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Evolution Maps and Admissibility

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Abstract. We show that the admissibility properties for a sequence of linear operators and the corresponding evolution maps are equivalent on various Banach spaces. We then use this information to obtain new descriptions for the hyperbolicity of a sequence of linear operators and the corresponding evolution maps.

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1. Introduction

Our main objective is to establish the equivalence of the admissibility properties for a sequence of linear operators on a Banach space and the corresponding evolution maps on various Banach spaces. This includes spaces of bounded sequences, of sequence vanishing at infinity, and ℓ^p spaces. We also consider the relation to hyperbolicity and, as an application, we use the correspondence of the admissibility properties to give new descriptions for the hyperbolicity of a sequence of linear operators.

The notion of admissibility goes back to Perron in [9]. A simple modification of his work for continuous time gives the following statement. Let $(A_m)_{m\in\mathbb{Z}}$ be a (two-sided) sequence of $n \times n$ matrices. If for each bounded sequence $(y_m)_{m\in\mathbb{Z}}$ in $\mathbb{R}^n$ there exists $x_0 \in \mathbb{R}^n$ such that the sequence

$$x_{m+1} = A_m x_m + y_{m+1} \tag{1}$$

is bounded for $m \in \mathbb{N}$, then any bounded sequence $A_m \cdots A_1 x$ tends to zero when $m \to \infty$. Related results for discrete time were first obtained by Li in [6]. For some early contributions we refer the reader to the books [5,7].

A general notion of admissibility can be introduced as follows. We say that a pair of Banach spaces (C, D) is *admissible* if for every sequence $(y_m)_{m\in\mathbb{Z}}$ in C there exists a unique sequence $(x_m)_{m\in\mathbb{Z}}$ in D satisfying (1). We consider this notion for various Banach spaces of sequences with values

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in a given Banach space X as well as their corresponding evolution maps. Namely, given a Banach space $Y \subset X^{\mathbb{Z}}$, we define a map $S\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ by

$$(Su)_n = A_{n-1}u_{n-1} \quad \text{for } n \in \mathbb{Z} \text{ and } u = (u_m)_{m\in\mathbb{Z}} \in X^{\mathbb{Z}}.$$

Provided that $S(Y) \subset Y$, the map S is called the *evolution map* of the sequence $(A_m)_{m\in\mathbb{Z}}$ on the Banach space Y.

As noted above, our main objective is to establish a faithful correspondence between the admissibility properties for some pairs of Banach spaces at the levels of sequences of linear operators and evolution maps, also on various Banach spaces. Evolution maps transfer the dynamics at the level of a sequence of linear maps to a dynamics on much larger space, although this new dynamics is autonomous, which often makes the approach much simpler. Furthermore, the properties of the dynamics are also transferred to those of the evolution map, and this often leads to much simpler proofs. In addition, the transference of properties is quite helpful in finding appropriate nonautonomous notions when they are not yet available in the nonautonomous case. An important example of such a correspondence is the study of hyperbolicity and its various variations that goes back to Mather in [8]. The theory of semigroups is nowadays an important tool in the theory of differential equations (see for example [10]).

To illustrate our results, we formulate a particular case of Theorem 3. Let $\ell_0^\infty(X)$ be the set of all sequences in X vanishing at infinity and define $D_0(X) = \ell_0^\infty(\ell_0^\infty(X))$.

Theorem 1. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a bounded sequence of linear maps. Then the following properties are equivalent:*

1. *for each $(y_m)_{m\in\mathbb{Z}} \in \ell_0^\infty(X)$ there exists a unique $(x_m)_{m\in\mathbb{Z}} \in \ell^\infty(X)$ satisfying*

$$x_{m+1} = A_m x_m + y_{m+1} \quad \textit{for } m \in \mathbb{Z};$$

2. *for each $(v_m)_{m\in\mathbb{Z}} \in D_0(X)$ there exists a unique $(u_m)_{m\in\mathbb{Z}} \in D_0(X)$ satisfying*

$$u_{m+1} = Su_m + v_{m+1} \quad \textit{for } m \in \mathbb{Z}.$$

Using the notion of admissibility, Theorem 1 can be reformulated as follows: for a bounded sequence A the following properties are equivalent:

1. the pair formed by the spaces $\ell_0^\infty(X)$ and $\ell^\infty(X)$ is admissible;
2. the pair formed by the spaces $D_0(X)$ and $D_0(X)$ is admissible.

Further pairs of spaces are considered in the paper. These include in particular spaces of sequences with bounded exponential growth and ℓ^p spaces.

More generally, we consider families of norms $\|\cdot\|_m$, for $m \in \mathbb{Z}$. These norms play an essential role for example in smooth ergodic theory in the presence of nonuniform exponential behavior. Moreover, we use our results on the equivalence of admissibility properties for sequence of linear operators and evolution maps to give new descriptions for the hyperbolicity of a sequence of linear operators (see Sect. 6).

2. Evolution Maps

Let $X = (X, \|\cdot\|)$ be a Banach space. Given a sequence $A = (A_m)_{m\in\mathbb{Z}}$ of continuous maps on X, we define

$$U(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n \end{cases}$$

for each $m, n \in \mathbb{Z}$ with $m \geq n$. We shall only consider sequences A that are exponentially bounded with respect to a sequence of norms. Namely, let $\|\cdot\|_m$, for $m \in \mathbb{Z}$, be a sequence of norms on X such that

$$\|x\| \leq \|x\|_m \leq R_m\|x\| \quad \text{for } m \in \mathbb{Z} \text{ and } x \in X$$

and some sequence $(R_m)_{m\in\mathbb{Z}}$ in $\mathbb{R}^+$. We say that the sequence A is *exponentially bounded with respect to the norms* $\|\cdot\|_m$ if there exist $\alpha, \kappa > 0$ such that

$$\|U(m,n)x\|_m \leq \kappa e^{\alpha(m-n)}\|x_n\|_n \quad \text{for } m \geq n \text{ and } x \in X.$$

To each sequence $A = (A_m)_{m\in\mathbb{Z}}$ of continuous maps on X, we associate maps $S = S|_Y$ on certain Banach spaces $Y \subset X^{\mathbb{Z}}$. Namely, we define the *evolution map* of A on a space Y by

$$(Su)_n = A_{n-1}u_{n-1} \quad \text{for } n \in \mathbb{Z} \text{ and } u = (u_m)_{m\in\mathbb{Z}} \in Y, \tag{2}$$

whenever $S(Y) \subset Y$.

