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On Affine Minimal Translation Surfaces and Ramanujan Identities

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Abstract. In this paper, using the Weierstrass–Enneper formula and the hodographic coordinate system, we find the relationships between the Ramanujan identity and a generalized class of minimal translation surfaces, known as affine minimal translation surfaces. We find the Dirichlet series expansion of the *affine Scherk* surface. We also obtain some of the probability measures of *affine Scherk* surface with respect to its logarithmic distribution. Next, we classify the affine minimal translation surfaces in $\mathbb{L}^3$ and remark the analogous forms in $\mathbb{L}^3$.

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1. Introduction

A surface in the Monge form represented by its height function $z(x, y)$ is a minimal surface whenever

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy} = 0. \tag{1.1}$$

The Weierstrass–Enneper (W–E) representation gives a general parametric form solution of a minimal surface in the neighbourhood of a non-umbilical interior point in the following form:

$$\begin{cases} x(\xi) = x_0 + \Re \int_{\xi_0}^{\xi} R(\omega)(1 - \omega^2)\mathrm{d}\omega \\ y(\xi) = y_0 + \Re \int_{\xi_0}^{\xi} \iota R(\omega)(1 + \omega^2)\mathrm{d}\omega \\ z(\xi) = z_0 + \Re \int_{\xi_0}^{\xi} 2R(\omega)\mathrm{d}\omega, \end{cases} \tag{1.2}$$

where ξ is a complex parameter, $\Re \int$ is the real part of the integral and $R(\omega)$ is a meromorphic function. In this parameterization (ξ_1, ξ_2) acts as isothermal parameterization, where $\xi = \xi_1 + \iota\xi_2$.

Now let $\varphi = \varphi(x, y)$ be the local form of a surface in Lorentz–Minkowski space ($\mathbb{L}^3$) with the metric $g = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2$. The mean curvature formula is given by

$$H = \epsilon \frac{EN - 2FM + GL}{2(EG - F^2)}, \tag{1.3}$$

where $\epsilon = 1$ for a spacelike surface and $\epsilon = -1$ for a timelike surface. The $H \equiv 0$ is equivalent to the following differential equation

$$(1 - z_y^2)z_{xx} + 2z_x z_y z_{xy} + (1 - z_x^2)z_{yy} = 0, \tag{1.4}$$

satisfying $z_x^2 + z_y^2 < 1$. Similar to the minimal surface, the W–E representation for a maximal surface is given by (for instance see [5])

$$\begin{cases} x(\xi) = x_0 + \Re \int_{\xi_0}^{\xi} R(\omega)(1 + \omega^2)\mathrm{d}\omega \\ y(\xi) = y_0 + \Re \int_{\xi_0}^{\xi} \iota R(\omega)(1 - \omega^2)\mathrm{d}\omega \\ z(\xi) = z_0 + \Re \int_{\xi_0}^{\xi} 2R(\omega)\mathrm{d}\omega, \end{cases} \tag{1.5}$$

In 3-dimensional Euclidean space ($\mathbb{E}^3$), a surface $\varphi(x, y)$ is called a translation surface if $\varphi(x, y)$ can be written as

$$\varphi(x, y) = f(x) + g(y), \tag{1.6}$$

where f and g are two planar (and/or non-planar) smooth curves lying in orthogonal (and/or non-orthogonal) planes, such that $f' \times g' \neq 0$. These two curves (f and g) are called as the generators of the surface. Depending upon the location and nature of generators, lots of research has been done with respect to such minimal translation surfaces.

In the case of planar curves where the generating curves lie in the orthogonal planes, Scherk proved that, apart from planes, the only minimal translation surfaces are the surfaces with the height function:

$$\varphi(x, y) = \log \frac{\cos y}{\cos x}. \tag{1.7}$$

Recently, López and Perdomo obtained a classification of the long-standing problem of minimal translation surfaces where both the curves are non-planar [12].

Kobayashi [9] studied the minimal translation surfaces in $\mathbb{L}^3$ and proved the following:

Every maximal spacelike translation surface in $\mathbb{L}^3$ is either a spacelike plane or is given by

$$\varphi(x, y) = \log \frac{\cosh y}{\cosh x}, \tag{1.8}$$

where $\tanh^2 x + \tanh^2 y < 1$. This surface is called the surface of Scherk of the first kind.

