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Bohr-Type Inequalities for Harmonic Mappings with a Multiple Zero at the Origin

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Abstract. In this paper, we first determine Bohr's inequality for the class of harmonic mappings $f = h + \overline{g}$ in the unit disk $\mathbb{D}$, where either both $h(z) = \sum_{n=0}^{\infty} a_{pn+m} z^{pn+m}$ and $g(z) = \sum_{n=0}^{\infty} b_{pn+m} z^{pn+m}$ are analytic and bounded in $\mathbb{D}$, or satisfies the condition $|g'(z)| \leq d|h'(z)|$ in $\mathbb{D}\backslash\{0\}$ for some $d \in [0, 1]$ and h is bounded. In particular, we obtain Bohr's inequality for the class of harmonic p-symmetric mappings. Also, we investigate the Bohr-type inequalities of harmonic mappings with a multiple zero at the origin and that most of results are proved to be sharp.

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1. Preliminaries and Some Basic Questions

The classical theorem of Bohr [14], examined a century ago, generates intensive research activity—what is called Bohr's phenomena. Determination of the Bohr radius for analytic functions in a domain [21], as well as for analytic functions from $\mathbb{D}$ into particular domains, such as the punctured unit disk, the exterior of the closed unit disk, and concave wedge-domains, has been discussed in the literature [1–3,5]. See also the recent survey articles [9,23,29] and [22, Chapter 8]. The interest in the Bohr phenomena was revived in the 90s due to the extensions to holomorphic functions of several complex variables and to more abstract settings. For example, Boas and Khavinson [13] found bounds for Bohr's radius in any complete Reinhard domains and showed that the Bohr radius decreases to zero as the dimension of the domain increases. This paper stimulated interests on Bohr-type questions in different settings. For example, Aizenberg [6,7], Aizenberg et al. [8], Defant and

Frerick [16], and Djakov and Ramanujan [18] have established further results on Bohr's phenomena for multidimensional power series. Several other aspects and generalizations of Bohr's inequality may be obtained from the literature. For instance, Defant [17] improved a version of the Bohnenblust–Hille inequality and Paulsen [39] proved a uniform algebra analogue of the classical inequality of Bohr concerning Fourier coefficients of bounded holomorphic functions in 2004. In Refs. [38,40], the authors demonstrated the classical Bohr inequality using different methods of operators. Abu Muhanna [1], and Kayumov and Ponnusamy [25] investigated Bohr's inequality for the class of analytic functions that are subordinate to univalent functions and odd univalent functions, respectively. On the other hand, Ali et al. [10] discussed Bohr's phenomenon for the classes of even and odd analytic functions and also for alternating series. In Refs. [11,30,34,36], the authors considered the Bohr radius for the family K-quasiconformal sense-preserving harmonic mappings and the class of all sense-preserving harmonic mappings, separately. Recently, the articles [35,41,42] presented a refined version of Bohr's inequality along with few other related improved versions of previously known results. In particular, after the appearance of the articles [9,26], several investigations and new problems on Bohr's inequality in the plane case appeared in the literature (cf. [4,12,27,33,35,41,42]).

One of our aims in this article is to address the harmonic analog of this question (see Problem 1) raised by Paulsen et al. [38] but with a refined formulation as in Ref. [42] (see Theorem A).

1.1. Classical Inequality of H. Bohr

Let $\mathcal{B}$ be the Schur class of all analytic functions f on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, such that $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| \leq 1$. Then, the classical inequality examined by Bohr [14] states that $1/3$ is the largest value of $r \in [0, 1)$ for which the following inequality holds:

$$B(f, r) := \sum_{k=0}^{\infty} |a_k| r^k \leq 1, \tag{1.1}$$

for every analytic function $f \in \mathcal{B}$ with the Taylor series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Bohr actually obtained that (1.1) is true when $r \leq 1/6$. Later, Riesz, Schur, and Wiener independently established the Bohr inequality (1.1) for $r \leq 1/3$ and that $1/3$ is the best possible constant. It is quite natural that the constant $1/3$ is called the Bohr radius for the space $\mathcal{B}$. Moreover, for:

$$\varphi_a(z) = \frac{a - z}{1 - az}, \quad a \in [0, 1),$$

it follows easily that $B(\varphi_a, r) > 1$ if and only if $r > 1/(1 + 2a)$, which for $a \to 1$ shows that $1/3$ is optimal. Bohr's and Wiener's proofs can be found in Ref. [14]. Other proofs of Bohr's inequality may be found from [44,45]. Then, it is worth pointing out that there is no extremal function in $\mathcal{B}$, such that the Bohr radius is precisely $1/3$ (cf. [22, Corollary 8.26]).