We also introduce a family of maps that will be used in the study of admissibility. Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X. Given a sequence $y = (y_n)_{n\in\mathbb{Z}}$ in X, we define a map $T_y \colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ by

$$(T_y u)_n = A_{n-1}u_{n-1} + y_n \quad \text{for } n \in \mathbb{Z} \text{ and } u = (u_m)_{m\in\mathbb{Z}} \in X^{\mathbb{Z}}.$$

3. Admissibility for Bounded Sequences

In this section we consider a certain admissibility property on a space of bounded sequences for evolution maps.

3.1. Evolution Maps

Let $\ell^\infty(X)$ be the set of all sequences $x = (x_m)_{m\in\mathbb{Z}}$ with values in X such that

$$\|x\|_\infty := \sup_{m\in\mathbb{Z}} \|x_m\|_m < \infty.$$

We note that $\ell^\infty(X)$ is a Banach space when equipped with the norm $\|\cdot\|_\infty$. We also consider the closed subspace $\ell_0^\infty(X) \subset \ell^\infty(X)$ of those $x \in \ell^\infty(X)$ such that

$$\lim_{|m|\to\infty} \|x_m\|_m = 0.$$

Proposition 1. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then the following properties hold:*

1. *for each* $y \in \ell^\infty(X)$ *we have*
$$T_y(\ell^\infty(X)) \subset \ell^\infty(X);$$
2. *for each* $y \in \ell_0^\infty(X)$ *we have*
$$T_y(\ell_0^\infty(X)) \subset \ell_0^\infty(X).$$

Proof. Take $y, u \in \ell^\infty(X)$. Then
$$\begin{aligned}\|T_y u\|_\infty &= \sup_{n\in\mathbb{Z}} \|A_{n-1}u_{n-1} + y_n\|_n \\ &\le \kappa e^\alpha \sup_{n\in\mathbb{Z}}\|u_{n-1}\|_{n-1} + \sup_{n\in\mathbb{Z}}\|y_n\|_n \\ &\le \kappa e^\alpha \|u\|_\infty + \|y\|_\infty < \infty\end{aligned}$$
and so $T_y u \in \ell^\infty(X)$. On the other hand, for $y, u \in \ell_0^\infty(X)$ we have
$$\begin{aligned}\|(T_y u)_n\|_n &= \|A_{n-1}u_{n-1} + y_n\|_n \\ &\le \kappa e^\alpha \|u_{n-1}\|_{n-1} + \|y_n\|_n \to 0\end{aligned}$$
when $|n| \to \infty$ and so $T_y u \in \ell_0^\infty(X)$. □

3.2. Admissibility Properties

We continue to consider a sequence of maps $A = (A_m)_{m\in\mathbb{Z}}$ that is exponentially bounded with respect to some norms $\|\cdot\|_m$. Taking $y = 0$ in Proposition 1 we find that A generates the evolution map $S = T_0$ on $\ell^\infty(X)$ given by
$$(Su)_n = A_{n-1}u_{n-1} \quad \text{for } n \in \mathbb{Z} \text{ and } u = (u_m)_{m\in\mathbb{Z}} \in \ell^\infty(X).$$
Moreover, let $D(X)$ be the set of all sequences $v = (v_m)_{m\in\mathbb{Z}}$ with values in $\ell^\infty(X)$ such that
$$\|v\|_D := \sup_{m\in\mathbb{Z}} \|v_m\|_\infty < \infty.$$
We note that $D(X)$ is a Banach space when equipped with the norm $\|\cdot\|_D$. We also consider the closed subspace $D_0(X) \subset D(X)$ of all sequences v with values in $\ell_0^\infty(X)$ such that
$$\lim_{|m|\to\infty} \|v_m\|_\infty = 0.$$

The following theorem is our main result.

Theorem 2. *Let* $A = (A_m)_{m\in\mathbb{Z}}$ *be a sequence of linear maps on* X *that is exponentially bounded with respect to the norms* $\|\cdot\|_m$*. Then the following properties are equivalent:*

1. *for each* $y \in \ell_0^\infty(X)$ *there exists a unique* $x \in \ell^\infty(X)$ *such that*
$$x_{m+1} = A_m x_m + y_{m+1} \quad \textit{for } m \in \mathbb{Z}; \tag{3}$$
2. *for each* $v \in D_0(X)$ *there exists a unique* $u \in D(X)$ *such that*
$$u_{m+1} = Su_m + v_{m+1} \quad \textit{for } m \in \mathbb{Z}. \tag{4}$$

Proof. We first prove an auxiliary result. For each $u = (u_m)_{m\in\mathbb{Z}} \in D(X)$, we shall denote by $u_{m,k}$ the kth term of the sequence $u_m \in \ell^\infty(X)$.

Lemma 1. *Given $u, v \in D(X)$, Property (4) holds if and only if*

$$u_{m+1-k,m+1} = A_m u_{m-k,m} + y_{k,m+1} \quad \textit{for } m, k \in \mathbb{Z}, \tag{5}$$

where $y_{k,n} = v_{n-k,n}$.

