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Oscillation Results Using Linearization of Quasi-Linear Second Order Delay Difference Equations

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Abstract. In this paper, the authors investigate the oscillatory behavior of quasilinear second order delay difference equations of the form

$$\Delta(b(n)(\Delta u(n))^{\alpha}) + p(n)u^{\beta}(n-\sigma) = 0.$$

By obtaining new monotonic properties of the nonoscillatory solutions and using them to linearize the equation leads to new oscillation criteria. The criteria obtained improve existing ones in the literature. Two examples are included to show the importance of the main results.

Mathematics Subject Classification. 39A10.

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1. Introduction

In this paper, we investigate the oscillatory and asymptotic behavior of solutions of the second order quasilinear delay difference equation

$$\Delta(b(n)(\Delta u(n))^{\alpha}) + p(n)u^{\beta}(n-\sigma) = 0,\ n \geq n_0 > 0, \tag{E}$$

where n_0 is a positive integer. We assume that the following conditions hold throughout this paper without further mention:

(C_1) $\{b(n)\}$ and $\{p(n)\}$ are positive real sequences;
(C_2) α and β are ratios of odd positive integers;
(C_3) σ is a positive integer;
(C_4) $B(n) = \sum_{s=n_0}^{n-1} b^{-1/\alpha}(s) \to \infty$ as $n \to \infty$.

By a *solution* of (E), we mean a real sequence $\{u(n)\}$ defied for $n \geq n_0 - \sigma$ satisfying equation (E) for all $n \geq n_0$. A nontrivial solution of (E) is called *oscillatory* if it is neither eventually negative nor eventually positive, and it is called *nonoscillatory* otherwise. Equation (E) is called oscillatory if all its solutions are oscillatory.

The problem of investigating oscillation criteria for various types of difference equations has been a very active research area over the past several decades. A large number of papers and monographs have been devoted to this problem; for a few examples, see [1–3,7–9,11–17] and the references contained therein. Koplatadze [11] obtained some very nice oscillatory criteria for the equation

$$\Delta^2 u(n) + p(n)u(n-\sigma) = 0 \tag{1.1}$$

based on the following monotonic properties of positive solutions:

$$u(n) \text{ is increasing and } \frac{u(n)}{n} \text{ is decreasing.} \tag{1.2}$$

The main aim of the paper is to establish some new comparison theorems for investigation of oscillatory behavior of solutions of (E). First, we will linearize equation (E). Then we will deduce the oscillation of (E) from that of its linearized forms. To achieve this, we obtain some results on the monotonic properties of nonoscillatory solutions of (E) that are new even for (1.1) and improves those in (1.2). We will demonstrate the usefulness of our main results via some examples. The technique of proof used here is based in part on some recent papers of Baculíková [4–6] on the oscillation of solutions of differential equations.

2. Auxiliary Results

We begin with some useful lemmas concerning monotonic properties of nonoscillatory solutions of (E).

Lemma 2.1. *Let* $\{u(n)\}$ *be a positive solution of* (E). *Then:*

(P_1*)* $\{u(n)\}$ *is eventually increasing and* $\{b(n)(\Delta u(n))\}$ *is eventually decreasing;*

(P_2*)* $\{\frac{u(n)}{B(n)}\}$ *is eventually decreasing.*

Moreover, if

$$\sum_{n=n_0}^{\infty} p(n)B^{\beta}(n-\sigma) = \infty, \tag{2.1}$$

then

(P_3*)* $\lim_{n\to\infty} \frac{u(n)}{B(n)} = 0.$

Proof. Let $\{u(n)\}$ be a positive solution of (E). Then $\Delta(b(n)(\Delta u(n))^{\alpha}) < 0$, and there is an integer $n_1 \geq n_0$ that $b(n)(\Delta u(n))^{\alpha}$ has a constant sign for all $n \geq n_1$. We claim that $b(n)(\Delta u(n))^{\alpha} > 0$ eventually. To show this, assume that $b(n)(\Delta u(n))^{\alpha} < 0$ for $n \geq n_2$ for some $n_2 \geq n_1$. Then there exists a constant $M > 0$ such that $b(n)(\Delta u(n))^{\alpha} < -M < 0$ for $n \geq n_2$. Summing the last inequality from n_2 to $n-1$ and using (C_4), we have

$$u(n) \leq u(n_2) - MB(n) \to -\infty \text{ as } n \to \infty$$

which is a contradiction and proves our claim. Employing the monotonicity of $b^{1/\alpha}(n)\Delta u(n)$, we obtain

$$u(n) \geq \sum_{s=n_2}^{n-1} \frac{b^{1/\alpha}(s)\Delta u(s)}{b^{1/\alpha}(s)} \geq B(n)b^{1/\alpha}(n)\Delta u(n), \tag{2.2}$$

which implies $\Delta\left(\frac{u(n)}{B(n)}\right) < 0$. Since $\frac{u(n)}{B(n)}$ is positive and decreasing, there exists $c \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{u(n)}{B(n)} = c \geq 0.$$

