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On Ulam Stability of an Operatorial Equation

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Abstract. An iterative method generates a sequence associated with an equation, that, under appropriate conditions, converges to a solution of that equation. A perturbation of the equation produces also a perturbation of the sequence. In this paper, we study the Ulam stability (the behavior of the solutions of the perturbed equation with respect to the solutions of the exact equation) of an operatorial equation of the form $x_{n+1} = T_n x_n + a_n$, where $T_n : X \to X$, $n \in \mathbb{N}$, are linear and bounded operators acting on a Banach space X. As applications we obtain some stability results for the case of Volterra, Fredholm and Gram–Schmidt operators. In this way, we improve and complement some results on this topic.

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1. Introduction

In what follows, by $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of all nonnegative integers and by X a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $T : X \to X$ be a bounded linear operator and consider the equation $x = Tx + y$, where $y \in X$ is a given element. By using the fixed point method to solve the equation, we get a sequence of successive approximations $(x_n)_{n\geq 0}$, satisfying the linear difference equation $x_{n+1} = Tx_n + y, n \geq 0, x_0 \in X$, converging to the solution. The Ulam stability of this difference equation was studied in [5].

In this paper, we consider a generalization of the previous difference equation, more precisely the linear difference equation

$$x_{n+1} = T_n x_n + a_n, \; n \in \mathbb{N}, \tag{1.1}$$

where $(T_n)_{n\geq 0}$ is a sequence of bounded linear operators, $T_n : X \to X$, and $(a_n)_{n\geq 0}$ a sequence in X. We study its Ulam stability, which concerns the behavior of the solutions of Eq. (1.1) under perturbations, with respect to the solutions of the unperturbed equation. A particular case of this equation was considered and studied in [12] for the case of matrix operators $(T_n)_{n\geq 0}$. For various results on difference equations we refer the reader to [14,15].

Definition 1.1. Equation (1.1) is called *Ulam stable* if there exists a constant $L \geq 0$ such that for every $\varepsilon > 0$ and every sequence $(x_n)_{n\geq 0}$ in X satisfying

$$\|x_{n+1} - T_n x_n - a_n\| \leq \varepsilon, \ n \in \mathbb{N}, \tag{1.2}$$

there exists a sequence $(y_n)_{n\geq 0}$ in X such that

$$y_{n+1} = T_n y_n + a_n, \ n \in \mathbb{N} \tag{1.3}$$

and

$$\|x_n - y_n\| \leq L\varepsilon, \ n \in \mathbb{N}. \tag{1.4}$$

The sequence $(x_n)_{n\geq 0}$ satisfying (1.2) for some $\varepsilon > 0$ is called *approximate solution* of Eq. (1.1).

So we can reformulate Definition 1.1 as follows: Eq. (1.1) is Ulam stable if for every approximate solution of it there exists an exact solution close to it. The number L from (1.4) is called an Ulam constant of Eq. (1.1). Further, we denote by L_R the infimum of all Ulam constants of (1.1). If L_R is an Ulam constant for (1.1), then we call it *the best Ulam constant* or *the Ulam constant* of Eq. (1.1). Generally, the infimum of all Ulam constants of an equation is not an Ulam constant of that equation (see [8,27]).

The origin of stability for functional equations is a question formulated by S. M. Ulam during a talk given to Madison University, Wisconsin, and concerns the approximate homomorphism of a metric group [29]. A first partial answer to Ulam's question was given a year later by D. H. Hyers for the Cauchy functional equation in Banach spaces [17]. The topic was intensively studied by many authors in the last 50 years; for results, various generalizations and extensions on Ulam stability we refer the reader to [2,8,18,22]. We recall also the results obtained in [7,20,21] on Ulam stability for some second-order linear functional equations in connection with Fibonacci and Lucas sequences.

Some results on Ulam stability for the linear difference equations in Banach spaces were obtained by Brzdek et al. [9–11,24]. Buse et al. [6,13] proved that a discrete system $X_{n+1} = AX_n, n \in \mathbb{N}$, where A is a $m \times m$ complex matrix, is Ulam stable if and only if A possesses a discrete dichotomy. Recently, Baias and Popa obtained results on Ulam stability of linear difference equations of order one and two and determined the best Ulam constant in [3,4]. Popa and Rasa obtained an explicit representation of the best Ulam constant of some classical operators in approximation theory in [25,26].

2. Main Results

In this section, we present some results on Ulam stability for Eq. (1.1). First, we give a result which will be useful in the sequel.

Lemma 2.1. *Suppose that the sequence* $(x_n)_{n\geq 0}$ *satisfies Eq.* (1.1). *Then*

$$x_n = T_{n-1}T_{n-2}\dots T_0x_0 + \sum_{k=1}^{n-1} T_{n-1}\dots T_k a_{k-1} + a_{n-1},\ n\geq 2.$$

If in addition T_n, $n\geq 0$, *are invertible operators, then*

$$x_n = T_{n-1}\dots T_0\left(x_0 + \sum_{k=1}^{n} T_0^{-1}T_1^{-1}\dots T_{k-1}^{-1}a_{k-1}\right),\ n\geq 1.$$

Proof. Induction. □

The first result on Ulam stability for Eq. (1.1) is contained in the following theorem.