Proof of the lemma. First assume that property (4) holds. By the definition of S we have

$$(Su_p)_{m+1} = A_m u_{p,m}$$

(where $u_{p,m}$ is the mth term of the sequence u_p) and so

$$u_{p+1,m+1} = (Su_p)_{m+1} + v_{p+1,m+1} = A_m u_{p,m} + v_{p+1,m+1}$$

for each $m, p \in \mathbb{Z}$. Taking $p = m - k$, we obtain

$$u_{m+1-k,m+1} = A_m u_{m-k,m} + v_{m+1-k,m+1} = A_m u_{m-k,m} + y_{k,m+1}. \tag{6}$$

Now assume that property (5) holds. Proceeding as in (6) and again by the definition of S, we have

$$\begin{aligned} u_{m+1-k,m+1} &= A_m u_{m-k,m} + y_{k,m+1} \\ &= A_m u_{m-k,m} + v_{m+1-k,m+1} \\ &= (Su_{m-k})_{m+1} + (v_{m+1-k})_{m+1}. \end{aligned}$$

Since m and k are arbitrary, this yields property (4). Indeed, replacing $m-k$ by p gives

$$\begin{aligned} (u_{p+1})_{m+1} &= u_{p+1,m+1} \\ &= (Su_p)_{m+1} + (v_{p+1})_{m+1} \\ &= (Su_p + v_{p+1})_{m+1}. \end{aligned}$$

Finally, since m is arbitrary, we obtain

$$u_{p+1} = Su_p + v_{p+1}$$

and property (4) follows from the arbitrariness of p (since k is arbitrary, for a given m one can choose k such that $p = m - k$ takes any desired value). □

We proceed with the proof of the theorem.

(1 ⇒ 2). Take $v \in D_0(X)$. For each $k \in \mathbb{Z}$, we define $y^{(k)} \in X^{\mathbb{Z}}$ by

$$y_m^{(k)} = v_{m-k,m} \quad \text{for } m \in \mathbb{Z} \tag{7}$$

(in a similar manner to that in Lemma 1). Since $v \in D_0(X)$, we have

$$\lim_{|m|\to\infty} \|y_m^{(k)}\|_m \le \lim_{|m|\to\infty} \|v_{m-k}\|_\infty = 0$$

and so $y^{(k)} \in \ell_0^\infty(X)$. By property 1, there exists a unique solution $x^{(k)} \in \ell^\infty(X)$ of Eq. (3) with $y = y^{(k)}$, for each $k \in \mathbb{Z}$.

By Lemma 1, if $u \in D(X)$ is a solution of Eq. (4), then $(u_{m-k,m})_{m\in\mathbb{Z}}$ is a solution of Eq. (3) with $y = y^{(k)}$, for each $k \in \mathbb{Z}$, that is,

$$u_{m+1-k,m+1} = A_m u_{m-k,m} + v_{m+1-k,m+1} \quad \text{for } m \in \mathbb{Z}.$$

Therefore, necessarily $u_{m-k,m} = x_m^{(k)}$ for $m, k \in \mathbb{Z}$, which is equivalent to

$$u_{m,n} = x_n^{(n-m)} \quad \text{for } m, n \in \mathbb{Z}. \tag{8}$$

This shows that any solution of Eq. (4) is given by (8) and so, in particular, it is unique. We show below that the sequence u defined by (8) belongs to $D(X)$.

Let E be the Banach space of all sequences $x \in \ell^\infty(X)$ for which there exists $y \in \ell_0^\infty(X)$ satisfying (3) and define a linear operator $R\colon E \to \ell_0^\infty(X)$ by

$$(Rx)_{m+1} = x_{m+1} - A_m x_m \quad \text{for } m \in \mathbb{Z}. \tag{9}$$

We show that R is closed. Let $(\bar{x}^{(i)})_{i\in\mathbb{N}}$ be a sequence in E converging to $x \in \ell^\infty(X)$ such that $\bar{y}^{(i)} = R\bar{x}^{(i)}$ converges to $y \in \ell_0^\infty(X)$. Then

$$\begin{aligned} x_{m+1} - A_m x_m &= \lim_{i\to\infty} \left(\bar{x}_{m+1}^{(i)} - A_m \bar{x}_m^{(i)}\right) \\ &= \lim_{i\to\infty} (R\bar{x}^{(i)})_{m+1} = y_{m+1} \end{aligned}$$

for $m \in \mathbb{Z}$. This shows that $Rx = y$ and so $x \in E$. Hence, the operator R is closed. By the closed graph theorem, R is bounded. Moreover, by property 1 the operator R is onto and invertible. It follows from the open mapping theorem that it has a bounded inverse.

Now we show that $u \in D(X)$. First observe that for a fixed m, replacing n by $m + k$ we have

$$\sup_{n\in\mathbb{Z}} \|x_n^{(n-m)}\|_n = \sup_{k\in\mathbb{Z}} \|x_{m+k}^{(k)}\|_{m+k} \le \sup_{k\in\mathbb{Z}} \|x^{(k)}\|_\infty.$$

Since $Rx^{(k)} = y^{(k)}$, we obtain

$$\|u_m\|_\infty = \sup_{n\in\mathbb{Z}} \|x_n^{(n-m)}\|_n \le \sup_{k\in\mathbb{Z}} \|x^{(k)}\|_\infty \le \|R^{-1}\| \sup_{k\in\mathbb{Z}} \|y^{(k)}\|_\infty.$$

Moreover,

$$\sup_{m\in\mathbb{Z}} \|v_{m-k,m}\|_m \le \sup_{m,n\in\mathbb{Z}} \|v_{m,n}\|_n \tag{10}$$

since the pairs $(m-k, m)$ with $m \in \mathbb{Z}$ form a subset of the pairs (m, n) with $m, n \in \mathbb{Z}$, and so

$$\begin{aligned} \|y^{(k)}\|_\infty = \sup_{m\in\mathbb{Z}} \|v_{m-k,m}\|_m &\le \sup_{m,n\in\mathbb{Z}} \|v_{m,n}\|_n \\ &= \sup_{m\in\mathbb{Z}} \|v_m\|_\infty = \|v\|_D < \infty. \end{aligned}$$

This shows that

$$\|u_m\|_\infty \le \|R^{-1}\| \cdot \|v\|_D < \infty$$

and so $u_m \in \ell^\infty(X)$. Finally, we also have

$$\sup_{m\in\mathbb{Z}} \|u_m\|_\infty \le \|R^{-1}\| \cdot \|v\|_D < +\infty$$

and so $u \in D(X)$.