If $c > 0$, then, $u(n) \geq cB(n)$ for $n \geq n_3 \geq n_2$. Using this in (E) and then summing from n_3 to $n-1$, we obtain

$$b(n_3)(\Delta u(n_3))^{\alpha} \geq c^{\beta} \sum_{s=n_3}^{n-1} p(s)B^{\beta}(s-\sigma),$$

which as $n \to \infty$ contradicts (2.1). Thus, $c = 0$, that is $\lim_{n\to\infty} \frac{u(n)}{B(n)} = 0$, which competes the proof of the lemma. □

Remark 2.2. In the case of equation (1.1) where $b(n) \equiv 1$ and $\beta = 1$, the three properties of nonoscillatory solutions described in Lemma 2.1 become:

(P_1) $\{u(n)\}$ is eventually increasing and $\{\Delta u(n)\}$ is eventually decreasing;

(P_2) $\{\frac{u(n)}{n-1}\}$ is eventually decreasing;

(P_3) $\lim_{n\to\infty} \frac{u(n)}{B(n)} = 0$.

Lemma 2.3. *Let $\{u(n)\}$ be an eventually increasing solution of (E). Then, $u^{\beta-\alpha}(n) \geq \eta(n)$, where $\eta(n)$ is given by*

$$\eta(n) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ a_1, & \text{if } \alpha < \beta, \\ a_2 B^{\beta-\alpha}(n), & \text{if } \alpha > \beta, \end{cases}$$

and a_1 and a_2 are positive constants.

Proof. Since $u(n)$ is a positive increasing solution of (E), there exists a constant $M > 0$ such that $u(n) \geq M$ for all $n \geq n_1$ for some $n_1 \geq n_0$. From (P_2), we see that $\frac{u(n)}{B(n)}$ is decreasing and so

$$\frac{u(n)}{B(n)} \leq \frac{u(n_1)}{B(n_1)} = M_1.$$

Thus,

$$u^{\beta-\alpha}(n) \geq \eta(n) = \begin{cases} 1, & \text{for } \alpha = \beta, \\ a_1, & \text{for } \alpha < \beta, \\ a_2 B^{\beta-\alpha}(n), & \text{for } \alpha > \beta, \end{cases}$$

where $a_1 = M^{\beta-\alpha}$ and $a_2 = M_1^{\beta-\alpha}$. This proves the lemma. □

Lemma 2.4. *Let* (2.1) *hold and assune there exists a constant* $\delta \in [0,1)$ *such that*

$$p(n)\eta(n-\sigma)B^{\alpha}(n-\sigma)B(n)b^{1/\alpha}(n) \geq \alpha\delta,\ n \geq n_0. \tag{2.3}$$

If $\{u(n)\}$ *is a positive solution of* (E), *then*

$$u(n) \geq \frac{B(n)b^{1/\alpha}(n)\Delta u(n)}{(1-\delta)} \text{ eventually} \tag{2.4}$$

and

$$\frac{u(n)}{B^{\delta_1}(n)} \text{ is eventually increasing, where } \delta_1 = \delta^{1/\alpha}. \tag{2.5}$$

Proof. In view of (2.1) and from Lemma 2.1, we see that (P_1) holds and

$$\Delta(b(n)(\Delta u(n))^{\alpha}B^{\alpha\delta}(n)) = \Delta(b(n)(\Delta u(n))^{\alpha})B^{\alpha\delta}(n+1) + b(n)(\Delta u(n))^{\alpha}\Delta B^{\alpha\delta}(n). \tag{2.6}$$

By the Mean-value Theorem,

$$B^{\alpha\delta}(n+1) - B^{\alpha\delta}(n) \leq \begin{cases} \alpha\delta B^{\alpha\delta-1}(n+1)\Delta B(n), & \text{if } \alpha\delta > 1, \\ \alpha\delta B^{\alpha\delta-1}(n)\Delta B(n), & \text{if } \alpha\delta < 1, \end{cases}$$

where $B(n) < B(n+1)$. Since $\Delta B(n) = b^{-1/\alpha}(n)$, we have

$$\Delta B^{\alpha\delta}(n) \leq \alpha\delta\frac{B^{\alpha\delta}(n+1)}{B(n)}b^{-1/\alpha}(n),$$

and using this in (2.6), we obtain

$$\Delta(b(n)(\Delta u(n))^{\alpha}B^{\alpha\delta}(n)) \leq -p(n)\eta(n-\sigma)u^{\alpha}(n-\sigma)B^{\alpha\delta}(n+1) + \alpha\delta b(n)(\Delta u(n))^{\alpha}\frac{b^{-1/\alpha}(n)B^{\alpha\delta}(n+1)}{B(n)}. \tag{2.7}$$

