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On the Exponential Diophantine Equation $F_{n+1}^x - F_{n-1}^x = F_m$

Bijan Kumar Patel and Ana Paula Chaves

Abstract. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. In this paper, we explicitly find all solutions of the title Diophantine equation using lower bounds for linear forms in logarithms and properties of continued fractions. Further, we use a version of the Baker–Davenport reduction method in Diophantine approximation due to Dujella and Pethö.

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1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1}+F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well known for possessing wonderful and amazing properties (consult [7] together with their very extensive annotated bibliography for additional references and history).

Diophantine equations related to sums of powers of two terms of a given linear recurrence sequence were studied by several authors. For instance, motivated by the naive identity

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \tag{1.1}$$

which tells us that the sum of the square of two consecutive Fibonacci numbers is still a Fibonacci number, Marques and Togbé [12] questioned what about such sums with higher powers, and showed that, if $x \geq 1$ is an integer such that $F_n^x + F_{n+1}^x$ is a Fibonacci number for all sufficiently large n, then $x \in \{1, 2\}$. In 2011, Luca and Oyono [10] solved this problem completely by showing that the Diophantine equation

$$F_m^s + F_{m+1}^s = F_n \tag{1.2}$$

has no solutions (m, n, s) with $m \geq 2$ and $s \geq 3$.

Subsequently, their result have been extended by considering k-generalized Fibonacci numbers [4,15]. Recently, Bednařík et al. [1] has proved that the Diophantine equation $(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 = F_m^{(l)}$ has no solution in positive integers n, m, k, l with $2 \leq k < l$ and $n > 1$. Hirata-Kohno and Luca [6] was revisited (1.2) and found that the solutions of the Diophantine equation $F_n^x + F_{n+1}^x = F_m^y$, in positive integers (m, n, x, y), are $(3, 1, x, 1), (n + 2, n, 1, 1), (2n + 1, n, 1, 1), (3, 4, 1, 3), (4, 2, 3, 2)$. Subsequently, Luca and Oyono [11] reversed the role of two exponents of the previous equation and studied whether $F_n^x + F_{n+1}^y = F_m^x$ or $F_n^y + F_{n+1}^x = F_m^x$. They proved that the only positive integer solution (m, n, x, y) of one of the mentioned equations with $n \geq 3$ and $x \neq y$, is $(5, 3, 2, 4)$, for which $F_4^3 + F_2^4 = F_2^5$. Miyazaki [14] showed that the only positive integer solutions (x, y, z, n) of the equation $F_n^x + F_{n+1}^y = F_{2n+1}^z$ are for $(x, y, z) = (2, 2, 1)$ (and for all positive integers n).

Among the several pretty algebraic identities involving Fibonacci numbers, we are interested in the following one:

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \quad \text{for all } n \geq 0. \tag{1.3}$$

In particular, the above identity tell us that, for two Fibonacci numbers whose positions in the sequence differ by two, the difference of their squares is still a Fibonacci number. So, in the same spirit of the previous works done on Eq. (1.1), one could ask: Does this property still holds for $F_{n+1}^3 - F_{n-1}^3$? And for $F_{n+1}^4 - F_{n-1}^4$? And so on?

The aim of this paper is to answer such questions, i.e. to know when $F_{n+1}^x - F_{n-1}^x$ is a Fibonacci number. More precisely, our main result is the following.

Theorem 1.1. *The only non-negative integer solutions* (m, n, x) *of the Diophantine equation*

$$F_{n+1}^x - F_{n-1}^x = F_m, \tag{1.4}$$

are $(2n, n, 2), (1, 1, x), (2, 1, x), (0, n, 0)$.

