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Blow-Up Phenomena for a Class of Parabolic or Pseudo-parabolic Equation with Nonlocal Source

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Abstract. In this paper, we consider a class of parabolic or pseudo-parabolic equation with nonlocal source term:

$$u_t - \nu\triangle u_t - \operatorname{div}(\rho(|\nabla u|)^2\nabla u) = u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)dy,$$

where $\nu \geq 0$ and $p > 0$. Using some differential inequality techniques, we prove that blow-up cannot occur provided that $q > p$, also, we obtain some finite-time blow-up results and the lifespan of the blow-up solution under some different suitable assumptions on the initial energy. In particular, we prove finite-time blow-up of the solution for the initial data at arbitrary energy level. Furthermore, the lower bound for the blow-up time is determined if blow-up does occur.

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Keywords. Nonblow-up, blow-up, upper bound, lower bound.

1. Introduction

In this paper, we deal with the following the initial boundary value problem of a class of parabolic or pseudo-parabolic equation with nonlocal source term:

$$\begin{cases} u_t - \nu\triangle u_t - \operatorname{div}(\rho(|\nabla u|)^2\nabla u) & \\ \qquad = u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)dy, & (x,t)\in\Omega\times(0,T) \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,t) = u_0(x) \geq 0, & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\nu \geq 0$, $p > 0$, $q > 0$ and $T \in (0,\infty]$ is the maximal existence time of the solution, $k(x,y)$ is an integrable, real-valued function satisfying:

$$k(x,y) = k(y,x), \quad \int_\Omega\int_\Omega k^2(x,y)dxdy < +\infty,$$

$$\int_\Omega \int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy > 0.$$

This type of equations describes a variety of important physical and biological phenomena, such as the analysis of heat conduction in materials with memory, the aggregation of population [12], and so on (see [1] and the references therein). In population dynamics theory, the nonlocal term indicates that the individuals are competing not only with others at their own point in space but also with individual at other points in the domain [11,12]. If $\nu = 0$, $\rho = 1$, Eq. (1.1) reduces to the following semilinear parabolic equation:

$$u_t - \triangle u = f(u). \tag{1.2}$$

The global existence, asymptotic behavior, and finite-time blow-up for the solutions to (1.2)(especially $f(u) = |u|^{p-1}u$) have been studied by many researchers, see [2,3,8]and the references therein. Recently, Eq. (1.2) with the nonlocal source $f(u) = \left(\frac{1}{|x|^{n-2}} * |u|^p\right)|u|^{p-2}u$ was considered in [5,7].

If $\nu > 0$ (for the sake of simplicity, $\nu = 1$ in this paper), $\rho = 1$, Eq. (1.1) reduces to the following semilinear pseudo-parabolic equation:

$$u_t - \triangle u_t - \triangle u = f(u). \tag{1.3}$$

There are many works about Eq. (1.3) with $f(u)$ being polynomial, such as the existence and uniqueness in [16], blow-up in [10,17–19], asymptotic behavior in [18], and so on.

Recently, Yang and Liang [20] considered a special case of (1.1), that is:

$$u_t - \triangle u_t - \triangle u = u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)dy. \tag{1.4}$$

They proved the finite-time blow-up result provided the initial energy is negative, as well as a nonblow-up criterion. We also mention the paper [1], where Di and Shang considered a four order pseudo-parabolic equation (i.e., an extra term $\triangle^2 u$ in RHS of (1.4)) and obtained a blow-up result of the solutions under suitable initial energy.

For the gentle case ρ:

$$u_t - \nu\triangle u_t - \operatorname{div}(\rho(|\nabla u|)^2\nabla u) = f(u), \tag{1.5}$$

where $f(u) \approx u^p$, the initial boundary problem of (1.5) was investigated in [14,15]. The blow-up results and the lifespan provided that the initial energy is negative as well as the nonblow-up criterion were established by Payne et al. [14] (parabolic case $\nu = 0$), Liu et al [6], and Peng et al. [15](pseudo-parabolic case $\nu = 1$).

Recently, Long and Chen [9] considered the following pseudo-parabolic equation with nonlocal source:

$$u_t - \triangle u_t - \operatorname{div}(|\nabla u|^{2q}\nabla u) = u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)dy. \tag{1.6}$$

Under $q \to 0$ and $|\nabla u| \neq 0$, the limit equation of (1.6) is (1.4). When (i) $q < p$ and $J(u_0) \leq 0$ or (ii) $q = p$ and $J(u_0) < 0$, they proved that the solutions blow up in finite time and the upper and the lower bound.

To our knowledge, no results have been obtained about finite-time blow-up and the lifespan for the solution of the gentle problem (1.1), especially, when the solution with high energy level. Moreover, there is little information about (1.1) under parabolic case ($\nu = 0$) with nonlocal source term. The aim of this paper is to present a comprehensive study for the finite blow-up and nonblow-up criterion of problem (1.1).

2. Preliminaries

Throughout this paper, the Banach spaces $L^p = L^p(\Omega)$ and $W_0^{1,p} = W_0^{1,p}(\Omega)$ are endowed with the norms by $\|\cdot\|_p = \big(\int_\Omega |\cdot|^p dx\big)^{\frac{1}{p}}$, and $\|\cdot\|_{W_0^{1,p}} = \big(\int_\Omega (|\cdot|^p + |\nabla\cdot|^p)dx\big)^{\frac{1}{p}}$ as usual. We assume that ρ is a positive C^1 function satisfying:

$$\rho(s) + 2s\rho'(s) \geq 0, \quad s > 0, \tag{2.1}$$

so that $\mathrm{div}(\rho(|\nabla\cdot|)^2\nabla\cdot)$ is elliptic. We also claim that ρ satisfies the condition:

$$\rho(s) \geq b_1 + b_2 s^q, \quad s > 0, \tag{2.2}$$

where $q > 0$ and b_1, b_2 are positive constants. Furthermore, we assume that u_0 satisfies the compatibility condition $u_0(x) = 0$ on $\partial\Omega$.

We first state the local existence theorem for the weak solution to problem (1.1) as follow. See similar result in [1,9,15].

Theorem 2.1. *Assume* $0 < p \leq \frac{2}{n-2}$, $u_0 \in W_0^{1,2q+2}(\Omega)$, *and* (2.1) *hold, there exists a* $T > 0$, *such that the problem* (1.1) *has a unique local solution* $u \in L^\infty(0,T;W_0^{1,2q+2}(\Omega))$ *with* $u_t \in L^2(0,T;H_0^1(\Omega))$ *(resp.* $u_t \in L^2(0,T;L^2(\Omega))$ *) for the case of* $\nu = 1$ *(resp.* $\nu = 0$*),*

(i) for a.e. $t \in [0,T]$, *the following identity:*

$$\langle u_t, v\rangle + \nu\langle\nabla u_t, \nabla v\rangle + \langle\rho(|\nabla u|^2\nabla u, \nabla v\rangle = \langle u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)dy, v\rangle \tag{2.3}$$

holds for all $v \in W_0^{1,2q+2}(\Omega)$.

(ii) $u(0) = u_0$.

Before processing our main results, we will make some calculations on the nonlocal term:

$$\begin{aligned} F(u) &= \int_0^1 (f(su), u)ds \\ &= \int_0^1 \int_\Omega s^p u^p(x,t)\Big(\int_\Omega k(x,y)s^{p+1}u^{p+1}(y,t)dy\Big)u(x,t)dxds \\ &= \frac{1}{2p+2}\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy. \end{aligned} \tag{2.4}$$

Differentiating (2.4) with respect to t, using the symmetry of $k(x, y)$, we have:

$$\frac{d}{dt}F(u) = \frac{1}{2p+2}\frac{d}{dt}\int\limits_{\Omega}\int\limits_{\Omega} k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy$$
$$= \int\limits_{\Omega}\int\limits_{\Omega} k(x,y)u^{p}(x,t)u^{p+1}(y,t)u_t(x,t)dxdy. \tag{2.5}$$

Now, we give some useful inequalities which will be used throughout the paper. Let λ_1 be the principal eigenvalue of the problem:

$$\begin{aligned} \triangle w + \lambda w = 0 \quad &\text{in} \quad \Omega, \\ w = 0 \quad &\text{on} \quad \partial\Omega, \\ w > 0 \quad &\text{in} \quad \Omega; \end{aligned} \tag{2.6}$$

then we have:

$$\lambda_1\|u\|_2^2 \leq \|\nabla u\|_2^2, \quad \|\nabla u\|_2^2 \geq \frac{\lambda_1}{1+\lambda_1}\|u\|_{H_0^1}^2, \quad u \in H_0^1(\Omega). \tag{2.7}$$

Using of Hölder's inequality and (2.7) , we get:

$$\|\nabla u(t)\|_{2q+2}^{2q+2} \geq |\Omega|^{-q}\Big(\frac{\lambda_1}{1+\lambda_1}\Big)^{q+1}\|u(t)\|_{H_0^1}^{2q+2}, \tag{2.8}$$

where $|\Omega|$ denotes the volume of Ω.