Since $b^{1/\alpha}(n)(\Delta u(n))$ is decreasing, from (2.2)

$$u^{\alpha}(n-\sigma) \geq B^{\alpha}(n-\sigma)b(n)(\Delta u(n))^{\alpha}. \tag{2.8}$$

Combining (2.7) and (2.8), and then using (2.3), we obtain

$$\Delta(b(n)(\Delta u(n))^{\alpha}B^{\alpha\delta}(n)) \leq -b(n)(\Delta u(n))^{\alpha}B^{\alpha\delta}(n+1)\left[p(n)\eta(n-\sigma)B^{\alpha}(n-\sigma) - \frac{\alpha\delta}{b^{1/\alpha}(n)B(n)}\right] < 0.$$

Hence, $\{b(n)(\Delta u(n))^{\alpha}B^{\alpha\delta}(n)\}$ is decreasing and thus there exists an integer $n_1 \geq n_0$ such that

$$\begin{aligned} u(n) &\geq b^{1/\alpha}(n)B^{\delta}(n)\Delta u(n)\sum_{s=n_1}^{n-1}\frac{B^{-\delta}(s)}{b^{1/\alpha}(s)} \\ &\geq b^{1/\alpha}(n)B^{\delta}(n)\Delta u(n)\sum_{s=n_1}^{n-1}\int_{B(s)}^{B(s+1)}\frac{dx}{x^{\delta}} \\ &= b^{1/\alpha}(n)B^{\delta}(n)\Delta u(n)\left(\frac{B^{1-\delta}(n)}{1-\delta}\right), \end{aligned}$$

which proves (2.4).

To prove (2.5) first note that (P_3) implies

$$\lim_{n\to\infty} b^{1/\alpha}(n)\Delta u(n) = 0. \tag{2.9}$$

Therefore, a summation of (E) yields

$$b^{1/\alpha}(n)\Delta u(n) \geq \left[\sum_{s=n}^{\infty} p(s)\eta(s-\sigma)u^{\alpha}(s-\sigma)\right]^{1/\alpha}. \tag{2.10}$$

Using (2.3) and the facts that $\frac{u(n)}{B(n)}$ is decreasing and $u(n)$ is increasing, it follows from (2.10) that

$$\begin{aligned} b^{1/\alpha}(n)\Delta u(n) &\geq u(n)\left[\sum_{s=n}^{\infty} \frac{p(s)\eta(s-\sigma)B^{\alpha}(s-\sigma)}{B^{\alpha}(s)}\right]^{1/\alpha} \\ &\geq u(n)\left[\sum_{s=n}^{\infty} \frac{\alpha\delta}{B^{\alpha+1}(s)b^{1/\alpha}(s)}\right]^{1/\alpha} \\ &\geq \frac{\delta_1 u(n)}{B(n)}. \end{aligned} \tag{2.11}$$

Now

$$\Delta\left(\frac{u(n)}{B^{\delta_1}(n)}\right) = \frac{B^{\delta_1}(n)\Delta u(n) - u(n)\Delta B^{\delta_1}(n)}{B^{\delta_1}(n)B^{\delta_1}(n+1)}. \tag{2.12}$$

By the Mean-Value Theorem,

$$B^{\delta_1}(n+1) - B^{\delta_1}(n) \leq \delta_1 B^{\delta_1 - 1}(n)\Delta B(n),$$

since $\delta_1 = \delta^{1/\alpha} < 1$ and $\Delta B(n) = b^{-1/\alpha}(n)$. Using this in (2.12), we obtain

$$\Delta\left(\frac{u(n)}{B^{\delta_1}(n)}\right) \geq \frac{B(n)b^{1/\alpha}(n)\Delta u(n) - \delta_1 u(n)}{B(n)B^{\delta_1}(n+1)b^{1/\alpha}(n)} \geq 0$$

in view of (2.11). This proves (2.5) and completes the proof of the lemma. □

Remark 2.5. The monotone increasing property of $\{\frac{u(n)}{B^{\delta_1}(n)}\}$ obtained in Lemma 2.4 improves that for $\{u(n)\}$. This is new even for equation (1.1) for which it takes the form $\frac{u(n)}{(n-1)^{\delta_1}}$.

The following lemma taken from [16] will also be needed in the proofs of our results.