1.2. The Bohr Radius for Functions Having Multiple Zeros at the Origin

Problem 1. In Ref. [38], the authors considered among others for $k \in \mathbb{N}$ the classes $\mathcal{B}_k := z^k\mathcal{B}$, that is:

$$\mathcal{B}_k = \left\{ f \in \mathcal{B} : \, f(0) = \cdots = f^{(k-1)}(0) = 0 \right\} := \left\{ z^k f : \, f \in \mathcal{B} \right\},$$

and asked for which $r_k \in (0,1)$, and:

$$f(z) = \sum_{n=k}^{\infty} a_n z^n \in \mathcal{B}_k, \tag{1.2}$$

we have the inequality:

$$\sum_{n=k}^{\infty} |a_n| r^n \leq 1 \quad \text{for}\, r \in [0, r_k], \tag{1.3}$$

and for each $r \in (r_k, 1)$, there exists a function $f_k(z) = \sum_{n=k}^{\infty} a_n^{(k)} z^n$ in $\mathcal{B}_k$ such that $\sum_{n=k}^{\infty} |a_n^{(k)}| r^n > 1$. Here, the constant r_k is referred to as the Bohr radius of order k.

Clearly, $\mathcal{B}_0 = \mathcal{B}$, and $\mathcal{B}_1 = \{f \in \mathcal{B} : \, f(0) = 0\}$. For $f \in \mathcal{B}_1$ (i.e., for $k = 1$), Tomić [45] proved that (1.3) holds for $0 \leq r \leq 1/2$ (also obtained by Landau independently, see [31]). Later, Ricci [43] established that this holds for $0 \leq r \leq 3/5$, and the largest value of r for which (1.3) holds would lie in the interval $(3/5, 1/\sqrt{2}]$. Later, Bombieri [15] found that the inequality (1.3) holds for $r \in [0, 1/\sqrt{2}]$, where the upper bound cannot be improved. An alternate proof of this result may be found from a recent paper of Kayumov and Ponnusamy [28] in which they solved an open problem of Djakov and Ramanujan on powered Bohr inequality. However, Problem 1 for $k \geq 2$ remains open. On the other hand, in connection with Problem 1, Ponnusamy and Wirths [42] proved the following sharp inequalities for $k \geq 2$, while the case $k = 1$ has been proved in Ref. [41]:

Theorem A. *For $k \geq 1$, let $f \in \mathcal{B}_k$ have an expansion (1.2) and:*

$$M_k^1(f,r) = \sum_{n=k}^{\infty} |a_n| \, r^n + \left(\frac{1}{1+|a_k|} + \frac{r}{1-r} \right) \sum_{n=k+1}^{\infty} |a_n|^2 \, r^{2n-k}$$

and

$$M_k(f,r) = \sum_{n=k}^{\infty} |a_n| \, r^n + \left(\frac{1}{1+|a_k|} + \frac{r}{1-r} \right) \sum_{n=k}^{\infty} |a_n|^2 \, r^{2n-k}.$$

Then, we have the following inequalities:

(1) *$M_k^1(f,r) \leq 1$ is valid for $r \in [0, R_k]$, where R_k is the unique root in $(0,1)$ of the equation:*

$$4(1-r) - r^{k-1}\left(1 - 2r + 5r^2\right) = 0.$$

The upper bound R_k cannot be improved.