A Born–Infeld soliton is a very close look alike of (1.4) satisfying the following equation [14]:

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (z_x^2 - 1)z_{yy} = 0. \tag{1.9}$$

The minimal (maximal) surfaces and the Born–Infeld solitons are very closely related in the sense that a minimal surface is obtained by a wick rotation in the variable $y \leftrightarrow \iota y$ in Born–Infeld soliton equation and vice-versa [3]. In the same way, wick rotating x by ιx in the Born–Infeld soliton equation, we get maximal surface equation and vice-versa [5].

Thus a soliton version of the height function of (1.8) is given by

$$\varphi(x,y) = \log \frac{\cosh y}{\cos x}. \tag{1.10}$$

Now coming towards the actual theme of the paper, in recent times, various relations between the minimal surfaces and the Ramanujan identities have been explored. The first such relations were found by Kamien [7], wherein using Ramanujan's identities, Kamien obtained a decomposition of the height function of the Scherk's first surface. At the moment, a minimal surface is equivalent to finding a solution of second-order partial differential equation (1.1). Dey simplified the study by using the hodographic coordinates and discovering a much simpler first-order analogous form of (1.1) in the following expression:

$$\xi^2 \bar{z}_\xi + z_\xi = 0. \tag{1.11}$$

Using (1.11), Dey rederived the W–E representation formula of minimal surfaces (see [1]).

In her next paper [2], Dey dedicated a major part of [2] to find non-trivial identities for Scherk surfaces using Ramanujan's identities, W–E representation and the technique developed by her in the earlier paper [1]. In the same paper, she also derived the series expressions for (1.10).

Then in [5], Dey and Singh, using the hodographic coordinates obtained the W–E formula for maximal surfaces. In particular, following their earlier paper [2], they obtained the series expression for (1.8) using Ramanujan identities. For similar studies, see [3,4].

Thus from the above surveys, we observe that the series expressions of the translations surfaces of the type (1.7), (1.8) and (1.10) has remained the primary focus of the authors.

In an attempt to generalize the notion of translation surfaces, Liu and Yu in [11] introduced a new class of translation surfaces given as a graph of

$$\varphi(x,y) = f(x) + g(y+ax)$$

for some non-zero real constant a. The generators lie in the non-orthogonal planes $x = 0$ and $y + ax = 0$, therefore acts as a natural generalization of (1.6). This type of surface is called as *affine translation surface*. In case of minimal affine translation surfaces, we have the following:

Theorem 1.1. [10,11] *Let φ be a minimal affine translation surface. Then φ is either a linear function or is given by a surface with the height function given by*

$$\varphi(x,y) = \log \frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)}. \tag{1.12}$$

The surface in (1.12) is called as *generalized Scherk* surface or an *affine Scherk* surface. For $a = 0$, we get (1.7).

Following the work of Dey and her collaborators, we extend the study by decomposing the height function of a generalized class of translation surfaces, i.e., affine minimal translation surface with the help of Ramanujan identity, W–E representation formula and hodographic coordinates. We also find their Dirichlet series expansion and find the probability mass function with the help logarithmic distribution.

In layered structures Scherk's first surfaces are used to express large twist-angle grain boundaries, however small twist-angle grain boundaries are often traced in the form of an array of screw dislocations. Kamien and Lubensky proved that there is no notable difference between these two notions, their relative energetics are generically dependent on the basic structure of their screw-dislocation topological defects (see [6]). In [7], Kamien derived a series decomposition of the height function of the Scherk's first surface using Ramanujan's identities. He showed that in layered structures Scherk's surfaces appear as infinite superposition in topological defects. Thus, looking at the physical importance of the decomposition of the height functions of minimal surfaces, in particular Scherk surfaces, we can say that the series decompositions obtained in this paper may be useful in understanding the topological defects as in Kamien and Lubensky work [6,7] .

The structure of the paper is as follows. In Sect. 2, we first derive the meromorphic function $R(\omega)$ needed in the W–E representation of *generalized Scherk* surfaces. Then we obtain the series decomposition of *generalized Scherk* surfaces using the Ramanujan identity. In Sect. 2.2, we obtain the Logarithmic distribution of the *affine Scherk* surface. We also comment on how to obtain its Born–Infeld soliton version. In Sect. 3, we first derive the analogous form of *generalized Scherk* surface in $\mathbb{L}^3$ and then revisit all the forms in this setting.