$(2 \Rightarrow 1)$. Take $y \in \ell_0^\infty(X)$ and define a sequence $(v_m)_{m\in\mathbb{Z}}$ with values in $\ell_0^\infty(X)$ by

$$v_{m,n} = \frac{y_n}{1+(m-n)^2} \quad \text{for } m, n \in \mathbb{Z}. \tag{11}$$

Note that

$$\|v_m\|_\infty = \sup_{n\in\mathbb{Z}} \frac{\|y_n\|_n}{1+(m-n)^2}.$$

Given $\varepsilon > 0$, take $\rho > 0$ such that $\|y_n\|_n < \varepsilon$ whenever $|n| \geq \rho$. Then

$$\|v_m\|_\infty \leq \sup_{n\in[-\rho,\rho]} \frac{\|y\|_\infty}{1+(m-n)^2} + \varepsilon \to \varepsilon$$

when $|m| \to \infty$. It follows from the arbitrariness of ε that $v \in D_0(X)$.

By property 2, there exists a unique $u \in D(X)$ satisfying (4). In view of Lemma 1, for each $k \in \mathbb{R}$ the sequence $x^{(k)} = (u_{m-k,m})_{m\in\mathbb{Z}}$ satisfies the equation

$$x_{m+1}^{(k)} = A_m x_m^{(k)} + y_{m+1}^{(k)} \quad \text{for } m \in \mathbb{Z},$$

where

$$y_n^{(k)} = v_{n-k,n} = \frac{y_n}{1+k^2} \quad \text{for } n \in \mathbb{Z}.$$

Therefore, $\bar{x}^{(k)} = (1+k^2)x^{(k)}$ satisfies Eq. (3) for each $k \in \mathbb{Z}$. Moreover, proceeding as in (10) we obtain

$$\begin{aligned}\|x^{(k)}\|_\infty &= \sup_{m\in\mathbb{Z}} \|u_{m-k,m}\|_m \leq \sup_{m,n\in\mathbb{Z}} \|u_{m,n}\|_n \\ &= \sup_{m\in\mathbb{Z}} \|u_m\|_\infty = \|u\|_D < \infty\end{aligned}$$

and so $\bar{x}^{(k)} \in \ell^\infty(X)$. We also show that $\bar{x}^{(k)}$ is independent of k. Given $p \in \mathbb{Z}$, we define a sequence $\bar{u} = (\bar{u}_m)_{m\in\mathbb{Z}}$ in $D(X)$ by

$$\bar{u}_{m,n} = \frac{\bar{x}_n^{(p)}}{1+(n-m)^2} \quad \text{for } m, n \in \mathbb{Z}. \tag{12}$$

Then $\bar{u}_{m-k,m} = \bar{x}_m^{(p)}/(1+k^2)$ satisfies equation (5) for all k and so by Lemma 1, $\bar{u}$ is a solution of Eq. (4). But by property 2, we must have $\bar{u} = u$. Therefore, for each $q \in \mathbb{Z}$ we have

$$x_m^{(q)} = u_{m-q,m} = \bar{u}_{m-q,m} = \frac{\bar{x}_m^{(p)}}{1+q^2}$$

for all $m \in \mathbb{Z}$ and so $\bar{x}^{(q)} = \bar{x}^{(p)}$. This shows that $\bar{x} := \bar{x}^{(k)} \in \ell^\infty(X)$, which is a solution of Eq. (3), is independent of k.

To establish property 1, it remains to show that Eq. (3) has a unique solution. Assume that $z \in \ell^\infty(X)$ was a solution different from $\bar{x}$. We define a sequence $w \in D(X)$ by

$$w_{m,n} = \frac{z_n}{1+(n-m)^2} \quad \text{for } m, n \in \mathbb{Z}.$$

Then $w_{m-k,m} = z_m/(1+k^2)$ satisfies Eq. (5) for all k, that is,

$$w_{m+1-k,m+1} = A_m w_{m-k,m} + y_{m+1}^{(k)} \quad \text{for } m,k \in \mathbb{Z}.$$

It follows from Lemma 1 that w is a solution of Eq. (4). But then both $u, w \in D(X)$ are solutions of Eq. (4), which by hypothesis has a single solution. Therefore, since $\bar{u} = u$ and $\bar{x}^{(p)} = \bar{x}$, it follows from (12) that

$$\frac{\bar{x}_n}{1+(n-m)^2} = u_{m,n} = w_{m,n} = \frac{z_n}{1+(n-m)^2}$$

for $m, n \in \mathbb{Z}$, which readily implies that $\bar{x} = z$. This contradiction shows that Eq. (3) has a unique solution. □

4. Admissibility with Exponential Growth

In this section we consider the same admissibility property as before but for spaces of sequences with bounded exponential growth.

