply (23) to get the desired estimate. When $n<m$, we apply again (23) and then use the fact $|n-m|>2|n-l|$, so the result follows. The case $n>l$ is similar and we omit the details.

Proof of Proposition 4.5. We focus on the proof for the bound (23). The one corresponding to (24) can be deduced in a similar way.

The key point in the proof is an appropriate decomposition of the difference of the involved kernels. By Lemma 4.1, we rewrite that difference by the expression

$$|K_\alpha^\beta(n,m)-K_\alpha^\beta(n+1,m)|=\frac{\beta^2-\alpha^2}{|\nu_m^\beta-\nu_n^\alpha||\nu_m^\beta-\nu_{n+1}^\alpha|}\left|S_1^{\alpha,\beta}(n,m)+S_2^{\alpha,\beta}(n,m)\right|,$$

where we have denote

$$S_1^{\alpha,\beta}(n,m)=(\nu_n^\alpha-\nu_{n+1}^\alpha)\mathcal{K}_\alpha^\beta(n,m)$$

and

$$S_2^{\alpha,\beta}(n,m)=(\nu_m^\beta-\nu_n^\alpha)\int_0^\infty\left(\mathcal{L}_n^\alpha(x)-\mathcal{L}_{n+1}^\alpha(x)\right)\mathcal{L}_m^\beta(x)\frac{dx}{x}.$$

Since $|\nu_m^\beta-\nu_n^\alpha|\simeq|\nu_m^\beta-\nu_{n+1}^\alpha|\simeq|m-n|$, provided $n\neq m$ and $n\neq m-1$, we have to check the uniform bound

$$\left|S_1^{\alpha,\beta}(n,m)+S_2^{\alpha,\beta}(n,m)\right|\lesssim 1 \tag{25}$$

for $m_0\le n\le m-2$ with $m\ge 6$, and $m+1\le n\le m_0^*$ with $m\ge 2$.

In the special case $n=m-1$, we can obtain (23) by showing that

$$|K_\alpha^\beta(n,n+1)-K_\alpha^\beta(n+1,n+1)|\lesssim 1,$$

but this is immediate by Cauchy-Schwarz's inequality and the orthonormality of the Laguerre functions.

Let us prove (25). On the one hand, it is easy to obtain the uniform bound

$$|S_1^{\alpha,\beta}(n,m)|\lesssim\mathcal{K}_\alpha^\beta(n,m)\lesssim 1$$

using the estimate (20). However, $S_2^{\alpha,\beta}(n,m)$ is more difficult to deal with and we have to split it in two parts, namely

$$S_2^{\alpha,\beta}(n,m)=I_1^{\alpha,\beta}(n,m)+I_2^{\alpha,\beta}(n,m),$$

with

$$I_1^{\alpha,\beta}(n,m)=(\nu_m^\beta-\nu_n^\alpha)w_n^{\alpha+1}\sqrt{n+\alpha+1}\\ \times\int_0^\infty\left(L_n^\alpha(x)-L_{n+1}^\alpha(x)\right)\mathcal{L}_m^\beta(x)x^{\alpha/2-1}e^{-x/2}\,dx$$

and

$$I_2^{\alpha,\beta}(n,m) = \frac{\nu_m^\beta - \nu_n^\alpha}{\sqrt{n+1}}(\sqrt{n+\alpha+1} - \sqrt{n+1})\mathcal{K}_\alpha^\beta(n,m)$$
$$= \frac{\alpha(\nu_m^\beta - \nu_n^\alpha)}{\sqrt{n+1}(\sqrt{n+\alpha+1} + \sqrt{n+1})}\mathcal{K}_\alpha^\beta(n,m).$$

Last expression can be bounded using the estimates in Lemma 4.2. Indeed,

$$|I_2^{\alpha,\beta}(n,m)| \lesssim \frac{|m-n|}{n+1} \lesssim 1.$$

Regarding $I_1^{\alpha,\beta}(n,m)$, we use the identity [15, Eq. 18.9.14]

$$L_n^\alpha(x) - L_{n+1}^\alpha(x) = \frac{x}{n+1}L_n^{\alpha+1}(x) - \frac{\alpha}{n+1}L_n^\alpha(x).$$

By means of it, we obtain the decomposition

$$|I_1^{\alpha,\beta}(n,m)| = |I_{1,1}^{\alpha,\beta}(n,m) - I_{1,2}^{\alpha,\beta}(n,m)|,$$

where

$$I_{1,1}^{\alpha,\beta}(n,m) = (\nu_m^\beta - \nu_n^\alpha)\frac{\sqrt{n+\alpha+1}}{n+1}\int_0^\infty \mathcal{L}_n^{\alpha+1}(x)\mathcal{L}_m^\beta(x)\frac{dx}{x^{1/2}}$$

and

$$I_{1,2}^{\alpha,\beta}(n,m) = (\nu_m^\beta - \nu_n^\alpha)\frac{\alpha}{n+1}\mathcal{K}_\alpha^\beta(n,m).$$

One more time, by (20), we obtain $|I_{1,2}^{\alpha,\beta}(n,m)| \lesssim 1$.