2. Affine Scherk Surface and Ramanujan Identity in $\mathbb{E}^3$

Lemma 2.1. *$R(\omega)$ for the affine Scherk surface is given by*

$$R(\omega) = \frac{\sqrt{1+a^2}(1+\iota a)}{(1+a^2)+(1+\iota a)^2\omega^2} + \frac{1}{1-\omega^2}, \tag{2.1}$$

where $a \in \mathbb{R}$.

Proof. In order to find $R(\omega)$, we use the method of hodographic coordinates. Let $z = x + \iota y$ and $\bar{z} = x - \iota y$ be the complex coordinates and $u = \varphi_{\bar{z}} = \frac{\varphi_x + \iota\varphi_y}{2}$, $v = \varphi_z = \frac{\varphi_x - \iota\varphi_y}{2}$. For the *affine Scherk* surface, we have

$$\varphi_x = \sqrt{1+a^2}\tan(\sqrt{1+a^2}\Re(x)) - a\tan[a\Re(z) + \Im(z)],\ \varphi_y = -\tan[a\Re(z) + \Im(z)].$$

This implies that

$$a\Re(z) + \Im(z) = \tan^{-1}\left(\iota(u-v)\right). \tag{2.2}$$

and

$$\Re(z) = \frac{1}{\sqrt{1+a^2}} \tan^{-1}\left[\frac{(u+v) - \iota a(v-u)}{\sqrt{1+a^2}}\right]. \tag{2.3}$$

Let $\xi = \frac{\sqrt{1+4uv}-1}{2v}$ and $\bar{\xi} = \frac{\sqrt{1+4uv}-1}{2u}$ be two new variables, with the inverse transformation $u = \frac{\xi}{1-\xi\bar{\xi}}$ and $v = \frac{\bar{\xi}}{1-\xi\bar{\xi}}$. In terms of ξ and $\bar{\xi}$, (2.2) and (2.3) reduces to

$$a\Re(z) + \Im(z) = \tan^{-1}(\iota\xi) - \tan^{-1}(\iota\bar{\xi}),$$

$$\Re(z) = \frac{1}{\sqrt{1+a^2}} \tan^{-1}\left[\frac{1}{\sqrt{1+a^2}}\left\{\frac{\xi(1+\iota a)}{1-\xi\bar{\xi}} + \frac{\bar{\xi}(1-\iota a)}{1-\xi\bar{\xi}}\right\}\right]. \tag{2.4}$$

Let $\varsigma(\xi) = \frac{\xi(1+\iota a)}{\sqrt{1+a^2}}$, $\bar{\varsigma}(\bar{\xi}) = \frac{\bar{\xi}(1-\iota a)}{\sqrt{1+a^2}}$, Eq. (2.4) is equivalent to

$$\Re(z) = \frac{1}{\sqrt{1+a^2}}\left(\tan^{-1}\varsigma + \tan^{-1}\bar{\varsigma}\right).$$

Thus, we can write

$$\bar{z} = F(\xi) + G(\bar{\xi}),$$

where

$$F(\xi) = \frac{1+\iota a}{\sqrt{1+a^2}} \tan^{-1}\xi - \iota\tan^{-1}(\iota\xi),$$

$$G(\bar{\xi}) = \frac{1+\iota a}{\sqrt{1+a^2}} \tan^{-1}\bar{\xi} + \iota\tan^{-1}(\iota\bar{\xi}).$$

We know that $R(\omega) = F'(\omega)$ (see [1]). This proves our claim in Eq. (2.1). □

Remark 2.2. In the above lemma, we see that the points $\pm 1, \pm\iota\frac{\sqrt{1+a^2}}{1+\iota a}$ are the poles of $R(\omega)$, which are precisely the umbilical points of minimal surface. Moreover, for $a = 0$, we get the $R(\omega)$ of the Scherk surface given in (1.7).