Lemma 2.6. ([16, Lemma 1]) *Let $F(n,u)$ be a continuous function defined on $\mathbb{N}_0 \times \mathbb{R}$ that is nondecreasing in u with $\operatorname{sgn} F(n,u) = \operatorname{sgn} u$, and let α and σ be as above. If the difference inequality*

$$\Delta((\Delta x_n)^{\alpha}) + F(n, x(n-\sigma) \leq 0$$

has an eventually positive solution, then so does the difference equation

$$\Delta((\Delta y_n)^{\alpha}) + F(n, y(n-\sigma)) = 0.$$

3. Comparison Results

In this section, we present new comparison principles that significantly simply the examination of (E).

Theorem 3.1. *Let conditions* (2.1) *and* (2.3) *hold. Then Eq.* (E) *is oscillatory provided that the equation*

$$\Delta w(n) + \frac{p(n)\eta(n-\sigma)B^{\alpha}(n-\sigma)}{(1-\delta)^{\alpha}} w(n-\sigma) = 0 \tag{3.1}$$

is oscillatory.

Proof. Assume to the contrary that $\{u(n)\}$ is a nonoscillatory solution of (E), say $u(n) > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. Then using (2.4) in (E), we obtain

$$\Delta(b(n)(\Delta u(n))^{\alpha}) + \frac{p(n)\eta(n-\sigma)B^{\alpha}(n-\sigma)b(n-\sigma)(\Delta u(n-\sigma))^{\alpha}}{(1-\delta)^{\alpha}} \leq 0.$$

Letting $w(n) = b(n)(\Delta u(n))^{\alpha}$, we see that $\{w(n)\}$ is a positive solution of the inequality

$$\Delta w(n) + \frac{p(n)\eta(n-\sigma)B^{\alpha}(n-\sigma)}{(1-\delta)^{\alpha}} w(n-\sigma) \leq 0.$$

By Lemma 2.6, the corresponding difference equation (3.1) also has a positive solution. This contradiction completes the proof of the theorem. □

Theorem 3.2. *Let $\alpha > 1$ and conditions* (2.1) *and* (2.3) *hold. Then Eq.* (E) *is oscillatory provided that*

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{(1-\delta)^{\alpha-1}}{\alpha} B^{\alpha-1}(n-\sigma)p(n)\eta(n-\sigma)u(n-\sigma) = 0 \tag{3.2}$$

is oscillatory.

Proof. Assume to the contrary that $\{u(n)\}$ is a positive solution of (E), say $u(n) > 0$ for $n \geq n_1 \geq n_0$. It is easy to see that by the Mean-Value Theorem,

$$\Delta(b(n)(\Delta u(n))^{\alpha}) \geq \alpha(b^{1/\alpha}(n)\Delta u(n))^{\alpha-1}\Delta(b^{1/\alpha}(n)\Delta u(n)),$$

or

$$-p(n)\eta(n-\sigma)u^{\alpha}(n-\sigma) \geq \alpha(b^{1/\alpha}(n)\Delta u(n))^{\alpha-1}\Delta(b^{1/\alpha}(n)\Delta u(n)),$$

which implies

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{1}{\alpha}(b^{1/\alpha}(n)\Delta u(n))^{1-\alpha}p(n)\eta(n-\sigma)u^{\alpha}(n-\sigma) \leq 0. \tag{3.3}$$

Using (2.4) in (3.3) and taking into account that $b^{1/\alpha}(n)\Delta u(n)$ is decreasing, we have

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{(1-\delta)^{\alpha-1}}{\alpha} B^{\alpha-1}(n-\sigma)p(n)\eta(n-\sigma)u(n-\sigma) \leq 0.$$

But by Lemma 2.6, the corresponding Eq. (3.2) has a positive solution, and so this contradiction completes the proof. □

Before stating our next theorem, first note that since $B(n)$ is increasing, there exists a constant $\lambda \geq 1$ such that

$$\frac{B(n)}{B(n-\sigma)} \geq \lambda. \tag{3.4}$$

Theorem 3.3. *Let* $0 < \alpha < 1$ *and conditions* (2.1) *and* (2.3) *hold. Then Eq.* (E) *is oscillatory provided the equation*

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{\delta^{\frac{1-\alpha}{\alpha}}\lambda^{(1-\alpha)}}{\alpha(1-\delta_1)^{\frac{1-\alpha}{\alpha}}} B^{\alpha-1}(n+1)p(n)\eta(n-\sigma)u(n-\sigma) = 0 \tag{3.5}$$

is oscillatory.