Finally, by Lemma 4.4, it is easy to obtain

$$|I_{1,1}^{\alpha,\beta}(n,m)| \lesssim \begin{cases} \frac{m-n}{m+1-n}\left(\frac{n+1}{m+1}\right)^{\alpha/2}, & m_0 \le n \le m-2, \\ \text{begineqnarray*10pt]}\frac{n-m}{n+1-m}\left(\frac{m+1}{n+1}\right)^{\beta/2}, & m+1 \le n \le m_0^*, \end{cases}$$

so $|I_{1,1}^{\alpha,\beta}(n,m)| \lesssim 1$, which concludes the proof. □

On the other hand, by taking $p=2$ and $w(n)=1$ for all $n \in \mathbb{N}$ in the inequalities (17), (21), and (22), it follows that P_α^β and both global operators O_α^β and $(O^*)_\alpha^\beta$ are bounded in $\ell^2(\mathbb{N})$. Since

$$Q_\alpha^\beta f(m) = T_\alpha^\beta f(m) - O_\alpha^\beta f(m) - (O^*)_\alpha^\beta f(m) - P_\alpha^\beta f(m),$$

then the operator Q_α^β is also bounded on $\ell^2(\mathbb{N})$. (Note that T_α^β is obviously a bounded operator from $\ell^2(\mathbb{N})$ into itself.)

Therefore, under the assumptions of Proposition 3.4, $K_\alpha^\beta(n,m)$ is a local standard kernel and the operator Q_α^β is a local Calderón–Zygmund operator. Therefore, by Theorem 2.1, for $1 < p < \infty$ the weighted norm inequality

$$\|Q_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}$$

holds provided w satisfies (A3). This fact finishes the proof of Proposition 3.4.

5. Proofs of Auxiliary Results

Most proofs showed in the present section are based on the connection formula for Laguerre polynomials [15, Eq. 18.18.18] given by

$$L_n^\alpha(x) = \sum_{j=0}^{n} \frac{(\alpha-\beta)_{n-j}}{(n-j)!} L_j^\beta(x),$$

which allow us to express a Laguerre polynomial of degree n and order α as a linear combination of other Laguerre polynomials of order β and degrees less or equal than n. Here, $(z)_\ell$ denotes the usual Pochhammer symbol [15, section 5.2(iii)] defined by

$$(z)_\ell = \begin{cases} 1, & \text{if } \ell = 0, \\ z(z+1)\cdots(z+\ell-1), & \text{if } \ell > 0. \end{cases}$$

Proof of Lemma 3.1. For the case $\beta = \alpha+2$, the kernel is given by

$$K_\alpha^{\alpha+2}(n,m) = \omega_n^\alpha \omega_m^{\alpha+2} \int_0^\infty L_n^\alpha(x) L_m^{\alpha+2}(x) x^{\alpha+1} e^{-x}\, dx.$$

Note that in these particular cases, we have the well-known identities

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x), \qquad n \geq 1,$$

and

$$L_m^{\alpha+2}(x) = \sum_{j=0}^{m} L_j^{\alpha+1}(x),$$

which can be deduced easily from the connection formula. Therefore, due to the orthogonality, the kernel is given by

$$K_\alpha^{\alpha+2}(n,m) = \omega_n^\alpha \omega_m^{\alpha+2} \sum_{j=0}^{m} \left(\delta_{j,n} \left(\omega_j^{\alpha+1}\right)^{-2} - \delta_{j,n-1} \left(\omega_j^{\alpha+1}\right)^{-2} \right), \qquad n \geq 1,$$

and, using that $L_0^\alpha(x) = L_0^{\alpha+1}(x) = 1$,

$$K_\alpha^{\alpha+2}(0,m) = (\alpha+1) \frac{\omega_m^{\alpha+2}}{\omega_0^\alpha}.$$

From these identities, the statement of the lemma is obtained in a straightforward way by checking the cases $m+1 < n$, $n = m+1$, and $1 \leq n \leq m$. □

Proof of Lemma 4.1. The proof relies on the direct application of the integration by parts formula over the kernel $K_\alpha^\beta(n,m)$, but it is more straightforward if the Laguerre functions of Hermite type $\varphi_n^\alpha(x) = \sqrt{x}\mathcal{L}_n^\alpha(x^2)/\omega_n^\alpha$ are considered. By means of them, the kernel can be rewritten as

$$K_\alpha^\beta(n,m) = 2\omega_n^\alpha \omega_m^\beta \int_0^\infty \varphi_n^\alpha(x) \varphi_m^\beta(x)\, dx.$$