4.1. Evolution Maps

Given $c \geq 0$, let $E^c(X)$ be the set of all sequences $(x_m)_{m\in\mathbb{Z}}$ with values in X such that the sequence $x^c = (x_m^c)_{m\in\mathbb{Z}}$ defined by $x_m^c = e^{-c|m|}x_m$ for $m \in \mathbb{Z}$ is in $\ell_0^\infty(X)$. We note that $E^c(X)$ is a Banach space when equipped with the norm

$$\|x\|_{E^c} := \|x^c\|_\infty.$$

Proposition 2. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then for each $c \geq 0$ and $y \in E^c(X)$ we have*

$$T_y(E^c(X)) \subset E^c(X).$$

Proof. Take $y, u \in E^c(X)$. Then

$$\begin{aligned} e^{-c|n|}\|(T_y u)_n\|_n &= e^{-c|n|}\,\|A_{n-1}u_{n-1} + y_n\|_n \\ &\leq \kappa e^{\alpha+c} e^{-c|n-1|}\|u_{n-1}\|_{n-1} + e^{-c|n|}\|y_n\|_n \to 0 \end{aligned}$$

when $|n| \to \infty$ and so $T_y u \in E^c(X)$. □

4.2. Admissibility Properties

As a preparation for the result relating admissibility properties using the spaces $E^c(X)$, we first establish a version of Theorem 2 in which we consider the same spaces for the perturbations and for the solutions.

Theorem 3. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then the following properties are equivalent:*

1. *for each $y \in \ell_0^\infty(X)$ there exists a unique $x \in \ell_0^\infty(X)$ satisfying (3);*
2. *for each $v \in D_0(X)$ there exists a unique $u \in D_0(X)$ satisfying (4).*

Proof. (1 ⇒ 2). Take $v \in D_0(X)$ and consider the sequences $y^{(k)} \in \ell_0^\infty(X)$ defined by (7) for each $k \in \mathbb{Z}$. By property 1, there exists a unique solution $x^{(k)} \in \ell_0^\infty(X)$ of equation (3) with $y = y^{(k)}$. Again we define $u_{m,n}$ as in (8). We will show that $u \in D_0(X)$. As in the proof of Theorem 2, u is then the unique solution of Eq. (4) in $D_0(X)$.

We already know from the proof of Theorem 2 that $u \in D(X)$ and so it remains to verify that $u_m \in \ell_0^\infty(X)$ for each $m \in \mathbb{Z}$ and that $\|u_m\|_\infty \to 0$ when $|m| \to \infty$. Since

$$\lim_{|m|\to\infty} \|v_m\|_\infty = 0,$$

for each $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that

$$\|v_{m,n}\|_n < \varepsilon \quad \text{whenever } |m| > M \text{ and } n \in \mathbb{Z}.$$

On the other hand, for each $m \in [-M, M] \cap \mathbb{Z}$ there exists $n_m \in \mathbb{N}$ such that

$$\|v_{m,n}\|_n < \varepsilon \quad \text{whenever } |n| \geq n_m.$$

Letting

$$N = \max\{n_{-M}, \ldots, n_M\},$$

we obtain

$$\|v_{m,n}\|_n < \varepsilon \quad \text{whenever } m \in [-M, M] \cap \mathbb{Z} \text{ and } |n| \geq N. \tag{13}$$

This readily implies that

$$\|y^{(k)}\|_\infty = \sup_{m\in\mathbb{Z}} \|v_{m-k,m}\|_m < \varepsilon \tag{14}$$

for any sufficiently large $|k|$ since then the line $\{(m-k, m) : m \in \mathbb{Z}\}$ does not intersect the rectangle $[-M, M] \times [-N, N]$. Hence, it follows from the arbitrariness of ε that

$$\lim_{|k|\to\infty} \|y^{(k)}\|_\infty = 0. \tag{15}$$

Let E_0 be the Banach space of all sequences $x \in \ell_0^\infty(X)$ for which there exists $y \in \ell_0^\infty(X)$ satisfying (3) and define a linear operator $R\colon E_0 \to \ell_0^\infty(X)$ by (9). One can show as in the proof of Theorem 2 that R has a bounded inverse. By (15) we have

$$\begin{aligned}\lim_{|n|\to\infty} \|u_{m,n}\|_n &= \lim_{|n|\to\infty} \|x_n^{(n-m)}\|_n \leq \lim_{|n|\to\infty} \|x^{(n-m)}\|_\infty \\ &\leq \|R^{-1}\| \lim_{|n|\to\infty} \|y^{(n-m)}\|_\infty = 0\end{aligned}$$

and so $u_m \in \ell_0^\infty(X)$ for each $m \in \mathbb{Z}$. Moreover, since

$$\|x^{(k)}\|_\infty \leq \|R^{-1}\| \cdot \|y^{(k)}\|_\infty,$$

it follows from (15) that for each $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$\|x_n^{(k)}\|_n < \varepsilon \quad \text{whenever } |k| > K \text{ and } n \in \mathbb{Z}.$$

Since $x^{(k)} \in \ell_0^\infty(X)$, for each $k \in [-K, K] \cap \mathbb{Z}$ there exist $n_k \in \mathbb{N}$ such that

$$\|x_n^{(k)}\|_n < \varepsilon \quad \text{whenever } |n| \geq n_k.$$

So, there exists $N \in \mathbb{N}$ such that

$$\|x_n^{(k)}\|_n < \varepsilon \quad \text{whenever } k \in [-K, K] \cap \mathbb{Z} \text{ and } |n| \ge N.$$

This implies that $\sup_{n\in\mathbb{Z}} \|x_n^{(n-m)}\|_n < \varepsilon$ for any sufficiently large $|m|$ since then the line $\{(n-m, n) : m \in \mathbb{Z}\}$ does not intersect $[-K, K] \times [-N, N]$. Hence, it follows from the arbitrariness of ε that

$$\lim_{|m|\to\infty} \|u_m\|_\infty = \lim_{|m|\to\infty} \sup_{n\in\mathbb{Z}} \|x_n^{(n-m)}\|_n = 0.$$

and so $u \in D_0(X)$.