Our method follows roughly the following steps: First, we use Matveev's result [13] on linear forms in logarithms to obtain an upper bound for x in terms of m and n. When n is small, say $n \leq 112$, we use Dujella and Pethö's result [5] to decrease the range of possible values and then let the computer check the non-existence of solutions in this case. To the case where $n \geq 113$, we use again linear forms in logarithms to obtain an absolute upper bound for x. In the final step, we use continued fractions to lower the bounds and then let the computer cover the range of possible values, showing that there are no solutions also in this case, which completes the proof.

2. Auxiliary Results

In this section, the main results used to "attack" our problem are presented. One basic fact about Fibonacci numbers is their closed-form, given by Binet's formula, which states that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad \text{for all } n \geq 0, \tag{2.1}$$

where $\alpha := (1+\sqrt{5})/2$, is the Golden Ratio, and $\beta = -\alpha^{-1}$, are the roots of $x^2 - x - 1$. This implies on the following well-known inequalities:

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}, \tag{2.2}$$

which holds for all $n \geq 1$. Also, it is easy to show that

$$\frac{F_{n-1}}{F_{n+1}} \leq \frac{2}{5}, \quad \text{holds for any } n \geq 3. \tag{2.3}$$

2.1. Lower Bounds for Linear Forms in Logarithms of Algebraic Numbers

To prove Theorem 1.1, we use a few times a Baker-type lower bound for a non-zero linear forms in logarithms of algebraic numbers. We state a result of Matveev [13] about the general lower bound for linear forms in logarithms, but first, recall some basic notations from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial

$$f(X) := a_0 X^d + a_1 X^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the a_is are relatively prime integers, $a_0 > 0$, and the $\eta^{(i)}$s are conjugates of η. Then

$$h(\eta) = \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \log \left(\max\{|\eta^{(i)}|, 1\} \right) \right) \tag{2.4}$$

is called the *logarithmic height* of η. Some properties of the logarithmic height, which will be used in the next section, are the following:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2. \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma). \\ h(\eta^s) &= |s| h(\eta) \ (s \in \mathbb{Z}). \end{aligned}$$

With the established notations, Matveev (see [13] or [2, Theorem 9.4]) proved the ensuing result.

Theorem 2.1. *Assume that $\gamma_1, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

is not zero. Then

$$|\Lambda| \geq \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \text{ for all } i = 1, \ldots, t.$$

2.2. Reduction Algorithm

Another result which will play an important role in our proof is due to Dujella and Pethö [5, Lemma 5 (a)]. It will be used to reduce the upper bounds of the variables on Eq. (1.4).

Lemma 2.2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-u},$$

in positive integers u and v with

$$u \leq M \quad \text{and} \quad u \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following result is known as Legendre's criterion of a rational r/s to be a convergent of α.

Lemma 2.3. [16, Lemma 5C] *If α is irrational and r/s is a rational number with $s > 0$ such that*

$$\left|\alpha - \frac{r}{s}\right| < \frac{1}{2s^2},$$

then r/s is a convergent to α.

3. The Proof of Theorem 1.1

Since the case $nx = 0$ is trivial, we assume $n, x \geq 1$. Observe that, for $x = 1$, the Diophantine equation (1.4) becomes $F_n = F_{n+1} - F_{n-1} = F_m$, which gives $m = n$, and for $x = 2$, $F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_m$, gives $m = 2n$. When $n = 1$, we have that $F_2^x - F_0^x = 1 = F_1 = F_2$ holds for all $x \geq 1$. For $n = 2$, we get $F_m + 1 = 2^x$, which has only the solutions $(m, x) = (0, 0)$ and $(4, 2)$, as it was showed in [3, Theorem 2].