Proposition 2.3. *Let φ be an affine non-planar minimal translation surface in $\mathbb{E}^3$ and ξ in a neighbourhood of $\mathbb{C}$ away from the umbilical points ($\xi \neq \pm 1, \pm\iota\frac{\sqrt{1+a^2}}{1+\iota a}$), we have the following identity*

$$\begin{aligned}
&\log\frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)}\\
&\quad = \log\prod_{k=1}^{\infty}\left(\frac{\left(k-\frac{1}{2}\right)\pi - [\Im(\lambda_3(\xi)) + 2\Re(\lambda_4(\xi)) + a(\Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)))]}{\left(k-\frac{1}{2}\right)\pi - \sqrt{1+a^2}(\Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)))}\right)\\
&\qquad \times\left(\frac{\left(k-\frac{1}{2}\right)\pi + [\Im(\lambda_3(\xi)) + 2\Re(\lambda_4(\xi)) + a(\Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)))]}{\left(k-\frac{1}{2}\right)\pi + \sqrt{1+a^2}(\Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)))}\right).
\end{aligned}$$

Proof. Using Lemma 2.1 and (1.2), the Weierstrass–Enneper data for (1.12) is given by

$$x(\xi) = \Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)) \tag{2.5}$$

$$y(\xi) = \Im(\lambda_3(\xi)) + 2\Re(\lambda_4(\xi)) \tag{2.6}$$

$$\varphi(\xi) = \Re(\lambda_5(\xi)), \tag{2.7}$$

where $\lambda_1(\xi) = -\frac{(\iota+a)}{\sqrt{1+a^2}}\xi$, $\lambda_2(\xi) = \xi - \frac{2\sqrt{1+a^2}\tan^{-1}\left(\frac{\sqrt{a-\iota}}{\iota\sqrt{\iota+a}}\xi\right)}{\iota\sqrt{\iota+a}(a-\iota)^{3/2}}$,

$$\lambda_3(\xi) = \left(\frac{1-\iota a}{\sqrt{1+a^2}} - 1\right)\xi + 2\tan^{-1}(\xi),\ \lambda_4(\xi) = \left(\frac{a\sqrt{1+a^2}\tan^{-1}\left(\frac{\sqrt{a-\iota}}{\iota\sqrt{\iota+a}\xi}\right)}{\iota\sqrt{\iota+a}(a-\iota)^{3/2}}\right),$$

$$\lambda_5(\xi) = \frac{1}{2\sqrt{1+a^2}}(\iota+a)\left[2\tan^{-1}\left(a - \frac{2a}{1+\xi^2}\right) - \iota\log(a^2(\xi^2-1)^2) + (\xi^2+1)^2\right] - \log(1-\xi^2).$$

Let X and A be complex with A not an odd multiple of $\frac{\pi}{2}$ (see [13]), we have

$$\frac{\cos(X+A)}{\cos(A)} = \prod_{k=1}^{\infty}\left[\left(1 - \frac{X}{(k-\frac{1}{2}\pi) - A}\right)\left(1 + \frac{X}{(k-\frac{1}{2}\pi) + A}\right)\right]$$

or

$$\log\frac{\cos(X+A)}{\cos(A)} = \log\prod_{k=1}^{\infty}\left(\frac{\left(k-\frac{1}{2}\right)\pi - (X+A)}{\left(k-\frac{1}{2}\right)\pi - A}\right)\left(\frac{\left(k-\frac{1}{2}\right)\pi + (X+A)}{\left(k-\frac{1}{2}\right)\pi + A}\right) \tag{2.8}$$

Let $X + A = y + ax$ and $A = \sqrt{1+a^2}x$ in (2.8), then if x is not an odd multiple of $\frac{\pi}{2\sqrt{1+a^2}}$, we have

$$\log\frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)} = \log\prod_{k=1}^{\infty}\left(\frac{\left(k-\frac{1}{2}\right)\pi - (y+ax)}{\left(k-\frac{1}{2}\right)\pi - \sqrt{1+a^2}x}\right)\left(\frac{\left(k-\frac{1}{2}\right)\pi + (y+ax)}{\left(k-\frac{1}{2}\right)\pi + \sqrt{1+a^2}x}\right) \tag{2.9}$$