3. Nonblow-Up Case

In this section, we prove that the solution $u(t)$ of problem (1.1) cannot blow-up at any finite time provided that $q > p > 0$.

We define the auxiliary function:

$$\varphi(t) = \varphi_\nu(t) = \|u(t)\|_2^2 + \nu\|\nabla u(t)\|_2^2, \quad \nu = 0, 1. \tag{3.1}$$

Differentiating (3.1) with respect to t, and using (1.1), (2.2) and (2.3), we obtain:

$$\begin{aligned} \varphi'(t) &= 2\int_\Omega u u_t dx + 2\nu\int_\Omega \nabla u\cdot\nabla u_t dx \\ &= -2\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 dx + 2\int_\Omega \\ &\quad \int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy \\ &\leq -2b_1\int_\Omega |\nabla u|^2 dx - 2b_2\int_\Omega |\nabla u|^{2q+2}dx + 2\int_\Omega \\ &\quad \int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy. \end{aligned} \tag{3.2}$$

Using the Hölder and Sobolev inequalities, we estimate the last term in the right-hand side of (3.2) as:

$$\begin{aligned}
&\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy\\
&\le \int_\Omega u^{p+1}(x,t)\left(\int_\Omega k^2(x,y)dy\right)^{\frac{1}{2}}\left(\int_\Omega u^{2p+2}(y,t)dy\right)^{\frac{1}{2}}dx\\
&= \|u\|_{2p+2}^{p+1}\int_\Omega u^{p+1}(x,t)\left(\int_\Omega k^2(x,y)dy\right)^{\frac{1}{2}}dx\\
&\le \|u\|_{2p+2}^{2p+2}\left(\int_\Omega\int_\Omega k^2(x,y)dydx\right)^{\frac{1}{2}}\\
&\le \kappa\|u\|_{2p+2}^{2p+2}\le \kappa C_*^{2p+2}\|u\|_{H_0^1}^{2p+2}, \qquad (3.3)
\end{aligned}$$

where $\kappa=\left(\int_\Omega\int_\Omega k^2(x,y)dydx\right)^{\frac{1}{2}}<\infty$, and C_* is the best embedding constant: $\|u\|_{2p+2}\le C_*\|u\|_{H_0^1}$.

Now, we will process our calculations in two cases: $\nu=1$ and $\nu=0$ respectively.

Pseudo-parabolic case: $\nu=1$. It follows from (2.7), (2.8),(3.2), and (3.3) that:

$$\varphi_1'(t)\le -A_1\varphi_1(t)-B_1[\varphi_1(t)]^{q+1}+C_1[\varphi_1(t)]^{p+1}, \qquad (3.4)$$

where:

$$A_1=\frac{2b_1\lambda_1}{1+\lambda_1},\quad B_1=2b_2|\Omega|^{-q}\left(\frac{\lambda_1}{1+\lambda_1}\right)^{q+1},\quad C_1=2\kappa C_*^{2p+2}.$$

We conclude from (3.4) and $q>p>0$ that the solution cannot blow-up in finite time. In fact, Let $h_1(s)=-A_1s-B_1s^{q+1}+C_1s^{p+1}$, and then, $h_1(0)=0$ and $\lim_{s\to+\infty}h_(s)=-\infty$, since $q>p$. By the continuity of $\varphi_1(t)$ and the properties of polynomials, we can deduce that (1) if $h_1(s)\le 0$ for all $s\ge 0$, then $\varphi_1(t)\le\varphi_1(0)$; (2) if there exists some finite time t_1, such that $h_1(t_1)>0$, we denote S_1 to be the largest positive root of $h_1(s)=0$, then $\varphi_1(t)$ could not be larger than the value S_1; otherwise, $\varphi_1'(t)$ would be negative which is impossible. Moreover:

$$\varphi_1(t)\le\max\{\varphi_1(0),S_1\}.$$

Parabolic case: $\nu=0$. Since

$$|\nabla u^{q+1}|^2=(q+1)^2u^{2q}|\nabla u|^2,$$

it follows from Hölder's inequality that:

$$\int_\Omega|\nabla u^{q+1}|^2dx\le(q+1)^2\left(\int_\Omega|\nabla u|^{2q+2}dx\right)^{\frac{1}{q+1}}\left(\int_\Omega u^{2q+2}dx\right)^{\frac{q}{q+1}}.$$

Letting $w=u^{q+1}$ in the Poincaré's inequality $\lambda_1\|w\|_2^2\le\|\nabla w\|_2^2$, we obtain that:

$$\int_\Omega u^{2q+2}dx\le\left[\frac{(q+1)^2}{\lambda_1}\right]^{q+1}\int_\Omega|\nabla u|^{2q+2}dx.$$

Using $q > p$ and Hölder's inequality again, we have:

$$\int_\Omega u^{2p+2}dx \le \left(\int_\Omega u^{2q+2}dx\right)^{\frac{p+1}{q+1}} |\Omega|^{\frac{q-p}{q+1}}.$$

and

$$\int_\Omega u^2 dx \le \left(\int_\Omega u^{2p+2}dx\right)^{\frac{1}{p+1}} |\Omega|^{\frac{p}{p+1}}.$$

It follows from (2.7),(3.2),(3.3) and the above inequalities that:

$$\begin{aligned}\varphi_0'(t) &\le -2b_1 \int_\Omega |\nabla u|^2 dx - 2b_2 \left[\frac{\lambda_1}{(q+1)^2}\right]^{q+1} \int_\Omega u^{2q+2}dx + 2\kappa \int_\Omega u^{2p+2}dx \\ &\le {}^{-}2b_1\lambda_1\varphi_0(t) - 2b_2\left[\frac{\lambda_1}{(q+1)^2}\right]^{q+1}\left(\int_\Omega u^{2p+2}dx\right) \\ &\quad \left(\int_\Omega u^{2p+2}dx\right)^{\frac{q-p}{p+1}} |\Omega|^{-\frac{q-p}{p+1}} + 2\kappa\int_\Omega u^{2p+2}dx \\ &\le -2b_1\lambda_1\varphi_0(t) - 2b_2\left[\frac{\lambda_1}{(q+1)^2}\right]^{q+1}\left(\int_\Omega u^{2p+2}dx\right) \\ &\quad \left(\int_\Omega u^2 dx\right)^{q-p} |\Omega|^{p-q} + 2\kappa\int_\Omega u^{2p+2}dx \\ &= -2b_1\lambda_1\varphi_0(t) + 2\int_\Omega u^{2p+2}dx\left(\kappa - b_2\left[\frac{\lambda_1}{(q+1)^2}\right]^{q+1}|\Omega|^{p-q}[\varphi_0(t)]^{q-p}\right).\end{aligned}$$

It follows from the above inequality that the solution $u(t)$ cannot blow-up in finite time. In fact, if $\varphi_0(t)$ were to be sufficiently large at some time t_0, then $\varphi_0'(t)$ would be negative, so that $\varphi_0(t)$ could not be larger than that value. Moreover:

$$\varphi_0(t) \le \max\left\{\varphi_0(0), \left[\frac{\kappa(q+1)^{2q+2}|\Omega|^{q-p}}{b_2\lambda_1^{q+1}}\right]^{\frac{1}{q-p}}\right\}.$$

We summarize the above discussions in the following theorem.

Theorem 3.1. *If* $0 < p < q$ *and* u *is the nonnegative solution of problem* (1.1), *then* u *cannot blow-up at finite time in* H_0^1*-norm for* $\nu = 1$ *(resp.* L^2*-norm for* $\nu = 0$*).*

4. Criterions of Blow-Up

In this section, we consider a specific class of problem (1.1), for which we can obtain the finite-time blow-up results provided that the initial energy satisfies different conditions. We also establish the lower and the upper bounds for the blow-up time. Let u be the nontrivial solution of:

$$\begin{cases} u_t - \nu\triangle u_t - b_1\triangle u - b_2\mathrm{div}(|\nabla u|^{2q}\nabla u) \\ \qquad = u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)dy, & (x,t)\in\Omega\times(0,T), \\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x)\geq 0, & x\in\Omega, \end{cases} \tag{4.1}$$

where b_1 and b_2 are positive constants and the parameters p and q satisfy the condition $0 \leq q \leq p$.