$(2 \Rightarrow 1)$. Take $y \in \ell_0^\infty(X)$ and consider the sequence $(v_m)_{m\in\mathbb{Z}} \in D_0(X)$ defined by (11). By property 2, there exists a unique $u \in D_0(X)$ satisfying (4). We already know from the proof of Theorem 2 that the sequence $x = (x_m)_{m\in\mathbb{Z}}$ with

$$x_m = (1 + k^2)u_{m-k,m} \quad \text{for } m \in \mathbb{Z}$$

is independent of $k \in \mathbb{Z}$ and that it is the unique solution of equation (3) in $\ell^\infty(X)$. It remains to verify that $x \in \ell_0^\infty(X)$.

As in the proof of the implication $1 \Rightarrow 2$ (see (13)), since $u \in D_0(X)$, for each $\varepsilon > 0$ there exist $M, N \in \mathbb{N}$ such that

$$\|u_{m,n}\|_n < \varepsilon \quad \text{whenever } m \in [-M, M] \cap \mathbb{Z} \text{ and } |n| \ge N.$$

Also as before (see (14)), this implies that $\sup_{m\in\mathbb{Z}} \|u_{m-k,m}\|_m < \varepsilon$ for any sufficiently large $|k|$. It thus follows from the arbitrariness of ε that $x \in \ell_0^\infty(X)$. This completes the proof of the theorem. $\square$

Using this result we are able to consider the space $E^c(X)$ for an arbitrary constant $c \ge 0$. Given $c \ge 0$ and taking $y = 0$ in Proposition 2, we find that A generates the evolution map $S = T_0$ on $E^c(X)$ given by

$$(Su)_n = A_{n-1}u_{n-1} \quad \text{for } n \in \mathbb{Z} \text{ and } u \in E^c(X).$$

Moreover, let $F^c(X)$ be the set of all sequences $(v_m)_{m\in\mathbb{Z}}$ with values in $E^c(X)$ such that

$$\lim_{|m|\to\infty} \|v_m\|_{E^c} = 0.$$

We note that $E^c(X)$ is a Banach space when equipped with the norm

$$\|v\|_{F^c} := \sup_{m\in\mathbb{Z}} \|v_m\|_{E^c} < \infty.$$

Theorem 4. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then for each $c \ge 0$ the following properties are equivalent:*

1. *for each $y \in E^c(X)$ there exists a unique $x \in E^c(X)$ satisfying (3);*
2. *for each $v \in F^c(X)$ there exists a unique $u \in F^c(X)$ satisfying (4).*

Proof. Take $y, x \in E^c(X)$. We consider the sequences $y^c, x^c \in \ell_0^\infty(X)$ defined by

$$y_m^c = e^{-c|m|}y_m \quad \text{and} \quad x_m^c = e^{-c|m|}x_m$$

for $m \in \mathbb{Z}$. Note that property (3) holds if and only if

$$x^c_{m+1} = A^c_m x^c_m + y^c_{m+1} \quad \text{for } m \in \mathbb{Z}, \tag{16}$$

where

$$A^c_m = e^{-c|m+1|+c|m|} A_m.$$

Therefore, property 1 holds if and only if for each $f \in E^c(X)$ there exists a unique $x \in E^c(X)$ satisfying (16) (using the definitions of y^c and x^c).

Notice that the sequence $A^c = (A^c_m)_{m\in\mathbb{Z}}$ is also exponentially bounded with respect to the norms $\|\cdot\|_m$. Since the maps $y \mapsto y^c$ and $x \mapsto x^c$ are bijections from $E^c(X)$ onto $\ell^\infty_0(X)$, it follows from Theorem 3 that property 1 holds if and only if for each $F \in D_0(X)$ there exists a unique $u \in D_0(X)$ satisfying

$$u_{m+1} = S^c u_m + v_{m+1} \quad \text{for } m \in \mathbb{Z}, \tag{17}$$

where

$$(S^c v)_m = A^c_{m-1} v_{m-1} \quad \text{for } m \in \mathbb{Z} \text{ and } v \in \ell^\infty_0(X).$$

We have

$$e^{c|m|}(S^c v)_m = A_{m-1} e^{c|m-1|} v_{m-1},$$

that is,

$$\gamma \circ S^c = S \circ \gamma, \quad \text{with } \gamma(v)_n = e^{c|n|} v_n.$$

Letting $u^c_m = \gamma(u_m)$, we obtain

$$\gamma(S^c u_m) = S(\gamma(u_m)) = S u^c_m$$

and so property (17) is equivalent to

$$u^c_{m+1} = S u^c_m + v^c_{m+1}, \tag{18}$$

where

$$(v^c_m)_n = \gamma(v_m)_n = e^{c|n|} v_{m,n}.$$

Since the maps $v \mapsto v^c = (v^c_m)_{m\in\mathbb{Z}}$ and $u \mapsto u^c = (u^c_m)_{m\in\mathbb{Z}}$ are bijections from $D_0(X)$ onto $F^c(X)$, it follows from (18) that property 1 holds if and only if property 2 holds. □

5. Admissibility on ℓ^p Spaces

In this section we consider once more an admissibility property, now for evolution maps on ℓ^p spaces.

5.1. Evolution Maps

For each $p \in [1, +\infty)$, let $\ell^p(X)$ be the set of all sequences $x = (x_m)_{m\in\mathbb{Z}}$ with values in X such that

$$\|x\|_{\ell^p} = \left(\sum_{m\in\mathbb{Z}} \|x_m\|_m^p\right)^{1/p} < \infty.$$

We note that $\ell^p(X)$ is a Banach space when equipped with the norm $\|\cdot\|_{\ell^p}$.