So, assume that $n \geq 3$. Since $x \geq 3$, we get that $F_m \geq F_4^3 - F_2^3 = 26$, so $m > 8$. Using (2.1) for F_m, we rewrite Eq. (1.4) as

$$\frac{\alpha^m}{\sqrt{5}} - F_{n+1}^x = -F_{n-1}^x + \frac{\beta^m}{\sqrt{5}} \in [-F_n^x, -F_n^x + 1]. \tag{3.1}$$

Dividing by F_{n+1}^x on both sides of (3.1) and taking absolute values, we get

$$\left|\alpha^m 5^{-1/2} F_{n+1}^{-x} - 1\right| < 2\left(\frac{F_{n-1}}{F_{n+1}}\right)^x < \frac{2}{2.5^x}. \tag{3.2}$$

To apply Theorem 2.1, we take

$$\gamma_1 := \alpha,\ \gamma_2 := \sqrt{5},\ \gamma_3 := F_{n+1},$$
$$b_1 := m,\ b_2 := -1,\ b_3 := -x,$$

and also $\Lambda_1 := \alpha^m 5^{-1/2} F_{n+1}^{-x} - 1$. Note that $\mathbb{Q}(\sqrt{5})$ is the algebraic number field containing γ_1, γ_2 and γ_3, so $D := 2$. If $\alpha^m = \sqrt{5} F_{n+1}^x$, then $\alpha^{2m} \in \mathbb{Z}$ which is false for all positive integers m, therefore $\Lambda_1 \neq 0$.

Since $h(\gamma_1) = \log \alpha/2$, we can choose $A_1 := 0.5 > \log \alpha$. Furthermore, since $h(\gamma_2) = \log 5/2$, and $h(\gamma_3) = \log F_{n+1}$, it follows that we can take $A_2 := 1.61 > \log 5$, and $A_3 := 2n \log \alpha > 2 \log F_{n+1}$. Lastly, using (2.2), the following chains of inequalities holds

$$\alpha^{(n-2)x+2} < \alpha^{(n-2)x} \cdot \underbrace{(\alpha^x - 1)}_{>\alpha^3 - 1 > \alpha^2} < F_{n+1}^x - F_{n-1}^x = F_m < \alpha^{m-1},$$

$$\alpha^{nx} > F_{n+1}^x > F_{n+1}^x - F_{n-1}^x = F_m > \alpha^{m-2}.$$

Thus, since $n \geq 3$ and $x > 2$,

$$x < (n-2)x + 3 < m < nx + 2 < (n+1)x, \tag{3.3}$$

so we can take $B := m$.

From Matveev's theorem, we have a lower bound for $|\Lambda_1|$, which combined with (3.2) gives

$$\exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) \cdot (1 + \log m) \cdot 0.5 \cdot 1.61 \cdot 2n \log \alpha\right) > \frac{2}{2.5^x}\ .$$

Hence, using (3.2), (3.3) and that $1 + \log m < 1.5 \log m$, is true for all $m \geq 8$, we get

$$x < \frac{\log 2}{\log 2.5} + 1.23 \cdot 10^{12} \cdot n \log m,$$

therefore,

$$x < 1.24 \cdot 10^{12} n \log((n+1)x). \tag{3.4}$$

3.1. The Case $n \in [3, 112]$

In this case,

$$x < 1.24 \cdot 10^{12} n \log((n+1)x) \leq 1.4 \cdot 10^{14} \log(113x),$$

providing $x < 5.8 \cdot 10^{15}$.

We set

$$\Gamma_1 := -x \log F_{n+1} + m \log \alpha - \log \sqrt{5}.$$

Thus, $\Lambda_1 = e^{\Gamma_1} - 1$. Recall that, from (3.1), we have $\Lambda_1 < 0$, which implies $\Gamma_1 < 0$. Now, since $|\Lambda_1| < 2/(2.5)^x \leq 0.128$, for $x \geq 3$, it follows that $e^{|\Gamma_1|} < 1.15$. Hence, we get

$$0 < |\Gamma_1| < e^{|\Gamma_1|} - 1 \leq e^{|\Gamma_1|} |e^{\Gamma_1} - 1| < \frac{2.3}{2.5^x}.$$