Theorem 2.2. *Suppose that* $(T_n)_{n\geq 0}$ *is a sequence of invertible operators such that*

$$\limsup \|T_n^{-1}\| < 1. \tag{2.1}$$

Then for every $\varepsilon > 0$ *and every sequence* $(x_n)_{n\geq 0}$ *in* X *satisfying*

$$\|x_{n+1} - T_nx_n - a_n\| \leq \varepsilon,\ n\in\mathbb{N}, \tag{2.2}$$

there exists a unique sequence $(y_n)_{n\geq 0}$ *in* X *such that*

$$y_{n+1} = T_ny_n + a_n,\ n\in\mathbb{N} \tag{2.3}$$

and

$$\|x_n - y_n\| \leq L\varepsilon,\ n\in\mathbb{N}, \tag{2.4}$$

where

$$L = \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \|T_n^{-1}\dots T_{n+k}^{-1}\| < \infty.$$

Proof. **Existence.** Suppose that $(x_n)_{n\geq 0}$ satisfies (2.2) and let

$$b_n := x_{n+1} - T_nx_n - a_n,\ n\in\mathbb{N}.$$

Then $\|b_n\| \leq \varepsilon,\ n\geq 0$, and, according to Lemma 2.1

$$x_n = T_{n-1}\dots T_0\left(x_0 + \sum_{k=1}^{n} T_0^{-1}\dots T_{k-1}^{-1}(a_{k-1}+b_{k-1})\right),\ n\geq 1.$$

Remark further that the series

$$\sum_{n=1}^{\infty} \|T_0^{-1}\|\cdots\|T_{n-1}^{-1}\|$$

is convergent. Indeed, denoting $c_n = \|T_0^{-1}\|\cdots\|T_{n-1}^{-1}\|,\ n\geq 1$, we get

$$\limsup \frac{c_{n+1}}{c_n} = \limsup \|T_n^{-1}\| < 1.$$

It follows that the series

$$\sum_{n=1}^{\infty} T_0^{-1}\dots T_{n-1}^{-1}b_{n-1}$$

is convergent, too. This is a simple consequence of the first comparison test, since

$$\begin{aligned}\|T_0^{-1}\dots T_{n-1}^{-1}b_{n-1}\| &\le \|T_0^{-1}\dots T_{n-1}^{-1}\|\cdot\|b_{n-1}\| \le \|T_0^{-1}\dots T_{n-1}^{-1}\|\varepsilon\\ &\le \|T_0^{-1}\|\cdots\|T_{n-1}^{-1}\|\varepsilon,\ n\ge 1.\end{aligned}$$

Put now

$$\sum_{n=1}^{\infty} T_0^{-1}\dots T_{n-1}^{-1}b_{n-1} = s, s\in X,$$

and define $(y_n)_{n\ge 0}$ by

$$y_{n+1} = T_n y_n + a_n,\ n\ge 1,\ y_0 = x_0 + s.$$

Then

$$y_n = T_{n-1}\dots T_0\left(y_0 + \sum_{k=1}^{n} T_0^{-1}\dots T_{k-1}^{-1}a_{k-1}\right),\ n\ge 1,$$

and

$$\begin{aligned}x_n - y_n &= T_{n-1}\dots T_0\left(x_0 - y_0 + \sum_{k=1}^{n} T_0^{-1}\dots T_{k-1}^{-1}b_{k-1}\right)\\ &= T_{n-1}\dots T_0\left(-\sum_{n=1}^{\infty} T_0^{-1}\dots T_{n-1}^{-1}b_{n-1} + \sum_{k=1}^{n} T_0^{-1}\dots T_{k-1}^{-1}b_{k-1}\right)\\ &= -T_{n-1}\dots T_0\left(\sum_{k=0}^{\infty} T_0^{-1}\dots T_{n+k}^{-1}b_{n+k}\right)\\ &= -\sum_{k=0}^{\infty} T_n^{-1}\dots T_{n+k}^{-1}b_{n+k},\ n\ge 1.\end{aligned}$$

Hence

$$\begin{aligned}\|x_n - y_n\| &\le \sum_{k=0}^{\infty}\|T_n^{-1}\dots T_{n+k}^{-1}\|\cdot\|b_{n+k}\|\\ &\le \varepsilon\sum_{k=0}^{\infty}\|T_n^{-1}\dots T_{n+k}^{-1}\| \le L\varepsilon,\ n\in\mathbb{N}.\end{aligned}\tag{2.5}$$

We prove now that $L < +\infty$. Indeed, since $\limsup\|T_n^{-1}\| < 1$, it follows that there exists a constant $q\in\mathbb{R}$ and $n_0\in\mathbb{N}$ such that

$$\|T_n^{-1}\| \le q < 1,\ n\ge n_0,$$

which implies that

$$\begin{aligned}\sum_{k=0}^{\infty}\|T_n^{-1}\dots T_{n+k}^{-1}\| &\le \sum_{k=0}^{\infty}\|T_n^{-1}\|\cdots\|T_{n+k}^{-1}\|\\ &\le \sum_{k=0}^{\infty} 2^{k+1} = \frac{q}{1-q},\ n\ge n_0.\end{aligned}$$

For $n = n_0 - 1$, $n_0 \geq 1$, we have

$$\begin{aligned}\sum_{k=0}^{\infty} \|T_n^{-1} \dots T_{n+k}^{-1}\| &= \sum_{k=0}^{\infty} \|T_{n_0-1}^{-1} \dots T_{n_0+k-1}^{-1}\| \\ &= \|T_{n_0-1}^{-1}\| + \|T_{n_0-1}^{-1} T_{n_0}^{-1}\| + \dots + \|\prod_{k=n_0-1}^{\infty} T_k^{-1}\| \\ &\leq \|T_{n_0-1}^{-1}\| \left(1 + \sum_{k=0}^{\infty} \|T_{n_0}^{-1} \dots T_{n_0+k}^{-1}\|\right) < \infty\end{aligned}$$

since the series

$$\sum_{k=0}^{\infty} \|T_{n_0}^{-1} \dots T_{n_0+k}^{-1}\|$$

is convergent. Analogously, the above series will be convergent for all $n \in \mathbb{N}$, $n < n_0$ and the proof is done.