Proposition 3. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then for each $y \in \ell^p(X)$, we have*

$$T_y(\ell^p(X)) \subset \ell^p(X).$$

Proof. Take $y, u \in \ell^p(X)$. By Minkowski's inequality we have

$$\begin{aligned}
\|T_y u\|_{\ell^p} &= \left(\sum_{n\in\mathbb{Z}} \|A_{n-1}u_{n-1} + y_n\|_n^p \, ds\right)^{1/p} \\
&\le \left(\sum_{n\in\mathbb{Z}} \|A_{n-1}u_{m-1}\|_n^p \, ds\right)^{1/p} + \left(\sum_{n\in\mathbb{Z}} \|y_n\|_p\right)^{1/p} \\
&\le \kappa e^\alpha \left(\sum_{n\in\mathbb{Z}} \|u_{n-1}\|_{n-1}^p\right)^{1/p} + \|y\|_{\ell^p} \\
&= \kappa e^\alpha \|u\|_{\ell^p} + \|y\|_{\ell^p} < \infty,
\end{aligned}$$

which shows that $T_y u \in \ell^p(X)$. □

5.2. Admissibility Properties

Taking $y = 0$ in Proposition 3, we find that A generates the evolution map $S = T_0$ on $\ell^p(X)$ given by

$$(Su)_n = A_{n-1}u_{n-1} \quad \text{for } n \in \mathbb{Z} \text{ and } u \in \ell^p(X).$$

Moreover, for each $p \in [1, +\infty)$ let $M^p(X) = \ell^p(\ell^p(X))$ be the set of all sequences $v = (v_n)_{n\in\mathbb{Z}}$ with $v_n \in \ell^p(X)$ such that

$$\|v\|_{M^p} := \left(\sum_{m\in\mathbb{Z}} \|v_m\|_{\ell^p}^p\right)^{1/p} = \left(\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \|v_{m,n}\|_n^p\right)^{1/p} < \infty.$$

We note that $M^p(X)$ is a Banach space when equipped with the norm $\|\cdot\|_{M^p}$.

Theorem 5. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then the following properties are equivalent:*

1. *for each $y \in \ell^p(X)$ there exists a unique $x \in \ell^p(X)$ satisfying (3);*
2. *for each $v \in M^p(X)$ there exists a unique $u \in M^p(X)$ satisfying (4).*

Proof. (1 ⇒ 2). Take $v \in M^p(X)$. For each $k \in \mathbb{R}$ we consider the sequence $y^{(k)} \in X^{\mathbb{Z}}$ defined by (7). We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \|y_n^{(k)}\|_{\ell^p}^p &= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \|v_{n-k,n}\|_n^p \\ &= \sum_{m\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \|v_{m,j}\|_j^p = \|v\|_{M^p}^p < \infty\end{aligned} \tag{19}$$

and so $y^{(k)} \in \ell^p(X)$. By property 1, there exists a unique solution $x^{(k)} \in \ell^p(X)$ of Eq. (3) with $y = y^{(k)}$. Again we define $u_{m,n}$ as in (8). By Lemma 1, u is a solution of Eq. (4) and as in the proof of Theorem 2 it is automatically unique. We will show that $u \in M^p(X)$.

Let F be the Banach space of all sequences $x \in \ell^p(X)$ for which there exists $y \in \ell^p(X)$ satisfying (3) and define a linear operator $R\colon F \to \ell^p(X)$ by (9). One can show as in the proof of Theorem 2 that R has a bounded inverse.

Now we show that $u \in M^p(X)$. We have

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \|u_m\|_{\ell^p}^p &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \|x_n^{(n-m)}\|_n^p \\ &= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \|x_j^{(k)}\|_j^p = \sum_{k\in\mathbb{Z}} \|x^{(k)}\|_{\ell^p}^p.\end{aligned} \tag{20}$$

Since

$$\|x^{(k)}\|_{\ell^p} \le \|R^{-1}\| \cdot \|y^{(k)}\|_{\ell^p},$$

it follows from (19) that

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \|u_m\|_{L^p}^p &= \sum_{k\in\mathbb{Z}} \|x^{(k)}\|_{L^p}^p \\ &\le \|R^{-1}\|^p \sum_{k\in\mathbb{Z}} \|y^{(k)}\|_{\ell^p}^p \\ &= \|R^{-1}\|^p \|v\|_{M^p}^p < \infty\end{aligned}$$

and so $u \in M^p(X)$.

(2 ⇒ 1). Take $y \in \ell^p(X)$ and define $v = (v_m)_{m\in\mathbb{Z}}$ by

$$v_{m,n} = \frac{y_n}{1 + (m-n)^2} \quad \text{for } m, n \in \mathbb{Z}.$$

Note that $v \in M^p(X)$. Indeed,

$$\begin{aligned}
\|v\|_{M^p}^p &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \|v_{m,n}\|_n^p \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \|v_{m,n}\|_n^p \\
&= \sum_{n\in\mathbb{Z}} \|y_n\|_n^p \sum_{m\in\mathbb{Z}} \frac{1}{(1+(m-n)^2)^p} \\
&= \sum_{n\in\mathbb{Z}} \|y_n\|_n^p \sum_{m\in\mathbb{Z}} \frac{1}{(1+m^2)^p} \\
&= c_p \|y\|_{\ell^p}^p < \infty
\end{aligned}$$

for some constant $c_p > 0$ that depends only on p. By property 2, there exists a unique $u \in M^p(X)$ satisfying (4). By Lemma 1, for each $k \in \mathbb{R}$ the sequence $x^{(k)}$ defined by

$$x_m^{(k)} = u_{m-k,m} \quad \text{for } m \in \mathbb{Z}$$

satisfies Eq. (3) with y replaced by $y^{(k)} = (y_m^{(k)})_{m\in\mathbb{Z}}$ with

$$y_m^{(k)} = v_{m-k,m} = \frac{y_m}{1+k^2} \quad \text{for } m \in \mathbb{Z}.$$