Dividing the last inequality by $\log \alpha$, and using from (3.3) that $m < (n+1)x$, we have the following

$$0 < x \left(\frac{\log F_{n+1}}{\log \alpha} \right) - m + \left(\frac{\log \sqrt{5}}{\log \alpha} \right) < 4.78 \cdot (2.5^{\frac{1}{113}})^{-m}. \tag{3.5}$$

To use the Reduction Method, take

$$\gamma_n := \frac{\log F_{n+1}}{\log \alpha},\ \mu := \frac{\log \sqrt{5}}{\log \alpha},\ A := 4.78,\ B := 2.5^{\frac{1}{113}}.$$

We claim that γ_n is irrational, for every $n \in [3, 112]$. Indeed, if $\gamma_n = p/q$ for some positive integers p and q, we would have $\alpha^q = F_{n+1}^p \in \mathbb{Z}$, which is an absurd. Let $q_{(146,n)}$ be the denominator of the 146th convergent of the continued fraction of γ_n.

Taking $M := 5.8 \cdot 10^{15}$, we use *Mathematica* to get

$$\min_{3\leq n\leq 112} q_{(146,n)} > 10^{22} > 6M \quad \text{and} \quad \max_{3\leq n\leq 112} q_{(146,n)} < 1.33 \cdot 10^{84}.$$

Also, for $\epsilon_n := ||\mu \cdot q_{(146,n)}|| - 5.8 \cdot 10^{15}||\gamma_n \cdot q_{(146,n)}||$, we obtain that

$$\min_{3\leq n\leq 112} \epsilon_n > 0.004,$$

which means that ϵ_n is always positive (this is not true if we consider the denominator of the 145th convergent). Notice that the conditions to apply Lemma 2.2 are fulfilled, and hence there is no solution to inequality (3.5) (and consequently no solution do the Diophantine equation (1.2)) for x and m satisfying

$$\frac{\log(A \cdot q_{(146,n)}/\epsilon_n)}{\log(B)} \leq m \quad \text{and} \quad x \leq M,$$

for all $n \in [3, 112]$. Then, the solutions, in this case, must be when

$$\begin{aligned} m &< \frac{\log(A \cdot q_{(146,n)}/\epsilon_n)}{\log(B)} \\ &< \frac{\log(4.78 \cdot 1.33 \cdot 10^{84}/0.004)}{2.5^{\frac{1}{113}}} \\ &< 24761. \end{aligned}$$

Therefore, $m \leq 24760$, and so $x < (m-3)/(n-2) < 24757/(n-2)$. Now, we prepare a simple routine in *Mathematica* in the range $n \in [3, 112]$, $m \in [8, 24761]$ and $x \in [3, 24757/(n-2)]$, which returns no solutions for (1.2). This completes the case $n \in [3, 112]$.

3.2. An Upper Bound for x in Terms of n

From now on, assume $n \geq 113$. Our short term goal is to find an explicit lower bound for x in terms of n. If $x \leq n+1$, we are done. Otherwise, if $x > n+2$, then from (3.4), we get

$$x < 2.48 \cdot 10^{12} n \log x,$$

which can be rewritten as

$$\frac{x}{\log x} < 2.48 \cdot 10^{12} n. \tag{3.6}$$

From the useful fact that, for all $x > e$, $x/\log x < A \Rightarrow x < 2A \log A$, holds whenever $A \geq 3$, and since $\log(2.48 \cdot 10^{12} n) < 7.05 \log n$, holds for all $n \geq 113$, it follows

$$\begin{aligned} x &< 2 \cdot (2.48 \cdot 10^{12} n) \log(2.48 \cdot 10^{12} n) \\ &< 4.96 \cdot 10^{12} n \cdot (7.04 \log n) \\ &< 3.5 \cdot 10^{13} n \log n. \end{aligned}$$

So finally, we have

$$x < 3.5 \cdot 10^{13} n \log n \tag{3.7}$$

for $n \geq 113$.