Proof. Let $\{u(n)\}$ be a nonoscillatory solution of (E) with $u(n) > 0$ for $n \geq n_1 \geq n_0$. From (2.9) and (E), we have

$$b^{1/\alpha}(n)\Delta u(n) = \left[\sum_{s=n}^{\infty} p(s)u^{\beta}(s-\sigma)\right]^{\frac{1}{\alpha}}.$$

Hence,

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) = \Delta\left(\left[\sum_{s=n}^{\infty} p(s)u^{\beta}(s-\sigma)\right]^{\frac{1}{\alpha}}\right).$$

By the Mean-Value Theorem,

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{1}{\alpha}\left[\sum_{s=n+1}^{\infty} p(s)\eta(s-\sigma)u^{\alpha}(s-\sigma)\right]^{\frac{1}{\alpha}-1} p(n)\eta(n-\sigma)u^{\alpha}(n-\sigma) \leq 0.$$

Since $\left\{\frac{u(n)}{B^{\delta_1}(n)}\right\}$ is increasing,

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{1}{\alpha}\frac{u^{1-\alpha}(n-\sigma)}{B^{\delta_1(1-\alpha)}(n-\sigma)}\left[\sum_{s=n+1}^{\infty} p(s)\eta(s-\sigma)B^{\alpha\delta_1}(s-\sigma)\right]^{\frac{1-\alpha}{\alpha}} p(n)\eta(n-\sigma)u^{\alpha}(n-\sigma) \leq 0.$$

Therefore, $\{u(n)\}$ satisfies the linear difference inequality

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{1}{\alpha}\frac{p(n)\eta(n-\sigma)}{B^{\delta_1(1-\alpha)}(n-\sigma)} \left[\sum_{s=n+1}^{\infty} p(s)\eta(s-\sigma)B^{\alpha\delta_1}(s-\sigma)\right]^{\frac{1-\alpha}{\alpha}} u(n-\sigma) \leq 0. \tag{3.6}$$

Since $\delta_1 < 1$, from (3.4) we obtain

$$B^{\alpha(\delta_1-1)}(n-\sigma) \geq \lambda^{\alpha(1-\delta_1)}B^{\alpha(\delta_1-1)}(n)$$

Using this and (2.3), we have

$$\begin{aligned}\sum_{s=n+1}^{\infty} p(s)\eta(s-\sigma)B^{\alpha\delta_1}(s-\sigma) &\geq \alpha\delta \sum_{s=n+1}^{\infty} \frac{B^{\alpha(\delta_1-1)}(s-\sigma)}{B(s)b^{1/\alpha}(s)} \\ &\geq \alpha\delta\lambda^{\alpha(1-\delta_1)} \sum_{s=n+1}^{\infty} \frac{B^{\alpha(\delta_1-1)-1}(s)}{b^{1/\alpha}(s)} \\ &= \alpha\delta\lambda^{\alpha(1-\delta_1)} \sum_{s=n+1}^{\infty} B^{\alpha(\delta_1-1)-1}(s)\Delta B(s) \\ &\geq \alpha\delta\lambda^{\alpha(1-\delta_1)} \sum_{s=n+1}^{\infty} \int_{B(s)}^{B(s+1)} \frac{du}{u^{\alpha(1-\delta_1)+1}} \\ &= \frac{\delta\lambda^{\alpha(1-\delta_1)}}{(1-\delta_1)} B^{\alpha(\delta_1-1)}(n+1).\end{aligned}$$

Substituting this into (3.6) gives

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{\delta^{\frac{1-\alpha}{\alpha}}\lambda^{(1-\delta_1)(1-\alpha)}}{\alpha(1-\delta_1)^{\frac{1-\alpha}{\alpha}}} \frac{B^{(\delta_1-1)(1-\alpha)}(n+1)}{B^{\delta_1(1-\alpha)}(n-\sigma)} p(n)\eta(n-\sigma)u(n-\sigma) \leq 0$$

which in view of (3.4) yields that $\{u(n)\}$ is a positive solution of the difference inequality

$$\Delta(b^{1/\alpha}(n)\Delta u(n)) + \frac{\delta^{\frac{1-\alpha}{\alpha}}\lambda^{1-\alpha}}{\alpha(1-\delta_1)^{\frac{1-\alpha}{\alpha}}} B^{(\alpha-1)}(n+1)p(n)\eta(n-\sigma)u(n-\sigma) \leq 0.$$

By Lemma 2.6, the corresponding difference equation (3.5) has also a positive solution, so the proof is complete. □

Remark 3.4. The comparison results presented in Theorems 3.1–3.3 reduce the examination of oscillatory properties of (E) to those of linear equations.

4. Oscillation Criteria

In this section, we apply the results from the previous section to establish new oscillation criteria for Eq. (E).