The functions φ_n^α are eigenfunctions of a second order differential operator

$$\mathfrak{L}^\alpha \varphi_n^\alpha(x) = -\nu_n^\alpha \varphi_n^\alpha(x), \qquad \mathfrak{L}^\alpha = \frac{d^2}{dx^2} + \frac{1/4-\alpha^2}{x^2} - x^2,$$

with its respective eigenvalue $\nu_n^\alpha = 4n + 2\alpha + 2$. Then, since $|\beta - \alpha| \neq 2k$ for every $k \in \mathbb{N}$ implies $\nu_m^\beta \neq \nu_n^\alpha$, the statement is obtained directly from the identity

$$\int_0^\infty \mathfrak{L}^\alpha \varphi_n^\alpha(x)\varphi_m^\beta(x)\,dx = \int_0^\infty \varphi_n^\alpha(x)\mathfrak{L}^\beta \varphi_m^\beta(x)\,dx + (\beta^2 - \alpha^2)\int_0^\infty \varphi_n^\alpha(x)\varphi_m^\beta(x)\frac{dx}{x^2}.$$

Note that the assumption $\alpha + \beta > 0$ is crucial to get previous identity by means of the integration by parts formula since the integrated term is $x^{\alpha+\beta}e^{-x^2}$ times a polynomial expression. Then, the mentioned condition ensures that this term vanishes at the origin (whereas it vanishes at infinity due to the exponential function). □

Proof of Lemma 4.2. In the spirit of the proof of Lemma 3.1, we use the connection formula to obtain

$$L_n^\alpha(x) = \sum_{j=0}^{n} \frac{(\alpha/2 - \beta/2 + 1)_{n-j}}{(n-j)!} L_j^{\alpha/2+\beta/2-1}(x)$$

and, similarly,

$$L_m^\beta(x) = \sum_{k=0}^{m} \frac{(\beta/2 - \alpha/2 + 1)_{m-k}}{(m-k)!} L_k^{\alpha/2+\beta/2-1}(x).$$

Therefore, due to the orthogonality,

$$\mathcal{K}_\alpha^\beta(n,m) = \sum_{j=0}^{\min\{n,m\}} \frac{(\alpha/2 - \beta/2 + 1)_{n-j}}{(n-j)!} \frac{(\beta/2 - \alpha/2 + 1)_{m-j}}{(m-j)!} \frac{\omega_n^\alpha \omega_m^\beta}{(\omega_j^{\alpha/2+\beta/2-1})^2}. \tag{26}$$

Then, we have to estimate previous sum. We can put the Pochhammer symbols in terms of the Gamma function and by [15, Eq. 5.11.12], we get the estimate

$$\frac{(\alpha/2 - \beta/2 + 1)_{n-j}}{(n-j)!} = \frac{\Gamma(n - j + \alpha/2 - \beta/2 + 1)}{\Gamma(n - j + 1)\Gamma(\alpha/2 - \beta/2 + 1)} \simeq (n + 1 - j)^{\alpha/2-\beta/2}$$

and the analogous for

$$\frac{(\beta/2-\alpha/2+1)_{m-j}}{(m-j)!} = \frac{\Gamma(m - j+\beta/2 - \alpha/2 + 1)}{\Gamma(m - j + 1)\Gamma(\beta/2 - \alpha/2 + 1)} \simeq (m + 1 - j)^{\beta/2-\alpha/2}.$$

Therefore, having in mind the equivalence $\omega_n^\alpha \simeq (n+1)^{-\alpha/2}$ we obtain that

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim (n + 1)^{-\alpha/2}(m + 1)^{-\beta/2} R_\alpha^\beta(n,m), \tag{27}$$

with

$$R_\alpha^\beta(n,m) = \sum_{j=0}^{\min\{n,m\}} (n + 1 - j)^{\alpha/2-\beta/2}(m + 1 - j)^{\beta/2-\alpha/2}(j + 1)^{\alpha/2+\beta/2-1}.$$

For $0 \le n \le m$, we have

$$R_\alpha^\beta(n,m) \lesssim (m+1)^{\beta/2-\alpha/2} \sum_{j=0}^{n} (n+1-j)^{\alpha/2-\beta/2}(j+1)^{\alpha/2+\beta/2-1}$$
$$\simeq (m+1)^{\beta/2-\alpha/2}(n+1)^{\alpha}.$$

Then, by (27),

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim \left(\frac{n+1}{m+1}\right)^{\alpha/2}, \qquad 0 \le n \le m,$$

and the proof of (18) is completed.