3.3. An Absolute Upper Bound on x

Set $y := x/\alpha^{2n}$. From (3.7) and $n \geq 113$, we have

$$y < \frac{3.5 \cdot 10^{13} n \log n}{\alpha^{2n}} < \frac{1}{\alpha^n}. \tag{3.8}$$

In particular, $y < \alpha^{-31} < 10^{-23}$. Now, we need to make a few algebraic manipulations apply Theorem 2.1 a second time. Rewrite Eq. (1.4) as follows:

$$\frac{\alpha^m}{\sqrt{5}} - \frac{\beta^m}{\sqrt{5}} = \frac{\alpha^{(n+1)x}}{5^{x/2}} - \frac{\alpha^{(n-1)x}}{5^{x/2}} + \left(F_{n+1}^x - \frac{\alpha^{(n+1)x}}{5^{x/2}}\right) - \left(F_{n-1}^x - \frac{\alpha^{(n-1)x}}{5^{x/2}}\right). \tag{3.9}$$

Since $y < 10^{-23}$ is very small, we have

$$\max\left\{\left|F_{n+1}^x - \frac{\alpha^{(n+1)x}}{5^{x/2}}\right|, \left|F_{n-1}^x - \frac{\alpha^{(n-1)x}}{5^{x/2}}\right|\right\} < \frac{2y\alpha^{(n+1)x}}{5^{x/2}} \quad (\text{Eq. } (16), [10]).$$

Taking absolute values, after a slight modification of (3.9), we obtain

$$\begin{aligned} \left|\frac{\alpha^m}{\sqrt{5}} - \frac{\alpha^{(n-1)x}}{5^{x/2}}(\alpha^{2x} - 1)\right| &= \left|\frac{\beta^m}{\sqrt{5}} + \left(F_{n+1}^x - \frac{\alpha^{(n+1)x}}{5^{x/2}}\right) - \left(F_{n-1}^x - \frac{\alpha^{(n-1)x}}{5^{x/2}}\right)\right| \\ &< \frac{1}{\alpha^m} + 2y\left(\frac{\alpha^{(n-1)x}(\alpha^{2x} - 1)}{5^{x/2}}\right). \end{aligned}$$

Now, dividing both sides of the above inequality by $\alpha^{(n+1)x}/5^{x/2}$, we get

$$\begin{aligned} \left|\alpha^{m-(n+1)x} 5^{(x-1)/2} - (1 - \alpha^{-2x})\right| &< \frac{5^{x/2}}{\alpha^{m+(n+1)x}} + 2y(1 - \alpha^{-2x}) \\ &< \frac{1}{2\alpha^n} + \frac{17y}{9} \\ &< \frac{43}{18\alpha^n}, \end{aligned} \tag{3.10}$$

where we used that $5^{x/2}/\alpha^{(n+1)x} \leq \left(\sqrt{5}/\alpha^{113}\right)^x < 1/2$, and $\alpha^{2x} \geq \alpha^6 > 18$. Thus,

$$\begin{aligned} \left|\alpha^{m-(n+1)x} 5^{(x-1)/2} - 1\right| &< \frac{43}{18\alpha^n} + \frac{1}{\alpha^{2x}} \\ &\leq \frac{61}{18\alpha^l}, \end{aligned}$$

where $l := \min\{n, 2x\}$. Hence, we deduce that

$$\left|\alpha^{m-(n+1)x} 5^{(x-1)/2} - 1\right| < \frac{61}{18\alpha^l}. \tag{3.11}$$