Theorem 4.1. *Let* (2.1) *and* (2.3) *hold. If*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} p(s)\eta(s-\sigma)B^{\alpha}(s-\sigma) > (1-\delta)^{\alpha} \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}, \qquad (4.1)$$

then (E) *is oscillatory.*

Proof. In view of (4.1) and Theorem 7.6.1 of [10], it is easy to see that Eq. (3.1) is oscillatory. Therefore, by Theorem 3.1, Eq. (E) is oscillatory. □

Theorem 4.2. *Let* $\alpha > 1$ *and conditions* (2.1) *and* (2.3) *hold. If*

$$\limsup_{n\to\infty}\left\{\frac{1}{B(n-\sigma)}\sum_{s=n_1}^{n-\sigma-1} p(s)\eta(s-\sigma)B(s)B^{\alpha}(s-\sigma) + \sum_{s=n-\sigma}^{n-1} p(s)\eta(s-\sigma)B^{\alpha}(s-\sigma) + B^{1-\delta_1}(n-\sigma)\sum_{s=n}^{\infty} p(s)\eta(s-\sigma)B^{\alpha+\delta_1-1}(s-\sigma)\right\} > \frac{1}{k} \tag{4.2}$$

for some $n_1 \geq n_0 + \sigma$*, where* $k = \frac{(1-\delta)^{\alpha-1}}{\alpha}$*, then equation* (E) *is oscillatory.*

Proof. Assume that (E) is not oscillatory. By Theorem 3.2, Eq. (3.2) is also nonoscillatory and we may assume that it possess an eventually positive solution $\{u(n)\}$ with $u(n) > 0$ for $n \geq n_1 \geq n_0+\sigma$ such that (4.2) holds. Summing (3.2) yields

$$\Delta u(n) \geq \frac{k}{b^{1/\alpha}(n)}\sum_{s=n}^{\infty} p(s)\eta(s-\sigma)B^{\alpha-1}(s-\sigma)u(s-\sigma).$$

Summing once more gives

$$\begin{aligned} u(n) &\geq k\sum_{s=n_1}^{n-1}\frac{1}{b^{1/\alpha}(s)}\sum_{t=s}^{\infty} p(t)\eta(t-\sigma)B^{\alpha-1}(t-\sigma)u(t-\sigma) \\ &= k\sum_{s=n_1}^{n-1}\frac{1}{b^{1/\alpha}(s)}\sum_{t=s}^{n-1} p(t)\eta(t-\sigma)B^{\alpha-1}(t-\sigma)u(t-\sigma) \\ &\quad + k\sum_{s=n_1}^{n-1}\frac{1}{b^{1/\alpha}(s)}\sum_{t=n}^{\infty} p(t)\eta(t-\sigma)B^{\alpha-1}(t-\sigma)u(t-\sigma). \end{aligned}$$

Using summation by parts,

$$\begin{aligned} u(n) &\geq k\sum_{s=n_1}^{n-1} p(s)\eta(s-\sigma)B(s+1)B^{\alpha-1}(s-\sigma)u(s-\sigma) \\ &\quad + kB(n)\sum_{t=n}^{\infty} p(t)\eta(t-\sigma)B^{\alpha-1}(t-\sigma)u(t-\sigma). \end{aligned}$$

Hence,

$$\begin{aligned} u(n-\sigma) &\geq k\sum_{s=n_1}^{n-\sigma-1} p(s)\eta(s-\sigma)B(s+1)B^{\alpha-1}(s-\sigma)u(s-\sigma) \\ &\quad + kB(n-\sigma)\sum_{s=n-\sigma}^{n-1} p(s)\eta(s-\sigma)B^{\alpha-1}(s-\sigma)u(s-\sigma) \\ &\quad + kB(n-\sigma)\sum_{s=n}^{\infty} p(s)\eta(s-\sigma)B^{\alpha-1}(s-\sigma)u(s-\sigma). \end{aligned}$$

In view of the fact that $\frac{u(n)}{B(n)}$ is decreasing and $\frac{u(n)}{B^{\delta_1}(n)}$ is increasing (see Lemmas 2.1 and 2.4), the previous inequality gives

$$\begin{aligned} u(n-\sigma) \geq k\frac{u(n-\sigma)}{B(n-\sigma)} &\sum_{s=n_1}^{n-\sigma-1} p(s)\eta(s-\sigma)B(s+1)B^{\alpha}(s-\sigma) \\ &+ ku(n-\sigma)\sum_{s=n-\sigma}^{n-1} p(s)\eta(s-\sigma)B^{\alpha}(s-\sigma) \\ &+ k\frac{B(n-\sigma)u(n-\sigma)}{B^{\delta_1}(n-\sigma)}\sum_{s=n}^{\infty} p(s)\eta(s-\sigma)B^{\alpha+\delta_1-1}(s-\sigma). \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned} &\left\{\frac{1}{B(n-\sigma)}\sum_{s=n_1}^{n-\sigma-1} p(s)\eta(s-\sigma)B(s+1)B^{\alpha}(s-\sigma) + \sum_{s=n-\sigma}^{n-1} p(s)\eta(s-\sigma)B^{\alpha}(s-\sigma)\right. \\ &\left. +B^{1-\delta_1}(n-\sigma)\sum_{s=n}^{\infty} p(s)\eta(s-\sigma)B^{\alpha+\delta_1-1}(s-\sigma)\right\} \leq \frac{1}{k}. \end{aligned}$$