(2) $M_k(f,r) \le 1$ *is valid for* $r \in [0, S_k]$, *where* S_k *is the unique root in* $(0,1)$ *of the equation:*

$$2(1-r) - r^k(3-r) = 0.$$

The upper bound S_k *cannot be improved. Also, as* $M_k^1(f,r) \le M_k(f,r)$, *it follows that* $S_k \le R_k$.

(3) *With* $|a_k| = a \in (0,1]$ *being fixed,* $M_k(f,r) \le 1$ *is valid for* $r \in [0, \rho_k(a)]$, *where* $\rho_k(a)$ *is the unique root in* $(0,1)$ *of the equation:*

$$(1+a)(1-r) - r^k\left[2a^2 + a + r\left(1-2a^2\right)\right] = 0.$$

The upper bound $\rho_k(a)$ *cannot be improved.*

Remark 1. We note that $\rho_k(1) = S_k$. As $M_k^1(f,r) \le M_k(f,r)$, it follows that $S_k \le R_k$.

1.3. The Bohr Radius for p-Symmetric Functions

Recently, Kayumov et al. [26] have obtained the following general result. As a corollary to this, an open problem raised by Ali et al. [10] about the determination of Bohr radius for odd functions from $\mathcal{B}$ has been settled affirmatively.

Theorem B. [26] *Let* $m, p \in \mathbb{N}$, $m \le p$, *and* $f \in \mathcal{B}$ *with* $f(z) = \sum_{k=0}^{\infty} a_{pk+m} z^{pk+m}$. *Then:*

$$B(f,r) = \sum_{k=0}^{\infty} |a_{pk+m}| r^{pk+m} \le 1 \quad \text{for } r \le r_{p,m},$$

where $r_{p,m}$ *is the maximal positive root of the equation* $-6r^{p-m} + r^{2(p-m)} + 8r^{2p} + 1 = 0$. *The extremal function has the form* $z^m(z^p - a)/(1 - az^p)$, *where:*

$$a = \left(1 - \frac{\sqrt{1 - r_{p,m}^{2p}}}{\sqrt{2}}\right) \frac{1}{r_{p,m}^p}.$$

Remark 2. We note that the case $m = 0$ is trivial as it follows from the classical theorem of H. Bohr with a change of variable $\zeta = z^p$. This gives the condition $r \le r_{p,0} = 1/\sqrt[p]{3}$. The case $p = 2$ and $m = 1$ corresponds to the question raised by Ali et al. [10].

1.4. The Bohr Radius for Harmonic Functions

In Ref. [30], the authors initiated the discussion on Bohr radius for the class of complex-valued function $f = u + iv$ harmonic in $\mathbb{D}$, where u and v are real-valued harmonic functions of $\mathbb{D}$. It follows that f admits the canonical representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$, such that $f(0) = 0 = g(0)$. The Jacobian $J_f(z)$ of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$, and we say that a locally univalent harmonic function f in $\mathbb{D}$ is said to be sense-preserving if $J_f(z) > 0$ in $\mathbb{D}$; or equivalently, its dilatation $\omega = g'/h'$ is an analytic function in $\mathbb{D}$ which maps $\mathbb{D}$ into itself (cf. [19] or [32]).

If a locally univalent and sense-preserving harmonic mapping $f = h + \overline{g}$ satisfies the condition $|\omega(z)| \le d < 1$ in $\mathbb{D}$, then f is called K-quasi-regular harmonic mapping on $\mathbb{D}$, where $K = \frac{1+d}{1-d} \ge 1$ (cf. [24,37]). Obviously, $d \to 1$ corresponds to the case $K \to \infty$.

For a harmonic function $f = h + \overline{g}$ in $\mathbb{D}$, where h and g admit power series expansions of the form $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, we denote the classical Bohr sum by:

$$B_H(f, r) := B(h, r) + B(g, r) = \sum_{n=0}^{\infty}(|a_n| + |b_n|)r^n.$$

A harmonic function $f = h + \overline{g}$ in $\mathbb{D}$ is said to be p-symmetric if h and g have the form $h(z) = \sum_{n=0}^{\infty} a_n z^{pn+m}$ and $g(z) = \sum_{n=0}^{\infty} b_n z^{pn+m}$ for some $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Harmonic extension of the classical Bohr theorem was established first in Ref. [30]. For example, they proved the following result (Theorem C). Furthermore, the Bohr radii for harmonic and starlike log harmonic mappings in $\mathbb{D}$ were investigated, for example, in Refs. [20,26,30,36], and in some cases in improved form.