Uniqueness. Suppose that for a sequence $(x_n)_{n\geq 0}$ satisfying the relation (2.2) there exist two sequences $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ satisfying the relations (2.3) and (2.4). Then, in view of (2.4)

$$\|x_n - y_n\| \leq L\varepsilon,\ \|x_n - z_n\| \leq L\varepsilon,\ n \geq 0,$$

hence

$$\|y_n - z_n\| \leq \|y_n - x_n\| + \|x_n - z_n\| \leq 2L\varepsilon,\ n \geq 0.$$

On the other hand, taking account of Lemma 2.1, one gets

$$y_n - z_n = T_{n-1} \dots T_0 (y_0 - z_0),\ n \geq 1,$$

or equivalently

$$y_0 - z_0 = T_0^{-1} \dots T_{n-1}^{-1} (y_n - z_n).$$

Thus

$$\begin{aligned}\|y_0 - z_0\| &\leq \|T_0^{-1} \dots T_{n-1}^{-1}\| \cdot \|y_n - z_n\| \\ &\leq 2L\varepsilon \cdot \|T_0^{-1} \dots T_{n-1}^{-1}\|, n \geq 1.\end{aligned} \tag{2.6}$$

Now, since the series $\sum_{n=1}^{\infty} \|T_0^{-1}\| \cdots \|T_{n-1}^{-1}\|$ is convergent, we get that the series

$$\sum_{n=1}^{\infty} \|T_0^{-1} \dots T_{n-1}^{-1}\|$$

is convergent too, and consequently

$$\lim_{n\to\infty} \|T_0^{-1} \dots T_{n-1}^{-1}\| = 0. \tag{2.7}$$

Finally, from (2.6) and (2.7), we get $y_0 = z_0$, so $y_n = z_n$, for all $n \in \mathbb{N}$. □

Theorem 2.3. *Suppose that $(T_n)_{n\geq 0}$ is a sequence of nonzero operators with*

$$\liminf \frac{1}{\|T_n\|} > 1. \tag{2.8}$$

Then there exists a constant $L \geq 0$ such that for every $\varepsilon > 0$ and every sequence $(x_m)_{m\geq 0}$ in X satisfying

$$\|x_{n+1} - T_n x_n - a_n\| \leq \varepsilon,\ n \in \mathbb{N}, \tag{2.9}$$

there exists a sequence $(y_n)_{n\geq 0}$ in X with the properties

$$y_{n+1} = T_n y_n + a_n,\ n \in \mathbb{N}, \tag{2.10}$$

$$\|x_n - y_n\| \leq L\varepsilon,\ n \geq 0. \tag{2.11}$$

Proof. Suppose that $(x_n)_{n\geq 0}$ satisfies (2.9) and let $b_n := x_{n+1} - T_n x_n - a_n,\ n \in \mathbb{N}$. Then $\|b_n\| \leq \varepsilon,\ n \in \mathbb{N}$, and

$$x_n = T_{n-1}\dots T_0 x_0 + \sum_{k=1}^{n-1} T_{n-1}\dots T_k(a_{k-1} + b_{k-1}) + a_{n-1} + b_{n-1},\ n \geq 2,$$

according to Lemma 2.1.

Consider now $(y_n)_{n\geq 0}$ given by (2.10) with $y_0 = x_0$. Then

$$y_n = T_{n-1}\dots T_0 y_0 + \sum_{k=1}^{n-1} T_{n-1}\dots T_k a_{k-1} + a_{n-1},\ n \geq 2.$$

Hence

$$\begin{aligned} \|x_n - y_n\| &= \|\sum_{k=1}^{n-1} T_{n-1}\dots T_k b_{k-1} + b_{n-1}\| \\ &\leq \varepsilon + \varepsilon \sum_{k=1}^{n-1} \|T_{n-1}\dots T_k\| \leq \varepsilon \left(1 + \sum_{k=1}^{n-1} \|T_{n-1}\|\cdots\|T_k\|\right). \end{aligned} \tag{2.12}$$

Taking account of (2.8), we find $q \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\|T_n\|} \geq q > 1,\ n \geq n_0.$$

Thus for every $n \geq n_0 + 1$ we have

$$\begin{aligned} 1 + \sum_{k=1}^{n-1} \|T_{n-1}\|\cdots\|T_k\| &= 1 + \sum_{k=1}^{n_0-1} \|T_{n-1}\|\cdots\|T_k\| + \sum_{k=n_0}^{n-1} \|T_{n-1}\|\cdots\|T_k\| \\ &\leq 1 + \sum_{k=1}^{n_0-1} \|T_{n-1}\|\cdots\|T_k\| + \sum_{k=n_0}^{n-1} \frac{1}{q^{n+k}} \\ &< 1 + \frac{1}{q-1} + \sum_{k=1}^{n_0-1} \|T_{n-1}\|\cdots\|T_k\| < \infty. \end{aligned}$$

Taking

$$L_1 := 1 + \frac{1}{q-1} + \sum_{k=1}^{n_0-1} \|T_{n-1}\|\cdots\|T_k\|$$

we get

$$\|x_n - y_n\| \leq L_1\varepsilon,\ \text{for all } n \in \mathbb{N}^*,\ n \geq n_0.$$

Finally, for

$$L := \max_{n \le n_0} \left\{ L_1, 1 + \sum_{k=1}^{n-1} \|T_{n-1}\| \cdots \|T_k\| \right\}$$

we obtain

$$\|x_n - y_n\| \le L\varepsilon, \ n \ge n_0.$$

□

Remark 2.4. The condition (2.8) in Theorem 2.3 can be replaced by the following: there exists $q \in (0,1)$ such that

$$\|T_n\| \le q, \text{ for all } n \in \mathbb{N}.$$

Thus, following the lines of the above proof, the Ulam constant can be chosen $L = \frac{1}{1-q}$.