This is a contradiction and proves the theorem. □

For our next and final result we set

$$L = \frac{\delta^{\frac{1-\alpha}{\alpha}}\lambda^{1-\alpha}}{\alpha(1-\delta_1)^{\frac{1-\alpha}{\alpha}}}.$$

Theorem 4.3. *Let $0 < \alpha < 1$ and conditions (2.1) and (2.4) hold. If*

$$\begin{aligned} \limsup_{n\to\infty} &\left\{\frac{1}{B(n-\sigma)}\sum_{s=n_1}^{n-\sigma-1} B^{\sigma}(s+1)B(s-\sigma)p(s)\eta(s-\sigma)\right. \\ &+ \sum_{s=n-\sigma}^{n-1} B^{\alpha-1}(s+1)B(s-\sigma)p(s)\eta(s-\sigma) \\ &\left. +B^{1-\delta_1}(n-\sigma)\sum_{s=n}^{\infty} B^{\alpha-1}(s+1)B^{\delta_1}(s-\sigma)p(s)\eta(s-\sigma)\right\} > \frac{1}{L}, \end{aligned} \tag{4.3}$$

then (E) is oscillatory.

Proof. Assume that equation (E) is not oscillatory. By Theorem 3.3, Eq. (3.5) is also nonoscillatory. Without loss of generality, we may assume that it possesses an eventually positive solution $\{u(n)\}$ for $n \geq n_1 \geq n_0$. Summing (3.5) gives

$$\Delta u(n) \geq \frac{L}{b^{1/\alpha}(n)}\sum_{s=n}^{\infty} B^{\alpha-1}(s+1)p(s)\eta(s-\sigma)u(s-\sigma).$$

Then,

$$u(n) \geq L\sum_{s=n_1}^{n-1}\frac{1}{b^{1/\alpha}(s)}\sum_{t=s}^{\infty} B^{\alpha-1}(t+1)p(t)\eta(t-\sigma)u(t-\sigma)$$

$$= L \sum_{s=n_1}^{n-1} \frac{1}{b^{1/\alpha}(s)} \sum_{t=s}^{n-1} B^{\alpha-1}(t+1)p(t)\eta(t-\sigma)u(t-\sigma)$$
$$+ L \sum_{s=n_1}^{n-1} \frac{1}{b^{1/\alpha}(s)} \sum_{t=n}^{\infty} B^{\alpha-1}(t-\sigma)p(t)\eta(t-\sigma)u(t-\sigma).$$

Thus,

$$u(n) \geq L \sum_{s=n_1}^{n-1} B^{\alpha}(s+1)p(s)\eta(s-\sigma)u(s-\sigma)$$
$$+ LB(n) \sum_{s=n}^{\infty} B^{\alpha-1}(s+1)p(s)\eta(s-\sigma)u(s-\sigma),$$

and so

$$u(n-\sigma) \geq L \sum_{s=n_1}^{n-\sigma-1} B^{\alpha}(s+1)p(s)\eta(s-\sigma)u(s-\sigma)$$
$$+ LB(n-\sigma) \sum_{s=n-\sigma}^{n-1} B^{\alpha-1}(s+1)p(s)\eta(s-\sigma)u(s-\sigma)$$
$$+ LB(n-\sigma) \sum_{s=n}^{\infty} B^{\alpha-1}(s+1)p(s)\eta(s-\sigma)u(s-\sigma).$$

Since $\frac{u(n)}{B(n)}$ is decreasing and $\frac{u(n)}{B^{\delta_1}(n)}$ is increasing, the last inequality implies that

$$u(n-\sigma) \geq L \frac{u(n-\sigma)}{B(n-\sigma)} \sum_{s=n_1}^{n-\sigma-1} B^{\alpha}(s+1)B(s-\sigma)p(s)\eta(s-\sigma)$$
$$+ Lu(n-\sigma) \sum_{s=n-\sigma}^{n-1} B^{\alpha-1}(s+1)p(s)\eta(s-\sigma)B(s-\sigma)$$
$$+ L \frac{B(n-\sigma)u(n-\sigma)}{B^{\delta_1}(n-\sigma)} \sum_{s=n}^{\infty} B^{\alpha-1}(s+1)B^{\delta_1}(s-\sigma)p(s)\eta(s-\sigma).$$