To obtain the blow-up results, we introduce the functions:

$$J(u(t)) = \frac{b_1}{2}\|\nabla u\|_2^2 + \frac{b_2}{2q+2}\|\nabla u\|_{2q+2}^{2q+2} - \frac{1}{2p+2}\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy \tag{4.2}$$

and

$$I(u(t)) = b_1\|\nabla u\|_2^2 + b_2\|\nabla u\|_{2q+2}^{2q+2} - \int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy,$$

where we have used (2.4).

Lemma 4.1. *Let u be the solution of the problem (4.1), and then, $J(u(t))$ is non-increasing function, that is, $\frac{d}{dt}J(u(t)) \leq 0$. Moreover, it holds that:*

$$J(u(t)) + \int_0^t (\|u_t\|^2 + \nu\|\nabla u_t\|^2)dt = J(u_0).$$

Proof. Similar to the proof of Lemma 2.1 in [1] and using (2.4) and (2.5), we can obtain the proof. □

4.1. Finite-Time Blow-Up for Nonpositive Initial Energy

In this subsection, we will establish the blow-up results for $J(u_0) \leq 0$ and the upper bounds for the maximal existence time T.

Theorem 4.1. *Let $0 \leq q \leq p$, u be the nonnegative solution of (4.1) with $J(u_0) < 0$, and then, u blows up in finite time T with:*

$$T \leq T_{11} = \frac{\|u_0\|_{H_0^1}^2}{-4p(p+1)J(u_0)}, \quad \textit{for } \nu = 1,$$

$$T \leq T_{10} = \frac{\|u_0\|_2^2}{-4p(p+1)J(u_0)}, \quad \textit{for } \nu = 0.$$

Proof. Given $\varphi(t)$ be the function defined in (3.1), since $0 \leq q \leq p$, we compute:

$$\begin{aligned} \varphi'(t) &= 2\int_\Omega uu_t dx + 2\nu\int_\Omega \nabla u\cdot\nabla u_t dx \\ &= -2b_1\|\nabla u\|_2^2 - 2b_2\|\nabla u\|_{2q+2}^{2q+2} + 2\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy \\ &\geq \psi(t), \end{aligned} \tag{4.3}$$

where:

$$\psi(t) = -4(p+1)J(u(t))$$

$$= -2(p+1)b_1\|\nabla u\|_2^2 - \frac{2(p+1)b_2}{q+1}\|\nabla u\|_{2q+2}^{2q+2}$$
$$+2\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy.$$

In view of Lemma 4.1, it holds that:

$$\psi'(t) = -4(p+1)\frac{d}{dt}J(u(t)) = 4(p+1)\int_\Omega(|u_t|^2+\nu|\nabla u_t|^2)dx \geq 0.$$

It follows from $J(u_0) < 0$ that $\psi(t) > 0$ for any $t \geq 0$. In view of Hölder's and Schwarz's inequalities and $\psi(t) > 0$, we conclude from the above inequalities that:

$$\begin{aligned}\varphi(t)\psi'(t) &= 4(p+1)\Big(\int_\Omega(|u|^2+\nu|\nabla u|^2)dx\Big)\left(\int_\Omega(|u_t|^2+\nu|\nabla u_t|^2)dx\right)\\ &\geq 4(p+1)\left(\int_\Omega(uu_t+\nu\nabla u\cdot\nabla u_t)dx\right)^2\\ &= (p+1)[\varphi'(t)]^2 \geq (p+1)\varphi'(t)\psi(t),\end{aligned}$$

which can be rewritten as:

$$\frac{\psi'(t)}{\psi(t)} \geq (p+1)\frac{\varphi'(t)}{\varphi(t)}. \tag{4.4}$$

Integrating (4.4) on $[0,t]$, noticing $\varphi'(t) \geq \psi(t)$, we have:

$$\frac{\psi(t)}{[\varphi(t)]^{p+1}} \geq \frac{\psi(0)}{[\varphi(0)]^{p+1}} \Rightarrow \frac{\varphi'(t)}{[\varphi(t)]^{p+1}} \geq \frac{\psi(0)}{[\varphi(0)]^{p+1}}. \tag{4.5}$$

Then, a further integration results in:

$$\frac{1}{\varphi^p(t)} \leq \frac{1}{\varphi^p(0)} - p\frac{\psi(0)}{[\varphi(0)]^{p+1}}t. \tag{4.6}$$

It is obvious that (4.6) cannot holds for all time t and u blows up at some finite T, i.e., $\lim_{t\to T^-}\varphi(t) = +\infty$, where:

$$T \leq T_1 = \frac{\varphi(0)}{p\psi(0)},$$

which implies the conclusions of this theorem for both $\nu = 1$ and $\nu = 0$ cases. □

Moreover, integrating the second inequality of (4.5) from t to T, we can obtain the following blow-up rate:

$$\varphi(t) \leq \left[\frac{p\psi(0)}{[\varphi(0)]^{p+1}}\right]^{-\frac{1}{p}}(T-t)^{-\frac{1}{p}},$$

that is:

$$\|u(t)\|_{H_0^1} \leq \left[\frac{-4p(p+1)J(u_0)}{[\|u_0\|_{H_0^1}^{2p+2}]}\right]^{-\frac{1}{2p}}(T-t)^{-\frac{1}{2p}}, \qquad \nu = 1;$$

$$\|u(t)\|_2 \leq \left[\frac{-4p(p+1)J(u_0)}{[\|u_0\|_2^{2p+2}]}\right]^{-\frac{1}{2p}} (T-t)^{-\frac{1}{2p}}, \qquad \nu = 0.$$

Now, we will give the blow-up result for the case $J(u_0) = 0$. Moreover, it is also valid for $J(u_0) \leq 0$ from the proof of the theorem.

Theorem 4.2. *Let* $0 < q < p$ *and* u *be the nonnegative solution of* (4.1) *with* $J(u_0) \leq 0$, *and then,* u *blows up in finite time* T *with:*

$$T \leq T_{21} = \int_{\|u_0\|_{H_0^1}^2}^{+\infty} \frac{d\eta}{A_2\eta + B_2\eta^{q+1}}, \quad \textit{for } \nu = 1,$$

$$T \leq T_{20} = \int_{\|u_0\|_2^2}^{+\infty} \frac{d\eta}{A_3\eta + B_3\eta^{q+1}}, \quad \textit{for } \nu = 0,$$

where the positive constants A_2, B_2 *and* A_3, B_3 *are defined in* (4.9) *and* (4.10), *respectively.*

Proof. Given $\varphi(t)$ be the function defined in (3.1), in view of(4.2) and Lemma 4.1, we have:

$$\begin{aligned} &\int_\Omega \int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy \\ &= 2(p+1)\bigg(\int_0^t (\|u_t\|_2^2 + \nu\|\nabla u_t\|_2^2)dt + \frac{b_1}{2}\|\nabla u\|_2^2 \\ &+ \frac{b_2}{2q+2}\|\nabla u\|_{2q+2}^{2q+2} - J(u_0)\bigg). \end{aligned} \tag{4.7}$$

Substituting (4.7) into $\varphi'(t)$ (see (4.3)), in view of $J(u_0) \leq 0$, we deduce that:

$$\varphi'(t) \geq 2pb_1\|\nabla u\|_2^2 + \frac{2(p-q)b_2}{q+1}\|\nabla u\|_{2q+2}^{2q+2}. \tag{4.8}$$

Pseudo-parabolic case: $\nu = 1$. It follows from (2.7) and (2.8), (4.8) reduces to:

$$\varphi_1'(t) \geq A_2\varphi_1(t) + B_2[\varphi_1(t)]^{q+1},$$

where:

$$A_2 = \frac{2pb_1\lambda_1}{1+\lambda_1}, \quad B_2 = \frac{2(q-p)b_2}{(q+1)|\Omega|^q}\left(\frac{\lambda_1}{1+\lambda_1}\right)^{q+1}. \tag{4.9}$$

On integrating the above inequality on $[0,t]$, we have:

$$t \leq \int_{\varphi_1(0)}^{\varphi_1(t)} \frac{d\eta}{A_2\eta + B_2\eta^{q+1}} \leq \int_{\varphi_1(0)}^{+\infty} \frac{d\eta}{A_2\eta + B_2\eta^{q+1}} < +\infty.$$

It follows from that the solution u blows up at some finite time in H_0^1-norm, since the above inequality cannot hold for all time t.