To prove (19), we estimate $R_\alpha^\beta(n,m)$ distinguishing the cases $m \le n \le m_0^*$ and $m_0^* < n$. When $m \le n \le m_0^*$, we have

$$\begin{aligned} R_\alpha^\beta(n,m) &\lesssim (m+1)^{\beta/2-\alpha/2}(n+1)^{\alpha/2-\beta/2} \sum_{j=0}^{[m/2]} (j+1)^{\alpha/2+\beta/2-1} \\ &\quad + (m+1)^{\beta-1} \sum_{j=[m/2]+1}^{m} (n+1-j)^{\alpha/2-\beta/2} \\ &\lesssim (m+1)^{\beta}(n+1)^{\alpha/2-\beta/2} + (m+1)^{\beta-1}\sum_{j=0}^{n}(n+1-j)^{\alpha/2-\beta/2} \\ &\lesssim (m+1)^{\beta}(n+1)^{\alpha/2-\beta/2} + (m+1)^{\beta-1}(n+1)^{\alpha/2-\beta/2+1} \\ &\lesssim (m+1)^{\beta}(n+1)^{\alpha/2-\beta/2}, \end{aligned}$$

where in the last step, we have used that $n \simeq m$ in this case. Now, for $m_0^* < n$, we have

$$R_\alpha^\beta(n,m) \lesssim (m+1)^{\beta/2-\alpha/2}(n+1-m)^{\alpha/2-\beta/2} \sum_{j=0}^{m} (j+1)^{\alpha/2+\beta/2-1}$$
$$\lesssim (m+1)^{\beta}(n+1)^{\alpha/2-\beta/2}.$$

Then, using (27),

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim \left(\frac{m+1}{n+1}\right)^{\beta/2}$$

and we finish the proof of (19).

Obviously, (20) is an immediate consequence of (18), (19) and the restriction $m_0 \le n \le m_0^*$. □

Proof of Lemma 4.4. Let us denote

$$J_\alpha^\beta(n,m) := \omega_n^{\alpha+1}\omega_m^\beta \int_0^\infty L_n^{\alpha+1}(x) L_m^\beta(x) x^{\alpha/2+\beta/2} e^{-x}\, dx.$$

By means of the connection formula, we have

$$L_n^{\alpha+1}(x) = \sum_{j=0}^{n} \frac{(\alpha/2-\beta/2+1)_{n-j}}{(n-j)!} L_j^{\alpha/2+\beta/2}(x)$$

and

$$L_m^\beta(x) = \sum_{j=0}^{m} \frac{(\beta/2-\alpha/2)_{m-j}}{(m-j)!} L_j^{\alpha/2+\beta/2}(x).$$

Therefore, proceeding in the same way as in the proof of Lemma 4.2, we obtain the estimate

$$J_\alpha^\beta(n,m) \simeq (n+1)^{-\alpha/2-1/2}(m+1)^{-\beta/2} S_\alpha^\beta(n,m),$$

where

$$S_\alpha^\beta(n,m) = \sum_{j=0}^{\min\{n,m\}} (n+1-j)^{\alpha/2-\beta/2}(m+1-j)^{\beta/2-\alpha/2-1}(j+1)^{\alpha/2+\beta/2}.$$

In case that $n \le m$, we have (see proof of Lemma 4.2)

$$\begin{aligned} S_\alpha^\beta(n,m) &\lesssim \frac{n+1}{m+1-n} \sum_{j=0}^{n} (n+1-j)^{\alpha/2-\beta/2}(m+1-j)^{\beta/2-\alpha/2}(j+1)^{\alpha/2+\beta/2-1} \\ &\lesssim \frac{(n+1)^{\alpha+1}(m+1)^{\beta/2-\alpha/2}}{m+1-n}, \end{aligned}$$

whereas, if $m \le n$,

$$\begin{aligned} S_\alpha^\beta(n,m) &\lesssim (n+1-m)^{\alpha/2-\beta/2} \sum_{j=0}^{m} (m+1-j)^{\beta/2-\alpha/2-1}(j+1)^{\alpha/2+\beta/2} \\ &\lesssim (n+1-m)^{\alpha/2-\beta/2}(m+1)^{\alpha/2+\beta/2} \sum_{j=0}^{m} (m+1-j)^{\beta/2-\alpha/2-1} \\ &\lesssim (n+1-m)^{\alpha/2-\beta/2}(m+1)^{\beta} \lesssim \frac{(n+1)^{\alpha/2-\beta/2+1}(m+1)^{\beta}}{n+1-m}. \end{aligned}$$

From these two estimates, the lemma follows immediately. □

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