Finally, we present a nonstability result for Eq. (1.1). Taking into account that the stability results presented above hold for $\|T_n\| < 1$ or $\|T_n^{-1}\| < 1, n \ge n_0$, we will consider for nonstability results the case $\|T_n\| = 1$, $n \in \mathbb{N}$.

Theorem 2.5. *Suppose that* $\|T_n\| = 1$*, for all* $n \in \mathbb{N}$ *and there exists* $u_0 \in \overline{B}(0_X, 1)$ *such that*

$$\lim_{n\to\infty} n\|T_{n-1}\dots T_0 u_0\| = +\infty. \tag{2.13}$$

Then for every $\varepsilon > 0$ *there exists a sequence* $(x_n)_{n\ge0}$ *in* X *satisfying*

$$\|x_{n+1} - T_n x_n - a_n\| \le \varepsilon, \ n \in \mathbb{N},$$

such that for every sequence $(y_n)_{n\ge0}$ *given by the recurrence*

$$y_{n+1} = T_n y_n + a_n, \ n \in \mathbb{N}, \ y_0 \in X,$$

we have

$$\sup_{n\in\mathbb{N}} \|x_n - y_n\| = +\infty,$$

i.e., Eq. (1.1) *is not Ulam stable.*

Proof. Let $\varepsilon > 0$ and consider the sequence $(x_n)_{n\ge0}$ defined by the relation

$$x_{n+1} = T_n x_n + a_n + \varepsilon T_n \dots T_0 u_0, \ n \in \mathbb{N}.$$

Then, according to Lemma 2.1, we get

$$\begin{aligned} x_n &= T_{n-1}\dots T_0 x_0 + \sum_{k=1}^{n-1} T_{n-1}\dots T_k(a_{k-1} + \varepsilon T_{k-1}\dots T_0 u_0) \\ &\quad + a_{n-1} + \varepsilon T_{n-1}\dots T_0 u_0, \ n \ge 2. \end{aligned}$$

On the other hand,

$$\|x_{n+1} - T_n x_n - a_n\| = \varepsilon \|T_n \dots T_0 u_0\| \le \varepsilon, \ n \in \mathbb{N},$$

hence $(x_n)_{n\geq 0}$ is an approximate solution of Eq. (1.1). Let $(y_n)_{n\geq 0}$ be an arbitrary sequence in X, $y_{n+1} = T_n y_n + a_n,\ n \in \mathbb{N},\ y_0 \in X$. Then

$$y_n = T_{n-1}\ldots T_0 y_0 + \sum_{k=1}^{n-1} T_{n-1}\ldots T_k a_{k-1} + a_{n-1},\ n \geq 1,$$

therefore

$$\begin{aligned} x_n - y_n &= T_{n-1}\ldots T_0(x_0 - y_0) + \varepsilon \sum_{k=1}^{n} T_{n-1}\ldots T_0 u_0, \\ &= T_{n-1}\ldots T_0(x_0 - y_0) + n\varepsilon T_{n-1}\ldots T_0 u_0,\ n \geq 1. \end{aligned}$$

It follows

$$\begin{aligned} \|x_n - y_n\| &= \|T_{n-1}\ldots T_0(x_0 - y_0) + \varepsilon n T_{n-1}\ldots T_0 u_0\| \\ &\geq \left| \|T_{n-1}\ldots T_0(x_0 - y_0)\| - \varepsilon n \|T_{n-1}\ldots T_0 u_0\| \right|,\ n \geq 1. \end{aligned}$$

The sequence $(T_{n-1}\ldots T_0(x_0 - y_0))_{n\geq 1}$ is bounded, since

$$\|T_{n-1}\ldots T_0(x_0 - y_0)\| \leq \|T_{n-1}\| \cdots \|T_0\| \|x_0 - y_0\| = \|x_0 - y_0\|,\ n \geq 1,$$

therefore $\lim_{m\to\infty} n\|\mathrm{x_m} - \mathrm{y_m}\| = +\infty$. □

Similar results can be obtained when we replace $(x_n)_{n\geq 0}$, $(T_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$ by $X_n := (x_1(n), x_2(n), \ldots, x_p(n))^T \in X^p$, $A_n \in \mathbb{K}^{p\times p}$ and $B_n \in X^p$, respectively. In this way we get Ulam stability results for systems of linear difference equations (see [30]). Note that on X^p, the following norm $\|Y\|_\infty := \max_{1\leq i\leq p} \|y_i\|$ $(Y = (y_1, y_2, \ldots, y_p)^T)$ is considered, alongside the matrix norm $\|A\|_\infty = \max_{1\leq i\leq p} \sum_{j=1}^{p} |a_{ij}|$ of $A \in \mathbb{K}^{p\times p}$ (which is simply the maximum absolute row sum of the matrix), induced by the vector norm $\|\cdot\|_\infty$ on $\mathbb{K}^p$. Also, one can easily verify that $\|AY\|_\infty \leq \|A\|_\infty \|Y\|_\infty$ and $\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty$, for any $A, B \in \mathbb{K}^{p\times p}$ and $Y \in X^p$. Finally, it is worth mentioning here that one can replace the above norms with some submultiplicative ones in order to obtain similar stability results for the equation