Theorem C. [30] *Let $p \in \mathbb{N}$ and $p \geq 2$. Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^{pn+1} + \sum_{n=0}^{\infty} \overline{b_n z^{pn+1}}$ is a harmonic p-symmetric function in $\mathbb{D}$, where h and g are bounded functions in $\mathbb{D}$. Then:*

$$B_H(f, r) = \sum_{n=0}^{\infty}(|a_n| + |b_n|)r^{pn+1} \leq \max\{\|h\|_\infty, \|g\|_\infty\} \quad \text{for } r \leq 1/2.$$

The number $1/2$ is sharp.

It is natural to raise the following.

Problem 2. Whether Theorem C holds under a weaker hypotheses, namely, by replacing the condition “boundedness of h and g” by “$|g'(z)| \leq |h'(z)|$ and h is bounded.”

In Theorem 1, we present an affirmative answer to this question in a more general setting.

The paper is organized as follows. In Sect. 2, we present the main results of this paper. In Theorem 1, we present an affirmative answer to Problem 2 in a general form, and Corollary 2 answers Problem 2. As consequence, generalization Theorem C (with of higher order zero at the origin) is established (see Theorem 2). In Sect. 3, we state and prove several lemmas. In addition, we present the proof of Bohr's inequalities for the class of harmonic mappings, which improve the first two items in Theorems A and Theorem C. In Sect. 4, we state and prove three theorems which extend three recent results of Ponnusamy et al. [42] from the case of analytic functions to the case of sense-preserving harmonic mappings.

2. Main Results

We now state a generalization of Theorem C in a general setting and the next result (Theorem 2) is a direct generalization of Theorem C.

Theorem 1. *Let $m, p \in \mathbb{N}$, $p \geq 2$. Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^{pk+m} + \sum_{k=0}^{\infty} \overline{b_k z^{pk+m}}$ is harmonic and p-symmetric in $\mathbb{D}$, such* *that* $|h^{(m)}(0)| = |g^{(m)}(0)|$ *and* $|g'(z)| \leq d|h'(z)|$ *in* $\mathbb{D}\backslash\{0\}$ *for some* $d \in [0,1]$, *where* h *is bounded. Then, the following holds:*

(1) *If* $\frac{p}{m} > \log_2(2+d)$, *then:*

$$B_H(f,r) = \sum_{k=0}^{\infty}(|a_k| + |b_k|)r^{pk+m} \leq \|h\|_\infty \quad \text{for}\, r \leq \sqrt[m]{\frac{1}{2}}.$$

When $d = 1$, *the extremal mapping has the form* $f(z) = h(z) + \overline{\lambda h(z)}$ *with* $h(z) = z^m$ *and* $|\lambda| = 1$.

(2) *If* $1 \leq \frac{p}{m} \leq \log_2(2+d)$, *then:*

$$B_H(f,r) \leq \|h\|_\infty \quad \text{for}\, r \leq r_{p,m,d},$$

where $r_{p,m,d}$ *is the maximal positive root of the equation:*

$$r^{2(p-m)} - (8+4d)r^{p-m} + 4(1+d)(3+d)r^{2p} + 4 = 0. \tag{2.1}$$

When $d = 1$, *the extremal function is given by* $f(z) = h(z) + \overline{\lambda h(z)}$, $|\lambda| = 1$, *where:*

$$h(z) = z^m\left(\frac{z^p - a}{1 - az^p}\right), \quad \text{with}\, a = \left(1 - \frac{\sqrt{1 - r_{p,m,1}^{2p}}}{\sqrt{2}}\right)\frac{1}{r_{p,m,1}^p}.$$

Corollary 1. *Suppose that* $m, p \in \mathbb{N}$, *and* $f(z) = \sum_{k=0}^{\infty} a_{pk+m}z^{pk+m} \in \mathcal{B}_{pk+m}$.