Parabolic case: $\nu = 0$. It follows from (2.7), (2.8) and Hölder's inequality , (4.8) reduces to:

$$\begin{aligned}\varphi_0'(t) &\geq 2pb_1\|\nabla u\|_2^2 + \frac{2(p-q)b_2}{(q+1)|\Omega|^q}\|\nabla u\|_2^{2q+2}\\ &\geq A_3\varphi_0(t) + B_3[\varphi_0(t)]^{q+1},\end{aligned}$$

where:

$$A_3 = 2pb_1\lambda_1, \quad B_3 = \frac{2(p-q)b_2}{(q+1)|\Omega|^q}\lambda_1^{q+1}. \tag{4.10}$$

Similarly, we have

$$t \leq \int_{\varphi_0(0)}^{\varphi_0(t)} \frac{d\eta}{A_3\eta + B_3\eta^{q+1}} \leq \int_{\varphi_0(0)}^{+\infty} \frac{d\eta}{A_3\eta + B_3\eta^{q+1}} < +\infty.$$

It follows from that the solution u blows up at some finite time in L^2-norm, since the above inequality cannot hold for all time t. □

Remark 4.1. (1) The similar result of Theorem 4.2 was obtained in [9] for the case $b_1 = 0, \nu = 1$.

(2) For the case $q = 0$ (assume $b_2 = 0$ for convenience), from the proof of Theorem 4.2, we can obtain that the solution $u(x,t)$ increases at least exponentially, that is:

$$\begin{aligned}\|u\|_{H_0^1} &\geq \|u_0\|_{H_0^1} e^{\frac{pb_1\lambda_1}{1+\lambda_1}t}, \quad \text{for } \nu = 1;\\ \|u\|_2 &\geq \|u_0\|_2 e^{pb_1\lambda_1 t}, \quad \text{for } \nu = 0.\end{aligned}$$

4.2. Finite-Time Blow-Up for Arbitrary Initial Energy

In this subsection, we will establish the blow-up results for arbitrary initial energy and the upper bounds for the maximal existence time T for both pseudo-parabolic case $\nu = 1$ and parabolic case $\nu = 0$.

Theorem 4.3. *Let $0 \leq q < p$ and u be the nonnegative solution of (4.1) with the initial energy satisfies:*

(1)

$$J(u_0) < \frac{pb_1\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2, \quad \textit{for } \nu = 1; \tag{4.11}$$

then, u blows up at some finite in H_0^1-norm. Moreover, the upper bound can be estimated by:

$$T \leq T_{31} = \frac{2(p+1)(1+\lambda_1)\|u_0\|_{H_0^1}^2}{p^3b_1\lambda_1\|u_0\|_{H_0^1}^2 - 2p^2(p+1)(1+\lambda_1)J(u_0)}. \tag{4.12}$$

(2)

$$J(u_0) < \frac{pb_1\lambda_1}{2(p+1)}\|u_0\|_2^2, \quad \textit{for } \nu = 0; \tag{4.13}$$

then, u blows up at some finite in L^2-norm. Moreover, the upper bound can be estimated by:

$$T \le T_{30} = \frac{2(p+1)\|u_0\|_2^2}{p^3 b_1 \lambda_1 \|u_0\|_2^2 - 2p^2(p+1)J(u_0)}. \tag{4.14}$$

Proof. Let $u(t)$ be the solution of the problem (4.1) with the initial energy satisfying (4.11) when $\nu = 1$ (resp. (4.13) when $\nu = 0$). We may assume $J(u(t)) \ge 0$; otherwise, there exists some $t_0 \ge 0$, such that $J(u(t_0)) < 0$, then $u(t)$ will blow up at some finite time by Theorem 4.1, the proof is complete. Therefore, in the following, we give our proof by contradiction, and assume that $u(t)$ exists globally and $J(u(t)) \ge 0$ for all $t \ge 0$.

In view of (4.1) and Lemma 4.1, we have the following equalities:

$$\begin{aligned} \frac{d}{dt} J(u(t)) &= -\|u_t\|_2^2 - \nu \|\nabla u_t\|_2^2, \\ \varphi'(t) = \frac{d}{dt}(\|u\|_2^2 + \nu\|\nabla u\|_2^2) &= -2I(u(t). \end{aligned}$$

Pseudo-parabolic case: $\nu = 1$. Since:

$$\begin{aligned} &\int_0^t \|u_s(s)\|_{H_0^1} ds \\ &\quad \ge \| \int_0^t u_s(s) ds\|_{H_0^1} = \|u(t) - u_0\|_{H_0^1} \ge \|u(t)\|_{H_0^1} - \|u_0\|_{H_0^1}, \quad t \ge 0, \end{aligned}$$

by Hölder's inequality and $J(u_0) \ge J(u(t)) \ge 0$, we obtain that:

$$\begin{aligned} \|u(t)\|_{H_0^1} &\le \|u_0\|_{H_0^1} + t^{\frac{1}{2}} [\int_0^t \|u_s(s)\|_{H_0^1}^2 ds]^{\frac{1}{2}} \\ &= \|u_0\|_{H_0^1} + t^{\frac{1}{2}} [J(u_0) - J(u(t))]^{\frac{1}{2}} \\ &\le \|u_0\|_{H_0^1} + t^{\frac{1}{2}} (J(u_0))^{\frac{1}{2}}, \quad t \ge 0. \end{aligned} \tag{4.15}$$

On other hand, in view of (2.7), (4.3), and $0 \le q < p$, we have:

$$\begin{aligned} \frac{d}{dt}\big(\|u(t)\|_{H_0^1}^2\big) &= 2pb_1 \|\nabla u\|_2^2 + \frac{2(p-q)b_2}{q+1} \|\nabla u\|_{2q+2}^{2q+2} - 4(p+1)J(u(t)) \\ &\ge \frac{2pb_1\lambda_1}{1+\lambda_1} \|u\|_{H_0^1}^2 - 4(p+1)J(u(t)) \\ &= \frac{2pb_1\lambda_1}{1+\lambda_1} \big[\|u\|_{H_0^1}^2 - \frac{2(p+1)(1+\lambda_1)}{pb_1\lambda_1} J(u(t))\big]. \end{aligned}$$

Since $\frac{d}{dt}(J(u(t))) \le 0$, then we can deduce that

$$\frac{d}{dt} H_1(t) \ge \frac{2pb_1\lambda_1}{1+\lambda_1} H_1(t)$$

for all $t \ge 0$, where:

$$H_1(t) = \|u\|_{H_0^1}^2 - \frac{2(p+1)(1+\lambda_1)}{pb_1\lambda_1} J(u(t)).$$

Using Gronwall's inequality, we obtain that:

$$\|u\|_{H_0^1}^2 \geq \frac{2(p+1)(1+\lambda_1)}{pb_1\lambda_1} J(u(t)) + e^{\frac{2pb_1\lambda_1}{1+\lambda_1}t} H_1(0),$$

where $H_1(0) = \|u_0\|_{H_0^1}^2 - \frac{2(p+1)(1+\lambda_1)}{pb_1\lambda_1} J(u_0) > 0$ due to (4.11). By the assumption $J(u(t)) \geq 0$ for all $t \geq 0$, we get:

$$\|u\|_{H_0^1} \geq \sqrt{H_1(0)} e^{\frac{pb_1\lambda_1}{1+\lambda_1}t},$$

which contradicts (4.15) for t sufficiently large. Hence, $u(t)$ blows up at some finite time, i.e., $T < \infty$.