$$X_{n+1} = A_n X_n + B_n,\ n \geq 0. \tag{2.14}$$

Corollary 2.6. *Suppose that $(A_n)_{n\geq 0}$ is a sequence of invertible matrices in $\mathbb{K}^{p\times p}$ with*

$$\limsup \|A_n^{-1}\|_\infty < 1. \tag{2.15}$$

Then for every $\varepsilon > 0$ and every sequence $(X_n)_{n\geq 0}$ in X^p with

$$\|X_{n+1} - A_n X_n - B_n\|_\infty \leq \varepsilon,\ n \in \mathbb{N}, \tag{2.16}$$

there exists a unique sequence $(Y_n)_{n\geq 0}$ in X^p such that

$$Y_{n+1} = A_n Y_n + B_n,\ n \in \mathbb{N} \tag{2.17}$$

and

$$\|X_n - Y_n\|_\infty \leq \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \|A_n^{-1}\|_\infty \cdots \|A_{n+k}^{-1}\|_\infty \varepsilon,\ n \in \mathbb{N}. \tag{2.18}$$

Corollary 2.7. *Suppose that* $(A_n)_{n\geq 0}$ *is a sequence of nonzero matrices in* $\mathbb{K}^{p\times p}$ *with*

$$\liminf \frac{1}{\|A_n\|_\infty} > 1. \tag{2.19}$$

Then, there exists a constant $L \geq 0$ *such that for every* $\varepsilon > 0$ *and every sequence* $(X_n)_n \geq 0$ *in* X^p *satisfying*

$$\|X_{n+1} - A_nX_n - B_n\|_\infty \leq \varepsilon,\ n \in \mathbb{N},$$

there exists a sequence $(Y_n)_{n\geq 0}$ *in* X^p *with the property*

$$Y_{n+1} = A_nY_n + B_n,\ n \in \mathbb{N} \tag{2.20}$$

such that

$$\|X_n - Y_n\|_\infty \leq L\varepsilon,\ n \geq 1. \tag{2.21}$$

The next nonstability result for Eq. (2.14) is a simple consequence of Theorem 2.5.

Corollary 2.8. *Suppose that* $(A_n)_{n\geq 0}$ *is a sequence of matrices in* $\mathbb{K}^{p\times p}$ *such that* $\|A_n\|_\infty = 1$*, for all* $n \in \mathbb{N}$ *and there exists* $U_0 \in X^p$*,* $\|U_0\|_\infty = 1$ *such that*

$$\lim_{n\to\infty} n\|A_{n-1}\dots A_0U_0\|_\infty = +\infty. \tag{2.22}$$

Then for every $\varepsilon > 0$ *there exists a sequence* $(X_n)_n \geq 0$ *in* X^p *satisfying*

$$\|X_{n+1} - A_nX_n - B_n\|_\infty \leq \varepsilon,\ n \in \mathbb{N},$$

such that for every sequence $(Y_n)_{n\geq 0}$ *given by the recurrence*

$$Y_{n+1} = A_nY_n + B_n,\ n \in \mathbb{N},\ y_0 \in X^p,$$

we have

$$\sup_{n\in\mathbb{N}} \|X_n - Y_n\|_\infty = +\infty,$$

i.e., Eq. (2.14) *is not Ulam stable.*

3. The Ulam Stability of a *p*-Order Linear Difference Equation with Variable Coefficients

In the sequel, we will investigate the Ulam stability of the following p-order linear recurrence with variable coefficients

$$x_{n+p} = a_{p-1}(n)x_{n+p-1} + \dots + a_0(n)x_n + b_n, \tag{3.1}$$

where $(a_k(n))_{n\geq 0}$, $0 \leq k \leq p-1$ are sequences in K and $(b_n)_{n\geq 0}$ is a sequence in X. If the recurrence has constant coefficients, we have a characterization of its Ulam stability. Namely, the equation is Ulam stable if and only if the characteristic equation has no roots on the unit circle (see [8]). Moreover, for $p = 1, 2, 3$, the best Ulam constant was obtained. But for equations with variable coefficients, there are few results on Ulam stability (see e.g., [23,30]). Let us remark that Eq. (3.1) can be rewritten as

$$X_{n+1} = A_nX_n + B_n,\ n \in \mathbb{N},$$

where, for some $k \in (0,\infty)$,

$$A_n = \begin{pmatrix} a_{p-1}(n) & ka_{p-2}(n) & \dots & k^{p-2}a_1(n) & k^{p-1}a_0(n) \\ \frac{1}{k} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{k} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{k} & 0 \end{pmatrix} \in \mathbb{K}^{p\times p},$$

$$X_n = (k^{p-1}x_{n+p-1}, \dots, kx_{n+1}, x_n)^T \in X^p \tag{3.2}$$

and

$$B_n = (k^{p-1}b_n, 0, \dots, 0)^T \in X^p.$$

Remark that the usual matriceal form for Eq. (3.1) is obtained for $k = 1$ (see [30]). The form considered in this paper for arbitrary $k > 0$ is more convenient to obtain good conditions under which the stability of Eq. (3.1) holds.