(1) *If* $1 \leq \frac{p}{m} \leq \log_2 3 \approx 1.58496$, *then:*

$$B(f,r) \leq \frac{1}{2} \quad \text{for}\, r \leq r_{p,m},$$

where $r_{p,m}$ *is the maximal positive root of the equation:*

$$-12r^{p-m} + r^{2(p-m)} + 32r^{2p} + 4 = 0. \tag{2.2}$$

The extremal function is given by:

$$f(z) = z^m\left(\frac{z^p - a}{1 - az^p}\right), \quad \textit{with}\, a = \left(1 - \frac{\sqrt{1 - r_{p,m}^{2p}}}{\sqrt{2}}\right)\frac{1}{r_{p,m}^p}. \tag{2.3}$$

(2) *If* $\frac{p}{m} > \log_2 3$, *then:*

$$B(f,r) \leq \frac{1}{2} \quad \text{for}\, r \leq \sqrt[m]{\frac{1}{2}}.$$

The extremal function has the form z^m.

Proof. Apply the method of the proof of Theorem 1 (by setting $d = 1$, $g(z) \equiv 0$). □

We now state a direct generalization of Theorem C.

Theorem 2. *Let* $m, p \in \mathbb{N}$, $p \geq 2$. *Suppose that* $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_{pk+m}z^{pk+m} + \sum_{k=0}^{\infty} \overline{b_{pk+m}z^{pk+m}}$ *is harmonic in* $\mathbb{D}$, *where* h *and* g *are bounded. The following holds:*

(1) *If* $\frac{p}{m} > \log_2 3 \approx 1.58496$*, then:*

$$B_H(f,r) \le \max\{\|h\|_\infty, \|g\|_\infty\} \quad \text{for } r \le \sqrt[m]{\frac{1}{2}}.$$

The extremal function is given by $f(z) = z^m + \overline{\lambda z^m}$, $|\lambda| = 1$.

(2) *If* $1 \le \frac{p}{m} \le \log_2 3$*, then:*

$$B_H(f,r) \le \max\{\|h\|_\infty, \|g\|_\infty\} \quad \text{for } r \le r_{p,m},$$

where $r_{p,m}$ *is the maximal positive root in* $(0,1)$ *of the Eq.* (2.2).

The extremal function is given by $f(z) = h(z) + \overline{\lambda h(z)}$, $|\lambda| = 1$, *where:*

$$h(z) = z^m \left(\frac{z^p - a}{1 - az^p}\right), \quad \text{with } a = \left(1 - \frac{\sqrt{1 - r_{p,m}^{2p}}}{\sqrt{2}}\right) \frac{1}{r_{p,m}^p}.$$

Remark 3. If we set $m = 1$ in Theorem 2(1), then we get Theorem C.

Note that the following corollary generalizes Theorem 2 under the conditions "$|h'(0)| = |g'(0)|$ and $|g'(z)| \le |h'(z)|$ in $\mathbb{D}\backslash\{0\}$" instead of "$h$ and g being bounded in $\mathbb{D}$."

Corollary 2. *Let* $m, p \in \mathbb{N}$, $p \ge 2$. *Suppose that* $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^{pk+m} + \sum_{k=0}^{\infty} \overline{b_k z^{pk+m}}$ *is harmonic and p-symmetric in* $\mathbb{D}$, *such that* $|h'(0)| = |g'(0)|$ *and* $|g'(z)| \le |h'(z)|$ *in* $\mathbb{D}\backslash\{0\}$, *where* h *is bounded. Then, the conclusions* (1) *and* (2) *of Theorem* 2 *continue to hold.*

Proof. Set $d = 1$ in Theorem 1 and let $r_{p,m} := r_{p,m,1}$. □

Because of its independent interest, let us next state the following result as a corollary to Theorems A. Indeed, applying the analogous methods as in the proofs of the three cases of Theorems A, we have the following. Therefore, we omit the details.