Observe that, by the same argument used previously, $\alpha^{m-(n+1)x}5^{(x-1)/2} - 1$ is non-zero. Since $x \geq 3$ and $n \geq 113$, we get

$$\left|\alpha^{m-(n+1)x}5^{(x-1)/2} - 1\right| \leq \frac{1}{\alpha^3} + \frac{1}{\alpha^{113}} < \frac{1}{2}, \tag{3.12}$$

so that $\alpha^{m-(n+1)x}5^{(x-1)/2} \in [1/2, 2]$. In particular,

$$1.6x - 4 < (n+1)x - m < 1.7x. \tag{3.13}$$

Now, we are about to use of Matveev's result one more time, to obtain a lower bound for the left-hand side of (3.11). For this, we take

$$t := 2, \gamma_1 := \alpha, \gamma_2 := \sqrt{5}, b_1 := m - (n+1)x, b_2 := x - 1,$$
$$D := 2, A_1 := \log\alpha, A_2 := \log 5, \text{ and } B := 1.7x > \max\{|b_1|, |b_2|\}.$$

Hence,

$$\log\left|\alpha^{m-(n+1)x}5^{(x-1)/2} - 1\right| > -1.4 \cdot 30^5 \cdot 2^{4.5} \cdot 2^2 \\ (1 + \log 2)(1 + \log(1.7x))(\log\alpha)(\log 5). \tag{3.14}$$

From (3.11) and (3.14), we deduce

$$l < 3.5 \times 10^9 \log x.$$

Treating separately the case $l = n$ and $l = 2x$, and performing the respective calculations, we arrive that $x < 5 \times 10^{36}$ and $x < 4.29 \times 10^{10}$ respectively. In any case we have that

$$x < 5 \times 10^{36}.$$

3.4. Reducing the Bound on x

Next, we take $\Gamma_2 := (x-1)\log\sqrt{5} - ((n+1)x - m)\log\alpha$. Observe that (3.12) becomes

$$\left|\alpha^{m-(n+1)x}5^{(x-1)/2} - 1\right| = |e^{\Gamma_2} - 1| < \frac{1}{2}.$$

Thus, we have that $e^{|\Gamma_2|} < 2$. Hence,

$$|\Gamma_2| \leq e^{|\Gamma_2|}\left|e^{\Gamma_2} - 1\right| < 2 \cdot \left|\alpha^{m-(n+1)x}5^{(x-1)/2} - 1\right| < 2\left(\frac{43}{18\alpha^n} + \frac{1}{\alpha^{2x}}\right).$$

This leads to

$$\left|\frac{\log\sqrt{5}}{\log\alpha} - \frac{(n+1)x - m}{x-1}\right| < \frac{1}{(x-1)\log\alpha}\left(\frac{43}{9\alpha^n} + \frac{2}{\alpha^{2x}}\right). \tag{3.15}$$

Assume that $x > 100$. Then $\alpha^{2x} > \alpha^{200} > 10^{41} > 10^4 x$. Hence, we deduce that

$$\frac{1}{(x-1)\log\alpha}\left(\frac{43}{9\alpha^n} + \frac{2}{\alpha^{2x}}\right) < \frac{8}{x(x-1)10^4\log\alpha} \\ < \frac{1}{600(x-1)^2}. \tag{3.16}$$

Estimates (3.15) and (3.16) lead to the following inequality:

$$\left|\frac{\log\sqrt{5}}{\log\alpha} - \frac{(n+1)x - m}{x-1}\right| < \frac{1}{600(x-1)^2}. \tag{3.17}$$

By virtue of Lemma 2.3, inequality (3.17) implies that the rational number

$$\frac{(n+1)x-m}{(x-1)}$$

is a convergent to $\gamma := \log(\sqrt{5})/\log\alpha$. Let $[a_0, a_1, a_2, a_3, \ldots] = [1, 1, 2, 19, \ldots]$ be the continued fraction of γ and p_t/q_t its tth convergent. Assume that $((n+1)x-m)/(x-1) = p_k/q_k$ for some k. Then $x-1 = dq_k$ for some positive integer d, where $d = \gcd[(n+1)x-m, x-1]$. On the other hand, using the *Mathematica*, we get that

$$q_{54} = 14014190203160504083256905054 > 1.4 \times 10^{28} > 3 \times 10^{27} - 1 > x - 1;$$

therefore, $0 \leq k \leq 53$. Again, using *Mathematica*, we observe that $a_t \leq 29$ for $t \in \{0, \ldots, 53\}$. From the properties of continued fractions, we have

$$\left|\gamma - \frac{(n+1)x-m}{x-1}\right| = \left|\gamma - \frac{p_k}{q_k}\right| > \frac{1}{(a_k+2)q_k^2} \geq \frac{1}{31(x-1)^2},$$

which contradicts (3.17). Hence, $x \leq 100$.