Substituting the W–E data in the above expression, we get

$$\begin{aligned}\Re\log(\lambda_5(\xi)) &= \log\frac{\cos[\Im(\lambda_3(\xi)) + 2\Re(\lambda_4(\xi))]}{\cos[\sqrt{1+a^2}\{\Im(\lambda_3(\xi)) + 2\Re(\lambda_4(\xi))\}]}\\ &= \log\prod_{k=1}^{\infty}\left(\frac{\left(k-\frac{1}{2}\right)\pi - [\Im(\lambda_3(\xi)) + 2\Re(\lambda_4(\xi)) + a(\Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)))]}{\left(k-\frac{1}{2}\right)\pi - \sqrt{1+a^2}(\Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)))}\right)\\ &\quad\times\left(\frac{\left(k-\frac{1}{2}\right)\pi + [\Im(\lambda_3(\xi)) + 2\Re(\lambda_4(\xi)) + a(\Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)))]}{\left(k-\frac{1}{2}\right)\pi + \sqrt{1+a^2}(\Im(\lambda_1(\xi)) + \Re(\lambda_2(\xi)))}\right).\end{aligned}$$

□

Remark 2.4. From (2.1), we observe that for $a = 0$, we get $R(\omega)$ for the Scherk surface, i.e., $R(\omega) = \frac{2}{1-\omega^4}$.

2.1. Dirichlet Series Expansion of Affine Scherk Surface

The first Dirichlet series expansion of minimal surfaces was obtained by Dey et al. (see [4]). In this section, with the help of Ramanujan identity, we obtain the Dirichlet series expansion of *affine Scherk* surface. We know that

$$\log\frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)} = \log\prod_{k=1}^{\infty}\left(\frac{\left(k-\frac{1}{2}\right)\pi - (y+ax)}{\left(k-\frac{1}{2}\right)\pi - \sqrt{1+a^2}x}\right)\left(\frac{\left(k-\frac{1}{2}\right)\pi + (y+ax)}{\left(k-\frac{1}{2}\right)\pi + \sqrt{1+a^2}x}\right).$$

Set $\left(k-\frac{1}{2}\right)\pi=\alpha_k$, we can write the above expression as

$$\log\frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)}=\log\prod_{k=1}^{\infty}\left(\frac{\alpha_k-(y+ax)}{\alpha_k-\sqrt{1+a^2}x}\right)\left(\frac{\alpha_k+(y+ax)}{\alpha_k+\sqrt{1+a^2}x}\right)$$
$$=\log\prod_{k=1}^{\infty}\left(1+\frac{(1+a^2)x^2}{\alpha_k^2-(1+a^2)x^2}\right)\left(1-\frac{(y+ax)^2}{\alpha_k^2}\right)$$

or

$$\log\frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)}=\sum_{k=1}^{\infty}\Big[\frac{(1+a^2)x^2}{\alpha_k^2-(1+a^2)x^2}-\frac{1}{2}\left(\frac{(1+a^2)x^2}{\alpha_k^2-(1+a^2)x^2}\right)^2+\frac{1}{3}\left(\frac{(1+a^2)x^2}{\alpha_k^2-(1+a^2)x^2}\right)^3-\cdots\Big]$$
$$-\sum_{k=1}^{\infty}\Big[\frac{(y+ax)^2}{\alpha_k^2}+\frac{1}{2}\left(\frac{(y+ax)^2}{\alpha_k^2}\right)^2+\frac{1}{3}\left(\frac{(y+ax)^2}{\alpha_k^2}\right)^3+\cdots\Big], \tag{2.10}$$

where $|\frac{(1+a^2)x^2}{\alpha_k^2-(1+a^2)x^2}|<1$ and $|\frac{(y+ax)^2}{\alpha_k^2}|<1$, $\forall$ integer k. This implies that

$$\log\frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)}=\sum_{k=1}^{\infty}\left[P_k\left(1,\frac{(1+a^2)x^2}{\alpha_k^2-(1+a^2)x^2}\right)-T_k\left(1,\frac{(y+ax)^2}{\alpha_k^2}\right)\right], \tag{2.11}$$

where $P_k(s,a)=\sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{a^n}{n^s}\right]$, $T_k(s,b)=\sum_{n=1}^{\infty}\frac{b^n}{n^s}$ are two Dirichlet series with real parameters a, b, respectively. Since ξ, x, y are related by (2.5)–(2.6) and $|\frac{(1+a^2)x^2}{\alpha_k^2-(1+a^2)x^2}|<1$ and $|\frac{(y+ax)^2}{\alpha_k^2}|<1$, $\forall$ integer k. Therefore for a considerably small bounded domain of ξ and on substituting the W-E data in (2.11), we can state the following:

Proposition 2.5. *For a small enough domain of ξ, the Dirichlet series expansion of the non-trivial affine minimal translation surface is given by*

$$\log\frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)}=\Re\log(\lambda_5(\xi))$$
$$=\sum_{k=1}^{\infty}\left[P_k\left(1,\frac{(1+a^2)[\Im(\lambda_1(\xi))+\Re(\lambda_2(\xi))]^2}{\alpha_k^2-(1+a^2)[\Im(\lambda_1(\xi))+\Re(\lambda_2(\xi))]^2}\right)-T_k\left(1,\frac{[\Im(\lambda_3(\xi))+2\Re(\lambda_4(\xi))+a(\Im(\lambda_1(\xi))+\Re(\lambda_2(\xi)))]^2}{\alpha_k^2}\right)\right].$$

2.2. Logarithmic Distribution of Affine Scherk Surface

It is easy to show the convergence of both the Dirichlet series P_k and T_k of (2.11), which in turn implies the convergence of both the series in RHS of

(2.10). Now we re-examine the RHS of (2.10) in the way:

$$\log \frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)} = \sum_{k=1}^{\infty}\left(-A_k - \frac{1}{2}A_k^2 - \frac{1}{3}A_k^3 - \cdots\right), \tag{2.12}$$

where $A_k^i = \left\{\frac{(y+ax)^2}{\alpha_k^2}\right\}^i + \left\{\frac{-(1+a^2)x^2}{\alpha_k^2-(1+a^2)x^2}\right\}^i$ and $0 < A_k < 1$. The RHS of the above converges for each k as it the sum of two convergent series. Supposing that the series in (2.12) converges to a constant s. From the RHS of (2.12), we have

$$\log(1-A_k) = \sum_{j=1}^{\infty} -\frac{A_k^j}{j}.$$

Since the above identity holds for each k, we can write

$$\sum_{k=1}^{n}\log(1-A_k) = \sum_{k=1}^{n}\sum_{j=1}^{\infty} -\frac{A_k^j}{j}, \tag{2.13}$$

where

$$\sum_{k=1}^{n}\sum_{j=1}^{\infty} -\frac{A_k^j}{j} = s_n$$

are the partial sums whose limiting sum tends to s, i,e.,

$$s = \log \frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(s_1^j + s_2^j + \cdots + s_k^j + \cdots\right). \tag{2.14}$$

Now we can write (2.13) as

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{n}\frac{-A_k^j}{j}\right) = \sum_{k=1}^{n}\log(1-A_k). \tag{2.15}$$

From the LHS of above expression, we get an indexed set of real values $\{s_n\}_{n\in\mathbb{N}}$ whose total limiting sum is s. The expression in (2.15) can be combined in a way such that:

$$\sum_{j=1}^{\infty}\left(\frac{\sum_{k=1}^{n}\left(\frac{-A_k^j}{j}\right)}{\sum_{k=1}^{n}\log(1-A_k)}\right) = 1.$$

This gives us the probability mass function $(f_k(j))$ of a $\log(A_k)$ distributed random variable which assumes the real values of the partial sums, i.e., $f_k(j) = \frac{\sum_{k=1}^{n}\left(\frac{-A_k^j}{j}\right)}{\sum_{k=1}^{n}\log(1-A_k)}$, $j \geq 1$ with the parameter $0 < A_k < 1$. Looking back at (2.12), we see that the values of the $\log(A_k)$ distributed random variable are the values assumed by the partial sums with the limiting sum equal to the height function of the *affine Scherk* surface. Thus we can state the following:

Proposition 2.6. *Let* $\log \frac{\cos(y+ax)}{\cos(\sqrt{1+a^2}x)} = s$ *be the height function of the affine Scherk surface, such that* s_n *are the convergent partial sum series with the limiting sum equal to the height function of affine Scherk surface. Then the* *probability mass function of the random variable assuming the values* s_n *is given by* $f_k(j) = \frac{\sum_{k=1}^{n}\left(\frac{-A_k^j}{j}\right)}{\sum_{k=1}^{n}\log(1-A_k)}$, $j \geq 1$, *with the parameter* $0 < A_k < 1$.