Proceeding as in (20), we obtain

$$\sum_{k\in\mathbb{Z}} \|x^{(k)}\|_{\ell^p}^p = \sum_{m\in\mathbb{Z}} \|u_m\|_{\ell^p}^p = \|u\|_{M^p}^p < \infty$$

and so $x^{(k)} \in \ell^p(X)$. One can then show in a similar manner to that in the proof of Theorem 2 that $x = (x_m)_{m\in\mathbb{Z}}$ with

$$x_m = (1+k^2)x_m^{(k)} \quad \text{for } m \in \mathbb{Z}$$

is independent of k and that it is the unique solution of Eq. (3) in $\ell^p(X)$. This concludes the proof of the theorem. □

6. Hyperbolicity

In this section we discuss the relation of hyperbolicity with the admissibility properties considered in the former sections.

Let $\|\cdot\|_m$, for $m \in \mathbb{Z}$, be a family of norms on a Banach space X. We say that a sequence $(A_m)_{m\in\mathbb{Z}}$ of linear maps on X is *hyperbolic with respect to the norms* $\|\cdot\|_m$ if:

1. there exist projections P_n for $n \in \mathbb{Z}$ such that $P_{n+1}A_n = A_nP_n$ and the map

$$A_n|_{\operatorname{Im} Q_n}\colon \operatorname{Im} Q_n \to \operatorname{Im} Q_{n+1},$$

where $Q_n = \mathrm{Id} - P_n$, is onto and invertible for each $n \in \mathbb{Z}$;

2. there exist constants $\lambda, N > 0$ such that for each $x \in X$ we have

$$\|U(m,n)P_n x\|_m \le N e^{-\lambda(m-n)} \|x\|_n \quad \text{for } m \ge n$$

and

$$\|\bar{U}(m,n)Q_n x\|_m \le N e^{-\lambda(n-m)} \|x\|_n \quad \text{for } m \le n,$$

where $\bar{U}(n,m) = (U(m,n)|_{\operatorname{Im} Q_n})^{-1}$.

The following proposition is a particular case of more general results in [1] that relate hyperbolicity with admissibility.

Proposition 4. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then the following properties are equivalent:*

1. *the sequence $(A_m)_{m\in\mathbb{Z}}$ is hyperbolic with respect to the norms $\|\cdot\|_m$;*
2. *for each $y \in \ell_0^\infty(X)$ there exists a unique $x \in \ell_0^\infty(X)$ satisfying (3);*
3. *for each $y \in \ell^p(X)$ there exists a unique $x \in \ell^p(X)$ satisfying (3).*

We refer the reader to [2] for a detailed list of references on further related results, including specifically for the family of norms $\|\cdot\|_m = \|\cdot\|$.

The following statement is a simple consequence of Theorems 3 and 5 together with Proposition 4.

Theorem 6. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then the following properties are equivalent:*

1. *the sequence $(A_m)_{m\in\mathbb{Z}}$ is hyperbolic with respect to the norms $\|\cdot\|_m$;*
2. *for each $v \in D_0(X)$ there exists a unique $u \in D_0(X)$ satisfying (4);*
3. *for each $v \in M^p(X)$ there exists a unique $u \in M^p(X)$ satisfying (4).*

One can also consider the hyperbolicity of the evolution map. We recall that a map T on a Banach space Y is said to be *hyperbolic* if:

1. there exists a projection P satisfying $PT = TP$ and the map

$$T|_{\operatorname{Im} Q} \colon \operatorname{Im} Q \to \operatorname{Im} Q,$$

where $Q = \mathrm{Id} - P$, is onto and invertible;

2. there exist $\lambda, N > 0$ such that

$$\|T^m P\| \le N e^{-\lambda m} \quad \text{and} \quad \|S^m Q\| \le N e^{-\lambda m}$$

for $m \ge 0$, where $S = (T|_{\operatorname{Im} Q})^{-1}$.

In particular, the equivalence of the notions of hyperbolicity for a sequence $(A_m)_{m\in\mathbb{Z}}$ and its evolution map on $\ell_0^\infty(X)$ and on $\ell^p(X)$ lead to further equivalences to the former admissibility properties.

In particular, we have the following result.

Theorem 7. [3] *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then $(A_m)_{m\in\mathbb{Z}}$ is hyperbolic with respect to the norms $\|\cdot\|_m$ if and only if the evolution map S on $Y = \ell_0^\infty(X)$ given by (2) is hyperbolic.*

In addition, one can replace the space Y in Theorem 7 by many other Banach spaces, including $\ell^p(X)$ with $p \in [1, +\infty)$ (see [4] for details).

The following statement is a simple consequence of the former results.

Theorem 8. *Let $A = (A_m)_{m\in\mathbb{Z}}$ be a sequence of linear maps on X that is exponentially bounded with respect to the norms $\|\cdot\|_m$. Then the following properties are equivalent:*

1. *the sequence $(A_m)_{m\in\mathbb{Z}}$ is hyperbolic with respect to the norms $\|\cdot\|_m$;*
2. *the pair formed by the spaces $\ell_0^\infty(X)$ and $\ell^\infty(X)$ is admissible;*
3. *the pair formed by the spaces $\ell^p(X)$ and $\ell^p(X)$ is admissible;*
4. *the evolution map S on $Y = \ell_0^\infty(X)$ or on $Y = \ell^p(X)$ is hyperbolic;*
5. *the pair formed by the spaces $D_0(X)$ and $D_0(X)$ is admissible;*
6. *the pair formed by the spaces $M^p(X)$ and $M^p(X)$ is admissible.*

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