Hence,

$$\left\{ \frac{1}{B(n-\sigma)} \sum_{s=n_1}^{n-\sigma-1} B^{\alpha}(s+1)B(s-\sigma)p(s)\eta(s-\sigma) \right.$$
$$+ \sum_{s=n-\sigma}^{n-1} B^{\alpha-1}(s+1)B(s-\sigma)p(s)\eta(s-\sigma)$$
$$\left. + B^{1-\delta_1}(n-\sigma) \sum_{s=n}^{\infty} B^{\alpha-1}(s+1)B^{\delta_1}(s-\sigma)p(s)\eta(s-\sigma) \right\} \leq \frac{1}{L}.$$

This contradicts (4.3), and completes the proof of the theorem. □

5. Examples

In this section, we illustrate the oscillation criteria obtained in the previous section with examples of Euler type difference equations.

Example 5.1. Consider the second order delay difference equation

$$\Delta((\Delta u(n))^3) + \frac{a}{n^4} u^3(n-2) = 0, \ n \geq 3, \tag{5.1}$$

with $a > 0$. Here we have $b(n) = 1$, $\sigma = 2$, and $\alpha = \beta = 3$. A simple calculation shows that $B(n) = n - 3$, and by taking $\lambda = 1$ and $\delta = \frac{1}{8}$, we have $\delta_1 = \frac{1}{2}$, $k = \frac{49}{192}$, and $\eta(n-\sigma) = 1$. Condition (2.1) becomes

$$\sum_{n=3}^{\infty} \frac{a}{n^4}(n-5)^3 = \infty$$

and (2.3) is satisfied if $a \geq 32$. Condition (4.2) becomes

$$\limsup_{n\to\infty} \left\{ \frac{1}{(n-3)} \sum_{s=6}^{n-3} \frac{a}{s^4}(s-1)(s-3)^3 + \sum_{s=n-2}^{n-1} a\frac{(s-3)^3}{s^4} \right.$$
$$\left. +(n-3)^{\frac{1}{2}} \sum_{s=n}^{\infty} \frac{a}{s^4}(s-3)^{\frac{5}{2}} \right\} = 3a > \frac{192}{49},$$

and hence (4.2) is satisfied for $n_1 = 6$ if $a > \frac{64}{49}$. Therefore, by Theorem 4.2, Eq. (5.1) is oscillatory if $a \geq 32$.

Example 5.2. Consider the second order delay difference equation

$$\Delta(n^{\frac{2}{9}}(\Delta u(n))^{\frac{1}{3}}) + \frac{a}{n^{\frac{10}{9}}} u^{\frac{1}{3}}(n-1) = 0, \ n \geq 1, \tag{5.2}$$

where $a > 0$. We have $b(n) = n^{2/9}$, $\sigma = 1$, and $\alpha = \beta = 1/3$. Then, with $\lambda = 1$ and $\delta = \frac{1}{2}$, we have $\delta_1 = \frac{1}{8}$, $\eta(n-\sigma) = 1$, $L = \frac{48}{49}$, and $B(n) \approx 3n^{1/3}$. Condition (2.1) becomes

$$\sum_{n=1}^{\infty} \frac{27a}{n^{\frac{1}{9}}} = \infty,$$

and (2.3) is satisfied for $a \geq \frac{1}{18}(2/27)^{\frac{1}{9}}$. Condition (4.3) reduces to

$$\limsup_{n\to\infty} \left\{ \frac{1}{3(n-1)^{\frac{1}{3}}} \sum_{s=n_1}^{n-2} 9(s+1)^{\frac{1}{3}}(s-1)^{\frac{1}{3}} \frac{a}{s^{\frac{10}{9}}} \right.$$
$$+ \sum_{s=n-1}^{n-1} [3(s+1)^{\frac{1}{3}}]^{-\frac{2}{3}} 3(s-1)^{\frac{1}{3}} \frac{a}{s^{\frac{10}{9}}}$$
$$\left. +[3(n-1)^{\frac{1}{3}}]^{\frac{7}{8}} \sum_{s=n}^{\infty} [3(s+1)^{\frac{1}{3}}]^{-\frac{2}{3}} [3(s-1)^{\frac{1}{3}}]^{\frac{1}{8}} \frac{a}{s^{\frac{10}{9}}} \right\} = \frac{7a}{72} > \frac{49}{48}.$$

Therefore, by Theorem 4.3, Eq. (5.2) is oscillatory if $a > \frac{21}{2}$.

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