The probability mass function gives the probability that a discrete random variable is exactly equal to some value (see [8]).

2.3. Wick Rotation of the Affine Minimal Translation Surface

By a wick rotation in the variable $y \to \iota y$ of Theorem 1.1, we get a Born–Infeld soliton analogue. We shall call a wick rotated affine minimal translation surface by affine wick minimal translation surface (AWMT). Thus from Theorem 1.1, one can observe:

Observation: Let φ be an AWMT surface. Then φ is either a linear function or is given by $\varphi(x, y) = \log\frac{\cos(\iota y+ax)}{\cos(\sqrt{1+a^2}x)}$.

3. Affine Scherk Surface and Ramanujan Identity in $\mathbb{L}^3$

In [5], Dey and Singh obtained the Weierstrass data and the corresponding Ramanujan identity expression of Scherk surface in $\mathbb{L}^3$. In this section, we generalize the notion to a more generalized class of Scherk surfaces, i.e., *affine Scherk* surfaces.

Lemma 3.1. *Let* φ *be a maximal affine translation surface. Then* φ *is either a linear function or up to dilation and translation is given by*

$$\varphi(x, y) = \log\left[\frac{\cosh(y + ax)}{\cosh(\sqrt{1 + a^2}x)}\right]. \tag{3.1}$$

Proof. Let $\varphi(x, y) = f(x) + g(y + ax)$ be an affine translation surface in $\mathbb{L}^3$ with the metric $g = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2$. The first and the second fundamental form coefficients are: $E = 1 - (f' + ag')^2$, $F = -g'(f' + ag')$, $G = 1 - g'$ and $L = -(f'' + a^2 g'')D^{-1}$, $M = -ag''D^{-1}$, $N = -g''D^{-1}$, where $f' = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$, $g' = \frac{\mathrm{d}g(y+ax)}{\mathrm{d}(y+ax)}$ and $D^2 = 1 - {g'}^2 - (f' + ag')^2$. Therefore, the maximal surface equation is given by

$$f''(1 - {g'}^2) + g''(1 + a^2 - {f'}^2) = 0. \tag{3.2}$$

By a direct computation of (3.2), we get the expression (3.1). □

We call the surface in (3.1) as the *affine Scherk* surface of first kind.

Lemma 3.2. $R(\omega)$ *for the affine Scherk surface of the first kind is given by*

$$R(\omega) = \frac{\sqrt{1 + a^2}(1 + \iota a)}{(1 + a^2) + (1 - \iota a)^2\omega^2} + \frac{1}{1 - \omega^2}, \tag{3.3}$$

where $a \in \mathbb{R}$.

Proof. Introducing the new variables $\zeta = \frac{1-\sqrt{1-4uv}}{2v}$, $\bar{\zeta} = \frac{1-\sqrt{1-4uv}}{2u}$ with the inverse transformations $u = \frac{\zeta}{1+\zeta\bar{\zeta}}$, $v = \frac{\bar{\zeta}}{1+\zeta\bar{\zeta}}$. Following the similar lines as in Lemma 2.1, we get the claim. □

Remark 3.3. The points $\pm 1, \pm\iota\frac{\sqrt{1+a^2}}{1-\iota a}$ act as umbilical points. So, the parameterization is to be considered away from these possible points.

Proposition 3.4. *Let φ be an affine non-planar minimal translation surface in $\mathbb{L}^3$ and ξ in a neighbourhood of $\mathbb{C}$ away from the umbilical points ($\xi \neq \pm 1, \pm\iota\frac{\sqrt{1+a^2}}{1-\iota a}$), we have the following identity*