Next, we establish an upper bound estimate of T. To this end, we first claim that:

$$\begin{aligned} I(u(t)) &= b_1\|\nabla u\|_2^2 + b_2\|\nabla u\|_{2q+2}^{2q+2} \\ &\quad - \int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)dxdy < 0, \quad t \in [0,T). \end{aligned}$$

Indeed, in view of the definitions of $J(u(t))$ and $I(u(t))$, after a simple calculation, we obtain:

$$\begin{aligned} J(u(t)) &= \frac{pb_1}{2(p+1)}\|\nabla u(t)\|_2^2 + \frac{(p-q)b_2}{2(q+1)(p+1)}\|\nabla u(t)\|_{2q+2}^{2q+2} \\ &\quad + \frac{1}{2(p+1)} I(u(t)), \quad t \in [0,T). \end{aligned} \tag{4.16}$$

It follows from (2.7), (4.11), and (4.16) that:

$$\frac{pb_1\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2 > J(u_0) \geq \frac{pb_1}{2(p+1)}\frac{\lambda_1}{1+\lambda_1}\|u_0\|_{H_0^1}^2 + \frac{1}{2(p+1)} I(u_0),$$

where we use $0 \leq q < p$, which implies $I(u_0) < 0$. We assume there exists a $t_0 \in (0,T)$, such that $I(u(t_0)) = 0$, $I(u(t)) < 0$, for $t \in [0,t_0)$. Hence, $\|u(t)\|_{H_0^1}^2$ is strictly increasing on $[0,t_0)$. Then, it follows from (4.11) that:

$$J(u_0) < \frac{pb_1\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2 < \frac{pb_1\lambda_1}{2(p+1)(1+\lambda_1)}\|u(t_0)\|_{H_0^1}^2. \tag{4.17}$$

On the other hand, since $J(u(t))$ is non-increasing with respect to t, and combining $0 \leq q < p$ and (4.16), we get:

$$\begin{aligned} J(u_0) \geq J(u(t_0)) &= \frac{pb_1}{2(p+1)}\|\nabla u(t_0)\|_2^2 \\ &\quad + \frac{(p-q)b_2}{2(q+1)(p+1)}\|\nabla u(t_0)\|_{2q+2}^{2q+2} + \frac{1}{2(p+1)} I(u(t_0)) \\ &\geq \frac{pb_1\lambda_1}{2(p+1)(1+\lambda_1)}\|u(t_0)\|_{H_0^1}^2, \end{aligned}$$

which contradicts (4.17). Hence, $I(u(t) < 0$ and $\|u(t)\|_{H_0^1}^2$ is strictly increasing on $[0,T)$.

For any $\tilde{T} \in (0, T)$, we define the functional:

$$F(t) = \int_0^t \|u(s)\|_{H_0^1}^2 ds + (T - t)\|u_0\|_{H_0^1}^2 + \beta(t + \gamma)^2, \quad t \in [0, \tilde{T}],$$

with two positive constants β, γ to be chosen later. Since $\|u(t)\|_{H_0^1}^2$ is strictly increasing, we have:

$$\begin{aligned} F'(t) &= \|u(t)\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + 2\beta(t + \gamma) \\ &= \int_0^t \frac{d}{ds}\|u(s)\|_{H_0^1}^2 ds + 2\beta(t + \gamma) > 0. \end{aligned} \tag{4.18}$$

In view of (2.7), Lemma 4.1, and $0 \le q < p$, we have:

$$\begin{aligned} F''(t) &= \frac{d}{dt}\|u(t)\|_{H_0^1}^2 + 2\beta \\ &= 2pb_1\|\nabla u\|_2^2 + \frac{2(p-q)b_2}{q+1}\|\nabla u\|_{2q+2}^{2q+2} - 4(p+1)J(u(t)) + 2\beta \\ &\ge \frac{2pb_1\lambda_1}{1+\lambda_1}\|u(t)\|_{H_0^1}^2 + 4(p+1)\int_0^t \|u_s\|_{H_0^1}^2 ds - 4(p+1)J(u_0). \end{aligned} \tag{4.19}$$

By the definition of $F(t)$, we have $F(0) = T\|u_0\|_{H_0^1}^2 + \beta\gamma^2 > 0$. Since $\|u(t)\|_{H_0^1}^2$ is strictly increasing on $[0, T)$, it follows from (4.18) that $F'(t) > 0$ for all $t \in [0, \tilde{T}]$, which implies that $F(t) > 0$ and $F(t)$ is strictly increasing for any $t \in [0, \tilde{T}]$.

Now, for any $t \in [0, \tilde{T}]$, we define:

$$\begin{aligned} \xi(t) := \Bigg(\int_0^t &\|u(s)\|_{H_0^1}^2 ds \\ &+\beta(t+\gamma)^2\Bigg)\left(\int_0^t \|u_s\|_{H_0^1}^2 ds + \beta\right) \\ &-\left(\int_0^t \frac{1}{2}\frac{d}{ds}\|u(s)\|_{H_0^1}^2 ds + \beta(t+\gamma)\right)^2. \end{aligned}$$

Using Hölder's inequality and the element algebraic inequality:

$$ab + cd \le \sqrt{a^2 + c^2}\sqrt{b^2 + d^2},$$

we can deduce:

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\|u(s)\|_{H_0^1}^2 =& \int_\Omega (uu_s + \nabla u \cdot \nabla u_s)dx \\ \le& \|u\|_2\|u_s\|_2 + \|\nabla u\|_2\|\nabla u_s\|_2 \\ \le& \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}}\left(\|u_s\|_2^2 + \|\nabla u_s\|_2^2\right)^{\frac{1}{2}} \\ =& \|u\|_{H_0^1}\|u_s\|_{H_0^1}. \end{aligned}$$

It follows from Hölder's inequality that:

$$\int_0^t \frac{1}{2}\frac{d}{ds}\|u(s)\|_{H_0^1}^2 ds \le \left(\int_0^t \|u\|_{H_0^1}^2 ds\right)^{\frac{1}{2}}\left(\int_0^t \|u_s\|_{H_0^1}^2 ds\right)^{\frac{1}{2}}.$$

In view of the above inequalities and the definition of $\xi(t)$ that:

$$\begin{aligned}\xi(t) \geq &\left(\int_0^t \|u\|_{H_0^1}^2 ds + \beta(t+\gamma)^2\right)\left(\int_0^t \|u_s\|_{H_0^1}^2 ds + \beta\right) \\ &- \left([\int_0^t \|u\|_{H_0^1}^2 ds]^{\frac{1}{2}} [\int_0^t \|u_s\|_{H_0^1}^2 ds]^{\frac{1}{2}} + \beta(t+\gamma)\right) \\ = \beta &\left([\int_0^t \|u\|_{H_0^1}^2 ds]^{\frac{1}{2}} - [\int_0^t \|u_s\|_{H_0^1}^2 ds]^{\frac{1}{2}}(t+\gamma)\right)^2 \geq 0.\end{aligned}$$

Hence, using the above inequality and (4.18), we have:

$$\begin{aligned}-(F'(t))^2 &= -4\left[\frac{1}{2}\int_0^t \frac{d}{ds}\|u(s)\|_{H_0^1}^2 ds + \beta(t+\gamma)\right]^2 \\ &= 4\Bigg(\xi(t) - \left(F(t) - (T-t)\|u_0\|_{H_0^1}^2\right) \\ &\quad \left(\int_0^t \|u_s(s)\|_{H_0^1}^2 ds + \beta\right)\Bigg) \\ &\geq -4F(t)\left(\int_0^t \|u_s(s)\|_{H_0^1}^2 ds + \beta\right).\end{aligned}$$

It follows from the above inequality and (4.19) that:

$$\begin{aligned}F(t)F''(t) - (p+1)(F'(t))^2 &\geq F(t)\left(F''(t) - 4(p+1)\int_0^t \|u_s(s)\|_{H_0^1}^2 ds - 4(p+1)\beta\right) \\ &\geq F(t)\left(\frac{2pb_1\lambda_1}{1+\lambda_1}\|u(t)\|_{H_0^1}^2 - 4(p+1)J(u_0) - 4(p+1)\beta\right) \\ &= F(t)\Bigg[4(p+1)\left(\frac{pb_1\lambda_1}{2(p+1)(1+\lambda_1)}\|u(t)\|_{H_0^1}^2 - J(u_0)\right) \\ &\quad -4(p+1)\beta\Bigg].\end{aligned}$$