3.5. The Final Step

From (3.10), we have

$$\left|\alpha^{m-(n+1)x}5^{(x-1)/2}(1-\alpha^{-2x})^{-1} - 1\right| < \frac{43}{18\alpha^n(1-\alpha^{-2x})} < \frac{3}{\alpha^n}.$$

Put $s := (n+1)x - m$. We computed all the numbers $\left|\alpha^{-s}5^{(x-1)/2}(1-\alpha^{-2x})^{-1}-1\right|$ for all $x \in [3, 100]$ and all $s \in [\lfloor 1.6x-4\rfloor, \lfloor 1.7x\rfloor]$. None of them ended up being zero, since if it were we would get the Diophantine equation

$$\alpha^{2x} - 1 = 5^{(x-1)/2}\alpha^{2x-s}. \tag{3.18}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$\beta^{2x} - 1 = 5^{(x-1)/2}\beta^{2x-s}. \tag{3.19}$$

Multiplying (3.18) and (3.19), we get

$$\begin{aligned} -\alpha^{2x} - \beta^{2x} + (-1)^s + 1 &= (\alpha^{2x}-1)(\beta^{2x-1}-1) \\ &= (\alpha\beta)^{2x-s}5^{(x-1)} \\ &= (-1)^s 5^{(x-1)}. \end{aligned} \tag{3.20}$$

If s is even, then (3.20) implies that $L_{2x} = 2 - 5^{x-1}$, where $(L_n)_{n\geq 0}$ be the Lucas sequence given by $L_{n+2} = L_{n+1} + L_n$, for $n \geq 0$, with $(L_0, L_1) = (2, 1)$. However, $5^{x-1} > 2$ for any $x \geq 2$. Thus, s is odd, then (3.20) implies that $L_{2x} = 5^{(x-1)}$. Using the identity $L_k^2 - 5F_k^2 = 4(-1)^k$, it is easy to check that $5 \nmid L_k$ for any positive integer k. This theorem is, therefore, proved.

Comments

It is evident from the title Diophantine equation that one can revisit the equation under some more general forms. A number of directions for future research are discussed below.

(i) The only positive integer solution (m, n, x, y) of $F_{n+1}^x - F_{n-1}^y = F_m^x$ with $m \geq 1, n \geq 3, x \geq 2, y \geq 1$ and $x \neq y$ is $(4, 4, 2, 4)$ for which $F_5^2 - F_3^4 = F_4^2$.

(ii) The only positive integer solutions (m, n, x, y) of $F_{n+1}^x - F_{n-1}^x = F_m^y$ with $n > 1$ are $(n, n, 1, 1)$, $(n, 2n, 2, 1)$, $(3, 3, 2, 3), (6, 3, 1, 3)$.

(iii) The only positive integer solutions (n, x, y, z) of $F_{n+1}^x - F_{n-1}^y = F_{2n}^z$ are $(x, y, z) = (2, y, 1)$ for $n = 2, 3$ and $(x, y, z) = (2, 2, 1)$ for $n \geq 4$.

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Bijan Kumar Patel
Department of Mathematics
Kalinga Institute of Industrial Technology
Bhubaneswar Odisha751 024
India
e-mail: bijan.patelfma@kiit.ac.in

Ana Paula Chaves
Instituto de Matemática e Estatística
Universidade Federal de Goiás
74690-612 Goiânia
Brazil
e-mail: apchaves@ufg.br

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