Corollary 3. *For* $k \ge 1$, $m, p \in \mathbb{N}$ *and* $m \le p$, *we let* $f(z) = \sum_{n=k}^{\infty} a_{pn+m} z^{pn+m} \in \mathcal{B}_{pk+m}$ *and:*

$$M^1_{pk+m}(f,r) = \sum_{n=k}^{\infty} |a_{pn+m}|\, r^{pn+m} + \left(\frac{1}{1 + |a_{pk+m}|} + \frac{r^p}{1 - r^p}\right) \times \sum_{n=k+1}^{\infty} |a_{pn+m}|^2\, r^{p(2n-k)+m}$$

and

$$M_{pk+m}(f,r) = \sum_{n=k}^{\infty} |a_{pn+m}|\, r^{pn+m} + \left(\frac{1}{1 + |a_{pk+m}|} + \frac{r^p}{1 - r^p}\right) \times \sum_{n=k}^{\infty} |a_{pn+m}|^2\, r^{p(2n-k)+m}.$$

Then, we have the following inequalities:

(1) *$M^1_{pk+m}(f,r) \le 1$ is valid for $r \in [0, v_k]$, where v_k is the unique root in $(0,1)$ of the equation:*

$$4\left(1-r^p\right)-r^{p(k-1)+m}\left(5r^{2p}-2r^p+1\right)=0.$$

The upper bound v_k cannot be improved.

(2) *$M_{pk+m}(f,r) \le 1$ is valid for $r \in [0, \omega_k]$, where ω_k is the unique root in $(0,1)$ of the equation:*

$$2\left(1-r^p\right)-r^{pk+m}\left(3-r^p\right)=0.$$

The upper bound ω_k cannot be improved.

(3) *With $|a_{pk+m}| = a \in (0,1]$ being fixed, $M_{pk+m}(f,r) \le 1$ is valid for $r \in [0, \eta_k]$, where η_k is the unique root in $(0,1)$ of the equation:*

$$(1+a)\left(1-r^p\right)-r^{pk+m}\left[2a^2+a+r^p\left(1-2a^2\right)\right]=0.$$

The upper bound η_k cannot be improved.

Proof. The desired conclusion follows if we write $f(z)$ as $f(z) = z^m t(z^p)$, where $t(z) = \sum_{n=k}^{\infty} a_{pn+m} z^n \in \mathcal{B}_k$, and apply the proof of Theorem A. □

Next, we generalize Theorem A or Corollary 3 by establishing Bohr-type inequalities for harmonic mappings with multiple zero at the origin.

Theorem 3. *Let $k \ge 1$, $m, p \in \mathbb{N}$, and $m \le p$. Suppose that $f = h + \overline{g}$ is harmonic in $\mathbb{D}$, where h and g are given by:*

$$h(z)=\sum_{n=k}^{\infty} a_{pn+m} z^{pn+m} \quad \text{and } g(z)=\sum_{n=k}^{\infty} b_{pn+m} z^{pn+m}. \tag{2.4}$$

In addition, let $|g'(z)| \le d|h'(z)|$ in $\mathbb{D}\backslash\{0\}$ for some $d \in [0,1]$ and $h \in \mathcal{B}_{pk+m}$. Define:

$$M_{pk+m}(h,r)=\sum_{n=k}^{\infty}\left|a_{pn+m}\right| r^{pn+m}+\left(\frac{1}{1+\left|a_{pk+m}\right|}+\frac{r^p}{1-r^p}\right) \times \sum_{n=k}^{\infty}\left|a_{pn+m}\right|^2 r^{p(2n-k)+m}, \tag{2.5}$$

and

$$N_{pk+m}(g,r)=\sum_{n=k}^{\infty}\left|b_{pn+m}\right| r^{pn+m}+\left(\frac{1}{1+\left|a_{pk+m}\right|}+\frac{r^p}{1-r^p}\right) \times \sum_{n=k}^{\infty}\left|b_{pn+m}\right|^2 r^{p(2n-k)+m}.$$