In view of (4.11), letting β sufficiently small, such that:

$$0 < \beta \leq \beta_0 := \frac{pb_1\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2 - J(u_0), \tag{4.20}$$

we get:

$$F(t)F''(t) - (p+1)(F'(t))^2 \geq 0, \quad t \in [0, \tilde{T}].$$

Define $G(t) = F^{-p}(t)$ for $t \in [0, \tilde{T}]$. After a simple calculation, by $F(t) > 0$, $F'(t) > 0$, and $p > 0$, we obtain:

$$\begin{aligned}G'(t) &= -pF^{-p-1}(t)F'(t) < 0, \\ G''(t) &= -pF^{-p-2}[F(t)F''(t) - (p+1)(F'(t))^2] \leq 0,\end{aligned}$$

holds for all $t \in [0, \tilde{T}]$, which means that $G(t)$ is concave on $[0, \tilde{T}]$. Hence, it holds that:

$$G(\tilde{T}) \leq G(0) + G'(0)\tilde{T}. \tag{4.21}$$

By the definition of $G(t)$, we obtain:

$$G'(0) = -pF^{-p-1}(0)F'(0) = -pG(0)\frac{F'(0)}{F(0)} < 0.$$

Hence, it follows from (4.21) that:

$$\tilde{T} \le -\frac{G(0)}{G'(0)} = \frac{T\|u\|_{H_0^1}^2 + \beta\gamma^2}{2p\beta\gamma} = \frac{\|u\|_{H_0^1}^2}{2p\beta\gamma}T + \frac{\gamma}{2p}$$

for any $\tilde{T} \in [0, T)$. Letting $\tilde{T} \to T^-$, we get:

$$T \le \frac{\|u\|_{H_0^1}^2}{2p\beta\gamma}T + \frac{\gamma}{2p}. \tag{4.22}$$

Fixing an arbitrary β satisfying (4.20), then let γ be sufficiently large, such that:

$$\frac{\|u_0\|_{H_0^1}^2}{2p\beta} < \gamma < +\infty.$$

Then, in view of (4.22), we have:

$$T \le \frac{\beta\gamma^2}{2p\beta\gamma - \|u_0\|_{H_0^1}^2}. \tag{4.23}$$

Define a function $T_\beta(\gamma)$ by:

$$T_\beta(\gamma) = \frac{\beta\gamma^2}{2p\beta\gamma - \|u_0\|_{H_0^1}^2}, \quad \gamma \in \left(\frac{\|u_0\|_{H_0^1}^2}{2p\beta}, +\infty\right).$$

It is easy to verify that the function $T_\beta(\gamma)$ has a unique minimum at:

$$\gamma_\beta := \frac{\|u_0\|_{H_0^1}^2}{p\beta} \in (\frac{\|u_0\|_{H_0^1}^2}{2p\beta}, +\infty).$$

Then, it follows from (4.23) that:

$$T \le \inf_{\gamma \in \left(\frac{\|u_0\|_{H_0^1}^2}{2p\beta}, +\infty\right)} T_\beta(\gamma) = T_\beta(\gamma_\beta) = \frac{\|u_0\|_{H_0^1}^2}{p^2\beta},$$

for any β satisfying (4.20). Hence, it holds that:

$$T \le \inf_{\beta \in (0, \beta_0]} \frac{\|u_0\|_{H_0^1}^2}{p^2\beta} = \frac{\|u_0\|_{H_0^1}^2}{p^2\beta_0} = \frac{2(p+1)(1+\lambda_1)\|u_0\|_{H_0^1}^2}{p^3 b_1 \lambda_1 \|u_0\|_{H_0^1}^2 - 2p^2(p+1)(1+\lambda_1)J(u_0)}.$$

This completes the proof of Theorem 4.3 for the pseudo-parabolic ($\nu = 1$) case.

Parabolic case: $\nu = 0$. In this case, we need to replace the norm H_0^1 by the norm L^2, and the inequality $\|\nabla u\|_2^2 \ge \frac{\lambda_1}{1+\lambda_1}\|u\|_{H_0^1}^2$ by the inequality $\|\nabla u\|_2^2 \ge \lambda_1\|u\|_2^2$. By modifying the previous proof, we can easily deduce the proof. □

Corollary 4.1. *For $0 \leq q < p$ and any $M \in \mathbb{R}$, then there exist initial data $u_{0M} \in W_0^{1,2q+2}(\Omega)$ satisfying $J(u_{0M}) = M$, such that the weak solution for the corresponding problem (4.1) will blows up at finite time in H_0^1-norm for $\nu = 1$ (resp. L^2-norm for $\nu = 0$).*

Proof. According to Theorem 4.1, it is easy to see that the result is valid for $M < 0$. We only need to verify the result for the case of $M \geq 0$. In view of Theorem 4.3, it is sufficiently to check that there exists $u_0 \in W_0^{1,2q+2}(\Omega)$ satisfying (4.11) for $\nu = 1$ (resp. (4.13) for $\nu = 0$).

Let Ω_1 and Ω_2 be two arbitrary disjoint open subdomains of Ω. We assume $v \in W_0^{1,2q+2}(\Omega_1) \subset W_0^{1,2q+2}(\Omega) \subset H_0^1(\Omega)$ be an arbitrary nonzero function, and then, we can take $\alpha_1 > 0$ sufficiently large, such that:

$$\alpha_1^2\left(\int_\Omega v^2dx + \int_\Omega \nu|\nabla v|^2dx\right) = \alpha_1^2\left(\int_{\Omega_1} v^2dx + \int_{\Omega_1} \nu|\nabla v|^2dx\right) > \frac{M}{C},$$

where $C = \frac{pb_1\lambda_1}{2(p+1)(1+\lambda_1)}$ when $\nu = 1$, and $C = \frac{pb_1\lambda_1}{2(p+1)}$ when $\nu = 0$.

We claim that there exists $w \in W_0^{1,2q+2}(\Omega_2) \subset W_0^{1,2q+2}(\Omega)$ and $\alpha_0 > \alpha_1$ such that $J(w) = M - J(\alpha_0 v)$. Indeed, we choose a function $w_k \in C_0^1(\Omega_2)$, such that $\|\nabla w_k\|_2 \geq k$ and $\|w_k\|_\infty \leq c_0$. In view of (3.3), we have:

$$\begin{aligned}
&\frac{b_1}{2}\int_{\Omega_2}|\nabla w_k|^2dx + \frac{b_2}{2q+2}\int_{\Omega_2}|\nabla w_k|^{2q+2}dx - \frac{1}{2p+2}\\
&\int_{\Omega_2}\int_{\Omega_2}k(x,y)w_k^{p+1}(x,t)w_k^{p+1}(y,t)dxdy\\
&\geq \frac{b_1}{2}\int_{\Omega_2}|\nabla w_k|^2dx + \frac{b_2}{2q+2}|\Omega_2|^{-q}\Big(\int_{\Omega_2}|\nabla w_k|^2dx\Big)^{q+1} - \frac{\kappa}{2p+2}c_0^{2p+2}|\Omega_2|.
\end{aligned}$$

On the other hand, since $0 \leq q < p$, it holds that:

$$\begin{aligned}
&M - J(\alpha v)\\
&\quad = M - \frac{b_1\alpha^2}{2}\int_{\Omega_1}|\nabla v|^2dx - \frac{b_2\alpha^{2q+2}}{2q+2}\int_{\Omega_1}|\nabla v|^{2q+2}dx\\
&+\frac{\alpha^{2p+2}}{2p+2}\int_{\Omega_2}\int_{\Omega_2}k(x,y)w_k^{p+1}(x,t)w_k^{p+1}(y,t)dxdy \to +\infty, \quad as \quad \alpha \to +\infty.
\end{aligned}$$

Therefore, there exist $k_0 > 0$ and $\alpha_0 > \alpha_1$, such that both are sufficiently large, such that:

$$\begin{aligned}
M - J(\alpha_0 v) = &\frac{b_1}{2}\int_{\Omega_2}|\nabla w_{k_0}|^2dx + \frac{b_2}{2q+2}\int_{\Omega_2}|\nabla w_{k_0}|^{2q+2}dx\\
&-\frac{1}{2p+2}\int_{\Omega_2}\int_{\Omega_2}k(x,y)w_{k_0}^{p+1}(x,t)w_{k_0}^{p+1}(y,t)dxdy.
\end{aligned}$$

Then, choosing $w = w_{k_0}$, and denoting $u_{0M} := \alpha_0 v + w$, we obtain:

$$\int_\Omega |u_{0M}|^2dx + \nu\int_\Omega|\nabla u_{0M}|^2dx = \alpha_0^2\int_{\Omega_1}v^2dx + \nu\alpha_0^2\int_{\Omega_1}|\nabla v|^2dx > \frac{M}{C},$$

and

$$M = J(\alpha_0 v) + J(w) = J(u_{0M}),$$

which implies that:

$$J(u_{0M}) < C\left(\int_\Omega |u_{0M}|^2 dx + \nu \int_\Omega |\nabla u_{0M}|^2 dx\right).$$