$$\begin{aligned}\log \frac{\cosh(y+ax)}{\cosh(\sqrt{1+a^2}x)} &= \log \prod_{k=1}^{\infty}\left(\frac{\left(k-\frac{1}{2}\right)\pi - \iota[\Re(\mu_3(\zeta)) + \Im(\mu_4(\zeta)) + a\{\Re(\mu_1(\zeta)) + \Im(\mu_2(\zeta))\}]}{\left(k-\frac{1}{2}\right)\pi - \iota\sqrt{1+a^2}(\Re(\mu_1(\zeta)) + \Im(\mu_2(\zeta)))}\right)\\ &\quad\times\left(\frac{\left(k-\frac{1}{2}\right)\pi + \iota[\Re(\mu_3(\zeta)) + \Im(\mu_4(\zeta)) + a\{\Re(\mu_1(\zeta)) + \Im(\mu_2(\zeta))\}]}{\left(k-\frac{1}{2}\right)\pi + \iota\sqrt{1+a^2}\{\Re(\mu_1(\zeta)) + \Im(\mu_2(\zeta))\}}\right).\end{aligned}$$

Proof. Using Lemma 3.2 and (1.2), the Weierstrass–Enneper data for (1.12) is given by

$$\begin{aligned}x(\zeta) &= \Re(\mu_1(\zeta)) + \Im(\mu_2(\zeta))\\ y(\zeta) &= \Re(\mu_3(\zeta)) + \Im(\mu_4(\zeta))\\ \varphi(\zeta) &= \Re(\mu_5(\zeta)) + \Re(\mu_6(\zeta)),\end{aligned}$$

where $\mu_1 = -\frac{\left(a^2+\iota\left(\sqrt{a^2+1}+2\right)a+\sqrt{a^2+1}-1\right)\zeta}{(\iota+a)^2} + \log\frac{1+\zeta}{1-\zeta}$,
$\mu_2 = -2\frac{a\sqrt{\iota-a}\sqrt{a^2+1}}{(\iota+a)^{5/2}}\tan^{-1}\left(\frac{\sqrt{\iota+a}}{\sqrt{\iota-a}}\zeta\right)$,
$\mu_3 = \frac{\sqrt{\iota-a}\sqrt{a^2+1}}{(\iota+a)^{5/2}}\left(\log\left(\frac{\sqrt{\iota-a}-\iota\sqrt{\iota+a}\zeta}{\sqrt{\iota-a}+\iota\sqrt{\iota+a}\zeta}\right)\right)$,
$\mu_4 = -\frac{\left(a^2+\iota\left(\sqrt{a^2+1}+2\right)a+\sqrt{a^2+1}-1\right)\zeta}{(\iota+a)^2}$, $\mu_5 = \log\frac{1+\zeta}{1-\zeta}$,
$\mu_6 = -2\frac{\sqrt{\iota-a}\sqrt{a^2+1}}{(\iota+a)^{3/2}}\tan^{-1}\left(\frac{\sqrt{\iota+a}}{\sqrt{\iota-a}}\zeta\right)$.

From the Ramanujan's identity, we have

$$\log\frac{\cos(X+A)}{\cos(A)} = \log\prod_{k=1}^{\infty}\left(\frac{\left(k-\frac{1}{2}\right)\pi - (X+A)}{\left(k-\frac{1}{2}\right)\pi - A}\right)\left(\frac{\left(k-\frac{1}{2}\right)\pi + (X+A)}{\left(k-\frac{1}{2}\right)\pi + A}\right). \tag{3.4}$$

Substituting $X+A = \iota(y+ax)$ and $A = \iota\sqrt{1+a^2}x$ in (3.4), where $\iota\sqrt{1+a^2}x$ is not an odd multiple of $\frac{\iota\pi}{2}$, we obtain

$$\log\frac{\cosh(y+ax)}{\cosh(\sqrt{1+a^2}x)} = \log\prod_{k=1}^{\infty}\left(\frac{\left(k-\frac{1}{2}\right)\pi - \iota(y+ax)}{\left(k-\frac{1}{2}\right)\pi - \iota\sqrt{1+a^2}x}\right)\left(\frac{\left(k-\frac{1}{2}\right)\pi + \iota(y+ax)}{\left(k-\frac{1}{2}\right)\pi + \iota\sqrt{1+a^2}x}\right).$$

Substituting the W–E data in the above identity, we get the claim. □

Remark 3.5. In case of *affine Scherk* surfaces in $\mathbb{L}^3$, we have now its W-E data and Proposition 3.4, so on the similar lines as in Sect. 2, we can easily find the analogous forms of Born–Infeld soliton, Dirichlet series expansion and the probability density function.

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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