Suppose that $a_0(n) \neq 0,\ n \in \mathbb{N}$, and let

$$e_n = \frac{1}{k^{p-1}|a_0(n)|} + \frac{|a_{p-1}(n)|}{k^{p-2}|a_0(n)|} + \cdots + \frac{|a_1(n)|}{|a_0(n)|}$$

and

$$f_n = |a_{p-1}(n)| + \cdots + k^{p-1}|a_0(n)|.$$

Corollary 3.1. *Suppose that there exists $k \in (0,1)$ such that*

$$\limsup e_n < 1.$$

Then for every $\varepsilon > 0$ and every sequence $(x_n)_{n\geq 0}$ in X satisfying

$$\|x_{n+p} - (a_{p-1}(n)x_{n+p-1} + \cdots + a_0(n)x_n + b_n)\| \leq \varepsilon,\ n \in \mathbb{N},$$

there exists a unique sequence $(y_n)_{n\geq 0}$ in X such that

$$y_{n+p} = a_{p-1}(n)y_{n+p-1} + \cdots + a_0(n)y_n + b_n,\ n \in \mathbb{N}$$

and

$$\|x_n - y_n\| \leq L\varepsilon,\ n \geq 0,$$

where

$$L = \sup_{n\in\mathbb{N}} \sum_{j=0}^{\infty} e_n e_{n+1} \dots e_{n+j}.$$

Proof. Consider the induced submultiplicative matrix norm $\|\cdot\|_\infty$, A_n, B_n and X_n as in relation (3.2) and observe that, since

$$A_n^{-1} = \begin{pmatrix} 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & k \\ \frac{1}{k^{p-1}a_0(n)} & -\frac{a_{p-1}(n)}{k^{p-2}a_0(n)} & -\frac{a_{p-2}(n)}{k^{p-3}a_0(n)} & \cdots & -\frac{a_1(n)}{a_0(n)} \end{pmatrix},$$

the property $\limsup \|A_n^{-1}\|_\infty < 1$ is equivalent to $\limsup e_n < 1$. Further, take an arbitrary $\varepsilon > 0$ and consider $\varepsilon_1 := k^{p-1}\varepsilon$. Then

$$\begin{aligned}\|X_{n+1} - A_nX_n - B_n\|_\infty &= \|(k^{p-1}(x_{n+p} - a_{p-1}(n)x_{n+p-1} - \cdots - b_n), \ldots, 0)^T\|_\infty \\ &= k^{p-1}\|x_{n+p} - (a_{p-1}(n)x_{n+p-1} + \cdots + a_0(n)x_n + b_n)\| \\ &\le \varepsilon_1,\end{aligned}$$

i.e.,

$$\|x_{n+p} - (a_{p-1}(n)x_{n+p-1} + \cdots + a_0(n)x_n + b_n)\| \le \varepsilon.$$

Then, in view of Corollary 2.6, there exists a unique sequence $Y_n \in X^p$, $Y_{n+1} = A_nY_n + B_n$, such that

$$\|X_n - Y_n\|_\infty \le L\varepsilon_1 \le L\varepsilon$$

with $L = \sup_{n\in\mathbb{N}} \sum_{j=0}^{\infty} e_n e_{n+1} \ldots e_{n+j}$. Finally, if we take $y_n := p_1(Y_n)$, where $p_1 : X^p \to X$ is given by $p_1(z_1, z_2, \ldots, z_p) = z_p$, one can easily check that y_n is a solution of (3.1) and that $\|x_n - y_n\| \le L\varepsilon$, $n \in \mathbb{N}$. □

Remark 3.2. In particular, if $e_n = k \in (0, 1)$ for every $n \in \mathbb{N}$, then $L = \frac{1}{1-k}$ in Corollary 3.1.

Example 3.3. Consider the recurrence

$$x_{n+2} = \frac{3n}{2n+1}x_{n+1} + \frac{7n+9}{n+1}x_n + b_n. \tag{3.3}$$

Then for every $\varepsilon > 0$ and every sequence $(x_n)_{n\ge0}$ in X satisfying

$$\|x_{n+2} - \frac{3n}{2n+1}x_{n+1} - \frac{7n+9}{n+1}x_n - b_n\| \le \varepsilon,\ n \in \mathbb{N}$$

there exists a solution $(y_n)_{n\ge0}$ of (3.3) such that

$$\|x_n - y_n\| \le 2\varepsilon,\ n \in \mathbb{N}.$$

Proof. If we choose $k = \frac{1}{2}$, then

$$e_n = \max\left\{\frac{1}{2}, \frac{2(n+1)}{7n+9} + \frac{3n(n+1)}{(2n+1)(7n+9)}\right\},$$

which implies that $e_n = \frac{1}{2}$, for every $n \in \mathbb{N}$. Further, taking into account Remark 3.2, we get the desired conclusion, i.e.

$$\|x_n - y_n\| \le 2\varepsilon,\ n \in \mathbb{N}.$$

□

Corollary 3.4. *Suppose that there exists $k > 1$ such that*

$$\liminf \frac{1}{f_n} > 1.$$

Then there exists a positive constant L such that for every $\varepsilon > 0$ and every sequence $(x_n)_{n\ge0}$ in X satisfying

$$\|x_{n+p} - (a_{p-1}(n)x_{n+p-1} + \cdots + a_0(n)x_n + b_n)\| \le \varepsilon,\ n \in \mathbb{N}$$

there exists a sequence $(y_n)_{n\ge0}$ in X satisfying (3.1) such that

$$\|x_n - y_n\| \le L\varepsilon.$$

Proof. Consider the induced submultiplicative matrix norm $\|\cdot\|_\infty$ and A_n, B_n and X_n given by (3.2) with k replaced by $k_1 := \frac{1}{k}$. Observe also that the property $\liminf \frac{1}{f_n} > 1$ is equivalent to the condition $\liminf \frac{1}{\|A_n\|_\infty} > 1$. Then applying Corollary 2.7 we get the desired conclusion. □

4. Other Applications

Consider first the Volterra operator, which is a bounded linear operator on the space $L^2[0,1]$ of complex-valued square-integrable functions on the interval [0,1]. The Volterra operator V is defined for a function $f \in L^2[0,1]$ by

$$V(f)(t) = \int_0^t f(s)\,\mathrm{d}s,\ t \in [0,1].$$

It is worth mentioning here that V is a quasinilpotent operator (that is, the spectral radius $\rho(V)$, is zero), but it is not nilpotent and the operator norm of V is exactly $\|V\| = \frac{2}{\pi}$ (see [16]).