In view of Theorem 4.3, the proof is complete. □

Remark 4.2. For the case $J(u_0) < 0$, the initial condition given in (4.11) (resp. (4.13)) is obviously satisfied. However, we obtain the upper bounds T_{11} and T_{31} (resp. T_{10} and T_{30}) for the blow-up time T in Theorem 4.1 and Theorem 4.3 using different methods. In fact, T_{11} is more accurate provided that $J(u_0) < \frac{p^2 b_1 \lambda_1 \|u_0\|_{H_0^1}}{2p(p+1)(3p+4)(1+\lambda_1)}$, and T_{31} is more accurate provided that $\frac{p^2 b_1 \lambda_1 \|u_0\|_{H_0^1}}{2p(p+1)(3p+4)(1+\lambda_1)} \leq J(u_0) < 0$; Resp. T_{10} is more accurate provided that $J(u_0) < \frac{p^2 b_1 \lambda_1 \|u_0\|_{L^2}}{2p(p+1)(3p+4)}$, and T_{30} is more accurate provided that $\frac{p^2 b_1 \lambda_1 \|u_0\|_{L^2}}{2p(p+1)(3p+4)} \leq J(u_0) < 0$;

4.3. Lower Bound for the Blow-Up Time

First, we consider problem (1.1) with $\nu = 1$, and determine a lower bound for T if the solution u blows up at finite time $t = T$ in H_0^1-norm.

Theorem 4.4. *Let $0 < q \leq p \leq \frac{2}{n-2}$, and u be the nonnegative solution of the problem (1.1)($\nu = 1$) which blows up at finite time T in H_0^1-norm, and then, T is bounded from below as:*

$$T \geq \frac{C_*^{-2(p+1)}}{2\kappa p \|u_0\|_{H_0^1}^{2p}}.$$

Proof. The proof is similar as that of Theorem 3.1 in [9]. It follows from (3.1) to (3.3) that:

$$\varphi_1'(t) \leq 2\kappa C_*^{-2(p+1)} [\varphi_1(t)]^{p+1},$$

which is equivalent to:

$$[\varphi_1^{-p}(t)]' \geq -2\kappa p C_*^{-2(p+1)}.$$

Integrating the above inequality from 0 to t leads to:

$$\varphi_1^{-p}(t) \geq \varphi_1^{-p}(0) - 2\kappa p C_*^{-2(p+1)} t.$$

Passing to the limit as $t \to T^-$, we obtain that the conclusion of Theorem 4.4 holds. □

Then, using the technique of differential inequality (see [13,14]), we obtain a lower bound for the blow-up time if the blow-up does occurs for the problem (1.1) with $\nu = 0$.

We define the auxiliary function $\chi(t)$:

$$\chi(t) = \int_\Omega u^{mp} dx, \tag{4.24}$$

for some positive constant m to be chosen later.

Theorem 4.5. *Let* $0 < q < p$, $m > \frac{2}{p} + 2$, *and* $u(x,t)$ *be the nonnegative solution of the problem* (1.1)$(\nu = 0)$ *which blows up at finite time* T *in the measure* χ *given in* (4.24), *and then* T *is bounded from below by:*

$$T \geq \int_{\chi(0)}^{\infty} \frac{d\chi}{K_1\chi^{\frac{2p+2}{mp}} + K_2\chi^{\frac{(n-2)\theta_1}{n\theta_1-2}} + K_3\chi^{\frac{(n-2)\theta_2}{n\theta_2-2}}}, \tag{4.25}$$

where K_1, K_2 *and* K_3 *are positive constants that will be determined in* (4.36).

Proof. In view of (1.1) and (2.2), we compute:

$$\begin{aligned}\chi'(t) &= \int_\Omega u^{mp-1}[\operatorname{div}(\rho(|\nabla u|^2)\nabla u) + u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)dy]dx\\ &= -mp(mp-1)\int_\Omega u^{mp-2}\rho(|\nabla u^2)|\nabla u|^2dx + mp\\ &\quad \int_\Omega\int_\Omega k(x,y)u^{mp-1+p}(x,t)u^{p+1}(y,t)dxdy\\ &\leq -mp(mp-1)\int_\Omega u^{mp-2}[b_1 + b_2|\nabla u|^{2q}]|\nabla u|^2dx\\ &\quad +mp\int_\Omega\int_\Omega k(x,y)u^{mp-1+p}(x,t)u^{p+1}(y,t)dxdy.\end{aligned}$$

By the similar argument as (3.3), we have:

$$\begin{aligned}&\int_\Omega\int_\Omega k(x,y)u^{mp-1+p}(x,t)u^{p+1}(y,t)dxdy\\ &\quad\leq \kappa\left(\int_\Omega u^{2p+2}(y,t)dy\right)^{\frac{1}{2}}\left(\int_\Omega u^{2mp-2+2p}(x,t)dx\right)^{\frac{1}{2}}\\ &\quad\leq \frac{\kappa}{2}\left(\int_\Omega u^{2p+2}dx + \int_\Omega u^{2mp-2+2p}dx\right).\end{aligned}$$

It follows from Hölder's inequality and $m > \frac{2}{p} + 2$ that:

$$\int_\Omega u^{2p+2}dx \leq |\Omega|^{\frac{mp-2-2p}{mp}}\left(\int_\Omega u^{mp}dx\right)^{\frac{2p+2}{mp}}. \tag{4.26}$$

Noticing that:

$$|\nabla u^\alpha|^{2(q+1)} = |\alpha u^{\alpha-1}\nabla u|^{2(q+1)} = \alpha^{2(q+1)}u^{mp-2}|\nabla u|^{2(q+1)},$$

where $\alpha = \frac{mp+2q}{2(q+1)}$. Denoting $\delta = \frac{mp-2+2(p-q)}{\alpha} > 0$ and $v = u^\alpha$, we obtain:

$$\int_\Omega u^{2mp-2+2p}dx = \int_\Omega v^{2(q+1)+\delta}dx,$$

Using Hölder's and Schwarz's inequalities, we have:

$$\begin{aligned}\int_\Omega|\nabla v^{q+1}|^2dx &\leq (q+1)^2\left(\int_\Omega|\nabla v|^{2(q+1)}dx\right)^{\frac{1}{q+1}}\left(\int_\Omega v^{2(q+1)}dx\right)^{\frac{q}{q+1}}\\ &\leq (q+1)\int_\Omega|\nabla v|^{2(q+1)}dx + q(q+1)\int_\Omega v^{2(q+1)}dx.\end{aligned}$$

Hence, in view of (4.26), by dropping the b_1 term, then we obtain:

$$\begin{aligned}\chi'(t) \leq & -\frac{mp(mp-1)b_2}{(q+1)\alpha^{2(q+1)}}\int_\Omega |\nabla v^{q+1}|^2 dx + \frac{mp\kappa}{2}|\Omega|^{\frac{mp-2-2p}{mp}}[\chi(t)]^{\frac{2p+2}{mp}} \\ & + \frac{qmp(mp-1)b_2}{\alpha^{2(q+1)}}\int_\Omega v^{2(q+1)}dx + \frac{mp\kappa}{2}\int_\Omega v^{2(q+1)+\delta}dx.\end{aligned} \tag{4.27}$$

To estimate the last two terms in (4.27), we will use the following Hölder's inequality:

$$\int_\Omega v^{r+s}dx \leq \left(\int_\Omega v^{\frac{r}{\theta}}dx\right)^{\theta}\left(\int_\Omega v^{\frac{s}{1-\theta}}dx\right)^{1-\theta}, \tag{4.28}$$

where $0 < \theta < 1$ and r, s are positive constants.

We choose r_1, s_1 and θ_1, such that

$$r_1 + s_1 = 2(q+1), \quad \frac{r_1}{\theta_1} = \frac{mp}{\alpha}, \quad \frac{s_1}{1-\theta_1} = (q+1)\frac{2n}{n-2},$$

that is:

$$\begin{aligned} r_1 &= \frac{mp}{\alpha}\frac{2(q+1)\frac{2}{n-2}}{2(q+1)\frac{n}{n-2} - \frac{mp}{\alpha}}, \\ s_1 &= 2(q+1) - \frac{mp}{\alpha}\frac{2(q+1)\frac{2}{n-2}}{2(q+1)\frac{n}{n-2} - \frac{mp}{\alpha}}, \\ \theta_1 &= \frac{2(q+1)\frac{2}{n-2}}{2(q+1)\frac{n}{n-2} - \frac{mp}{\alpha}}. \end{aligned}$$

Then, it follows from (4.28) that:

$$\int_\Omega v^{2(q+1)}dx \leq \left(\int_\Omega v^{\frac{mp}{\alpha}}dx\right)^{\theta_1}\left(\int_\Omega v^{(q+1)\frac{2n}{n-2}}dx\right)^{1-\theta_1}. \tag{4.29}$$

By the same argument, we choose r_2, s_2 and θ_2, such that:

$$r_2 + s_2 = 2(q+1) + \delta, \quad \frac{r_2}{\theta_2} = \frac{mp}{\alpha}, \quad \frac{s_2}{1-\theta_2} = (q+1)\frac{2n}{n-2},$$

that is:

$$\begin{aligned} r_2 &= \frac{mp}{\alpha}\frac{2(q+1)\frac{2}{n-2} - \delta}{2(q+1)\frac{n}{n-2} - \frac{mp}{\alpha}}, \\ s_2 &= 2(q+1) + \delta - \frac{mp}{\alpha}\frac{2(q+1)\frac{2}{n-2} - \delta}{2(q+1)\frac{n}{n-2} - \frac{mp}{\alpha}}, \\ \theta_2 &= \frac{2(q+1)\frac{2}{n-2} - \delta}{2(q+1)\frac{n}{n-2} - \frac{mp}{\alpha}}, \end{aligned}$$

and obtain:

$$\int_\Omega v^{2(q+1)+\delta}dx \leq \left(\int_\Omega v^{\frac{mp}{\alpha}}dx\right)^{\theta_2}\left(\int_\Omega v^{(q+1)\frac{2n}{n-2}}dx\right)^{1-\theta_2}. \tag{4.30}$$

Letting c_* be the best embedding constant: $\|w\|_{\frac{2n}{n-2}} \leq c_*\|\nabla w\|_2$, we obtain:

$$\|v^{q+1}\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}(1-\theta_1)} \leq c_*^{\frac{2n}{n-2}(1-\theta_1)}\|\nabla v^{q+1}\|_2^{\frac{2n}{n-2}(1-\theta_1)}, \tag{4.31}$$

and

$$\|v^{q+1}\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}(1-\theta_2)} \leq c_*^{\frac{2n}{n-2}(1-\theta_2)}\|\nabla v^{q+1}\|_2^{\frac{2n}{n-2}(1-\theta_2)}. \tag{4.32}$$

Combining (4.29) and (4.31), using Schwarz's inequality, we get:

$$\begin{aligned}\int_\Omega v^{2(q+1)}dx &\leq c_*^{\frac{2n}{n-2}(1-\theta_1)}\Big(\int_\Omega v^{\frac{mp}{\alpha}}dx\Big)^{\theta_1}\left(\int_\Omega |\nabla v^{q+1}|^2dx\right)^{\frac{n}{n-2}(1-\theta_1)} \\ &\leq \frac{n\theta_1-2}{n-2}(c_*^2\varepsilon_1^{-1})^{\frac{n(1-\theta_1)}{n\theta_1-2}}\left(\int_\Omega v^{\frac{mp}{\alpha}}\right)^{\frac{(n-2)\theta_1}{n\theta_1-2}} \\ &\quad+\frac{n(1-\theta_1)}{n-2}\varepsilon_1\int_\Omega |\nabla v^{q+1}|^2dx,\end{aligned} \tag{4.33}$$

where ε_1 is a positive constant to be chosen later. Similarly, it follows from (4.30) and (4.32) that:

$$\begin{aligned}\int_\Omega v^{2(q+1)+\delta}dx \leq \frac{n\theta_2-2}{n-2}&(c_*^2\varepsilon_2^{-1})^{\frac{n(1-\theta_2)}{n\theta_2-2}} \\ &\left(\int_\Omega v^{\frac{mp}{\alpha}}\right)^{\frac{(n-2)\theta_2}{n\theta_2-2}} \\ &+\frac{n(1-\theta_2)}{n-2}\varepsilon_2\int_\Omega |\nabla v^{q+1}|^2dx,\end{aligned} \tag{4.34}$$

where ε_1 is a positive constant to be chosen later.

Combining (4.33) and (4.34) with (4.27) gives:

$$\begin{aligned}\chi'(t) \leq -\Bigg[&\frac{mp(mp-1)b_2}{(q+1)\alpha^{2(q+1)}} - \frac{qmp(mp-1)b_2}{\alpha^{2(q+1)}} \\ &\frac{n(1-\theta_1)}{n-2}\varepsilon_1 - \frac{mp\kappa}{2}\frac{n(1-\theta_2)}{n-2}\varepsilon_2\Bigg]\int_\Omega |\nabla v^{q+1}|^2dx \\ &+\frac{mp\kappa}{2}|\Omega|^{\frac{mp-2-2p}{mp}}[\chi(t)]^{\frac{2p+2}{mp}} \\ &\quad+\frac{qmp(mp-1)b_2}{\alpha^{2(q+1)}}\frac{n\theta_1-2}{n-2} \\ &\quad(c_*^2\varepsilon_1^{-1})^{\frac{n(1-\theta_1)}{n\theta_1-2}}[\chi(t)]^{\frac{(n-2)\theta_1}{n\theta_1-2}} \\ &+\frac{mp\kappa}{2}\frac{n\theta_2-2}{n-2} \\ &\quad(c_*^2\varepsilon_2^{-1})^{\frac{n(1-\theta_2)}{n\theta_2-2}}[\chi(t)]^{\frac{(n-2)\theta_2}{n\theta_2-2}}.\end{aligned}$$

By choosing ε_1 and ε_2 sufficiently small, such that:

$$\frac{mp(mp-1)b_2}{(q+1)\alpha^{2(q+1)}} - \frac{mp(mp-1)b_2}{\alpha^{2(q+1)}}\frac{n(1-\theta_1)}{n-2}\varepsilon_1 - \frac{mp\kappa}{2}\frac{n(1-\theta_2)}{n-2}\varepsilon_2 \geq 0,$$

then, we can obtain the differential inequality:

$$\chi'(t) \leq K_1[\chi(t)]^{\frac{2p+2}{mp}} + K_2[\chi(t)]^{\frac{(n-2)\theta_1}{n\theta_1-2}} + K_3[\chi(t)]^{\frac{(n-2)\theta_2}{n\theta_2-2}},$$

or equivalently:

$$\frac{d\chi}{K_1\chi^{\frac{2p+2}{mp}} + K_2\chi^{\frac{(n-2)\theta_1}{n\theta_1-2}} + K_3\chi^{\frac{(n-2)\theta_2}{n\theta_2-2}}} \leq dt, \tag{4.35}$$

where:

$$K_1 = \frac{mp\kappa}{2}|\Omega|^{\frac{mp-2-2p}{mp}}; \quad K_2 = \frac{mp(mp-1)b_2}{\alpha^{2(q+1)}}\frac{n\theta_1-2}{n-2}(c_*^2\varepsilon_1^{-1})^{\frac{n(1-\theta_1)}{n\theta_1-2}};$$
$$K_3 = \frac{mp\kappa}{2}\frac{n\theta_2-2}{n-2}(c_*^2\varepsilon_2^{-1})^{\frac{n(1-\theta_2)}{n\theta_2-2}}. \tag{4.36}$$

Integrating of the differential inequality (4.35) from 0 to t leads to:

$$\int_{\chi(0)}^{\chi(t)} \frac{d\chi}{K_1\chi^{\frac{2p+2}{mp}} + K_2\chi^{\frac{(n-2)\theta_1}{n\theta_1-2}} + K_3\chi^{\frac{(n-2)\theta_2}{n\theta_2-2}}} \leq t.$$

Passing to the limit as $t \to T^-$, we obtain:

$$\int_{\chi(0)}^{\infty} \frac{d\chi}{K_1\chi^{\frac{2p+2}{mp}} + K_2\chi^{\frac{(n-2)\theta_1}{n\theta_1-2}} + K_3\chi^{\frac{(n-2)\theta_2}{n\theta_2-2}}} \leq T.$$

Thus, the proof is complete. □

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