Taking into account Remark 2.4 we get the following stability result for the linear difference equation

$$x_{n+1} = Vx_n + a_n,\ x_0 \in L^2[0,1],\ n \in \mathbb{N},$$

where $(a_n)_{n\geq 0}$ is a sequence in $L^2[0,1]$.

Corollary 4.1. *For every $\varepsilon > 0$ and every sequence $(x_n)_n \geq 0$ in $L^2[0,1]$ satisfying*

$$\|x_{n+1} - Vx_n - a_n\| \leq \varepsilon,\ n \in \mathbb{N},$$

there exists a sequence $(y_n)_{n\geq 0}$ in $L^2[0,1]$ such that

$$y_{n+1} = Vy_n + a_n,\ y_0 \in L^2[0,1],\ n \in \mathbb{N}$$

and

$$\|x_n - y_n\| \leq \frac{\pi\varepsilon}{\pi - 2},\ n \geq 0. \tag{4.1}$$

Further, given a domain Ω in R^m, we consider a sequence of Hilbert–Schmidt kernels, that is, a sequence of functions $(k_n)_{n\geq 0}$, $k_n : \Omega \times \Omega \to \mathbb{C}$ with

$$\int_\Omega \int_\Omega |k_n(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y < \infty,\ n \geq 0,$$

which means that the $L^2(\Omega\times\Omega;\mathbb{C})$ norm of each k_n is finite. Further, we associated the following sequence of Hilbert–Schmidt integral operators $(K_n)_{n\geq 0}$, $K_n : L^2(\Omega;\mathbb{C}) \to L^2(\Omega;\mathbb{C})$ defined by

$$(K_n u)(x) = \int_\Omega k_n(x,y)u(y)\,\mathrm{d}y,\ u \in L^2(\Omega;\mathbb{C}).$$

Then K_n is a Hilbert–Schmidt operator for every $n \in \mathbb{N}$ and its Hilbert–Schmidt norm is

$$\|K_n\|_{HS} = \|k_n\|_{L^2}.$$

Hilbert–Schmidt integral operators are continuous (and hence bounded) and compact (see [28]).

Taking into account Theorem 2.3, we get the following stability result for the linear difference equation

$$x_{n+1} = K_n x_n + a_n,\ x_0 \in L^2(\Omega;\mathbb{C}),\ n \in \mathbb{N},$$

where $(a_n)_{n\geq 0}$ is a sequence in $L^2(\Omega;\mathbb{C})$.

Corollary 4.2. *Suppose that $(K_n)_{n\geq 0}$ is a sequence of nonzero Hilbert–Schmidt integral operators and*

$$\liminf \frac{1}{\|k_n\|_{L^2}} > 1.$$

Then, there exists a constant $L \geq 0$, such that for every $\varepsilon > 0$ and every sequence $(x_n)_n \geq 0$ in $L^2(\Omega;\mathbb{C})$ satisfying

$$\|x_{n+1} - K_n x_n - a_n\| \leq \varepsilon,\ n \in \mathbb{N},$$

there exists a sequence $(y_n)_{n\geq 0}$ in $L^2(\Omega;\mathbb{C})$ with the property

$$y_{n+1} = K_n y_n + a_n,\ y_0 \in L^2(\Omega;\mathbb{C}),\ n \in \mathbb{N}$$

and

$$\|x_n - y_n\| \leq L\varepsilon,\ n \geq 0. \tag{4.2}$$

Remark 4.3. An example of nonstable equation is given below. Take now the Bernstein operator (see [1]) which assigns to each continuous, real-valued function $f \in C[0,1]$ (where $C[0,1]$ is endowed with the supremum norm) the polynomial function $B_n f$ defined by

$$B_n f(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right),\ n \in \mathbb{N}.$$

It is well known that B_n preserves affine functions.

Further, we consider the sequence $(x_n)_{n\geq 0}$ defined by the recurrence

$$x_{n+1} = B_n x_n + a_n,\ n \in \mathbb{N}, \tag{4.3}$$

and we show that Eq. (4.3) is not Ulam stable. Indeed, choosing $u_0(t) = 1,\ t \in [0,1]$, one can easily observe that $\|u_0\| = 1$ and $\|B_{n-1}\dots B_0 u_0\| = 1,\ \forall n \geq 1$. Using now Theorem 2.5, one gets the desired conclusion.

Another example concerns the Ulam stability of Eq. (1.1) for operators T_n, $n \geq 0$, acting on a finite dimensional Banach space X.

Suppose in what follows that a vector norm $\|\cdot\|$ on $X = \mathbb{K}^p$ is given. Then any square matrix A of order p with entries in $\mathbb{K}$ induces a linear operator $T : \mathbb{K}^p \to \mathbb{K}^p$, $Tx = Ax$, with respect to the standard basis, and one defines the corresponding induced norm on the space $\mathbb{K}^{p\times p}$ of all $p \times p$ matrices as follows:

$$\begin{aligned}\|A\| &= \sup\{\|Ax\| : x \in \mathbb{K}^p \text{ with } \|x\| = 1\}\\ &= \left\{\frac{\|Ax\|}{\|x\|} : x \in \mathbb{K}^p \text{ with } x \neq 0\right\}.\end{aligned}$$

If we consider $\mathbb{K}^p$ to be endowed with the Euclidean norm (i.e. $\|x\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$, where $x = (x_1, \ldots, x_n) \in \mathbb{K}^p$), then the induced matrix norm is the spectral norm. Let us recall also here that the spectral norm of a matrix A is the largest singular value of A, i.e., the square root of the largest eigenvalue of the matrix A^*A, where A^* denotes the conjugate transpose of A.

Take now a sequence $(A_n)_{n\geq 0}$ of $p \times p$ matrices, $T_n : \mathbb{K}^p \to \mathbb{K}^p$, $T_n x = A_n x$ and denote by $\lambda_1^{(n)}, \ldots, \lambda_p^{(n)}$ and $\Lambda_1^{(n)}, \ldots, \Lambda_p^{(n)}$ the eigenvalues of A_n and $A_n^* A_n$, respectively. If we suppose that

$$|\lambda_1^{(n)}| \leq |\lambda_2^{(n)}| \leq \cdots \leq |\lambda_p^{(n)}| \text{ and } \Lambda_1^{(n)} \leq \Lambda_2^{(n)} \leq \cdots \leq \Lambda_p^{(n)},$$

then (see [19]) $\|T_n\| = \sqrt{\Lambda_p^{(n)}}$ and, if A_n are additionally invertible matrices, then $\|T_n^{-1}\| = \frac{1}{\sqrt{\Lambda_1^{(n)}}}$. Moreover, if A_n are normal and invertible matrices, then $\|T_m\| = |\lambda_p^{(m)}|$ and $\|T_m^{-1}\| = \frac{1}{|\lambda_1^{(m)}|}$.

The following results on Ulam stability follow easily, if we take also into account Theorems 2.2 and 2.3.

Theorem 4.4. *Let $(a_n)_{n\geq 0}$ be a sequence in X and suppose that A_n are invertible matrices and that there exist $q > 1$ and $n_0 \in \mathbb{N}$ such that*

$$\Lambda_1^{(n)} \geq q > 1, \;\; n \geq n_0. \tag{4.4}$$

Then for every $\varepsilon > 0$ and every sequence $(x_n)_{n\geq 0}$ in $\mathbb{K}^p$ with

$$\|x_{n+1} - A_n x_n - a_n\| \leq \varepsilon, \;\; n \in \mathbb{N},$$

there exists a unique sequence $(y_n)_{n\geq 0}$ in $\mathbb{K}^p$ such that

$$y_{n+1} = A_n y_n + a_n, \;\; n \in \mathbb{N}$$

and

$$\|x_n - y_n\| \leq \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{\Lambda_1^{(n)} \ldots \Lambda_1^{(n+k)}}} \varepsilon, \;\; n \in \mathbb{N}.$$

Theorem 4.5. *Suppose that A_n are nonzero matrices and that there exist $q < 1$ and $n_0 \in \mathbb{N}$ such that*

$$\Lambda_p^{(n)} \leq q < 1, \;\; n \geq n_0. \tag{4.5}$$

Then, there exists $L \geq 0$ such that for every $\varepsilon > 0$ and every sequence $(x_n)_n \geq 0$ in $\mathbb{K}^p$ satisfying

$$\|x_{n+1} - A_n x_n - a_n\| \leq \varepsilon, \;\; n \in \mathbb{N},$$

there exists a sequence $(y_n)_{n\geq 0}$ in $\mathbb{K}^p$ with the property

$$y_{n+1} = A_n y_n + a_n, \;\; n \in \mathbb{N}$$

and

$$\|x_n - y_n\| \leq L\varepsilon, \;\; n \geq 1.$$

Remark 4.6. In case A_n are normal and invertible matrices, the above results hold with $|\lambda_1^{(n)}|$ and $|\lambda_p^{(n)}|$ instead of $\sqrt{\Lambda_1^{(n)}}$ and $\sqrt{\Lambda_p^{(n)}}$, respectively.

Remark 4.7. If we consider now $\mathbb{K}^p$ to be endowed with the Taxicab norm (i.e. $\|x\|_1 = \sum_{i=1}^p |x_i|$, where $x = (x_1, \ldots, x_p) \in \mathbb{K}^p$) or with the maximum norm (i.e. $\|x\|_\infty = \max\limits_{1\le i\le p} |x_i|$) then the induced matrix norms are $\|A\|_1 = \max_{1\le j\le p} \sum_{i=1}^p |a_{ij}|$ and $\|A\|_\infty = \max_{1\le i\le p} \sum_{j=1}^p |a_{ij}|$, respectively. Further, take $A_n = (a_{ij}^{(n)}) \in \mathbb{K}^{p\times p}$, for all $n \in \mathbb{N}$. Then the condition (4.5) above can be replaced by

$$\max_{1\le j\le p} \sum_{i=1}^p |a_{ij}^{(n)}| \le q < 1, \text{ for all } n \ge n_0,$$

or by

$$\max_{1\le i\le p} \sum_{j=1}^p |a_{ij}^{(n)}| \le q < 1, \text{ for all } n \ge n_0,$$

in order to obtain the desired conclusion, since the following inequalities

$$\frac{1}{\sqrt{p}}\|A\|_1 \le \|A\|_2 \le \sqrt{p}\|A\|_1$$

and

$$\frac{1}{\sqrt{p}}\|A\|_\infty \le \|A\|_2 \le \sqrt{p}\|A\|_\infty$$

hold always true for any square matrix A of order p. It is worth mentioning also that neither the following implications

$$\liminf \frac{1}{\|A\|_1} > 1 \Longrightarrow \liminf \frac{1}{\|A\|_2} > 1$$

and

$$\liminf \frac{1}{\|A\|_\infty} > 1 \Longrightarrow \liminf \frac{1}{\|A\|_2} > 1$$

nor the opposite ones seam to be valid.

The results obtained in this section generalize the results obtained in [5,12].

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