Then, the inequality:

$$M_{pk+m}(h,r)+N_{pk+m}(g,r) \le 1 \tag{2.6}$$

is valid for $r \in [0, r_k]$, where $r_k = \min\{r_k', 1/\sqrt[p]{3}\}$, and r_k' is the unique root in $(0,1)$ of the equation $t_k(r) = 0$, where:

$$t_k(r)=\frac{2}{d+1}\left(1-r^p\right)-r^{pk+m}\left(3-r^p\right). \tag{2.7}$$

Theorem 4. *Let $k \geq 2$, $m, p \in \mathbb{N}$, and $m \leq p$. Suppose that $f = h + \overline{g}$ is harmonic in $\mathbb{D}$, where h and g are given by (2.4), In addition, let $h, g \in \mathcal{B}_{pk+m}$ and define:*

$$M^1_{pk+m}(h,r) = \sum_{n=k}^{\infty} |a_{pn+m}| \, r^{pn+m} + \left(\frac{1}{1+|a_{pk+m}|} + \frac{r^p}{1-r^p} \right) \times \sum_{n=k+1}^{\infty} |a_{pn+m}|^2 \, r^{p(2n-k)+m}.$$

Then, the inequality

$$M^1_{pk+m}(h,r) + M^1_{pk+m}(g,r) \leq 1 \tag{2.8}$$

is valid for $r \in [0, \tau_k]$, where τ_k is the unique root in $(0,1)$ of the equation:

$$2\left(1 - r^p\right) - r^{p(k-1)+m}\left(5r^{2p} - 2r^p + 1\right) = 0.$$

The upper bound τ_k cannot be improved.

Theorem 5. *Let $k \geq 2$, $m, p \in \mathbb{N}$ and $m \leq p$. Suppose that $f = h + \overline{g}$ is harmonic in $\mathbb{D}$, where h and g are given by (2.4), and $h, g \in \mathcal{B}_{pk+m}$. Then, the inequality:*

$$M_{pk+m}(h,r) + M_{pk+m}(g,r) \leq 1 \tag{2.9}$$

is valid for $r \in [0, \theta_k]$, where $M_{pk+m}(h,r)$ is given by (2.5) and θ_k is the unique root in $(0,1)$ of the equation:

$$\left(1 - r^p\right) - r^{pk+m}\left(3 - r^p\right) = 0.$$

The upper bound θ_k cannot be improved as $f(z) = h(z) + \overline{\lambda h(z)}$ shows, where $h(z) = z^{pk+m}$ and $|\lambda| = 1$.

Our next result is similar to Theorem 4, but for fixed initial coefficients a_{pk+m} and b_{pk+m} having same modulus value.

Theorem 6. *Let $k \geq 2$, $m, p \in \mathbb{N}$, and $m \leq p$. Suppose that $f = h + \overline{g}$ is harmonic in $\mathbb{D}$, where h and g are given by (2.4), and $h, g \in \mathcal{B}_{pk+m}$. Let $|a_{pk+m}| = |b_{pk+m}| = a \in (0,1]$ be fixed. Then, the inequality:*

$$M_{pk+m}(h,r) + M_{pk+m}(g,r) \leq 1 \tag{2.10}$$

is valid for $r \in [0, \varsigma_k]$, where $\varsigma_k = \varsigma_k(a)$ is the unique root in $(0,1)$ of the equation:

$$(1+a)\left(1 - r^p\right) - 2r^{pk+m}\left[2a^2 + a + r^p\left(1 - 2a^2\right)\right] = 0.$$

The upper bound ς_k cannot be improved.

Remark 4. Clearly, $\varsigma_k(1) = \theta_k$. Also, it is possible to fix both $|a_{pk+m}|$ and $|b_{pk+m}|$ separately and obtain an analogous general result than Theorem 6.

3. Key Lemmas and Their Proofs

To establish our main results, we need the following lemmas.

Lemma 1. *Suppose that m, $p \in \mathbb{N}$, $m \le p$, $d \in (0,1]$, and $r = r_{p,m,d}$ is as in Theorem 1, i.e., the root of (2.1) in $(0,1)$. Then:*

$$r^{p+m} \le \frac{1}{3+d}.$$

Proof. Let $y = r_{p,m,d}^{p+m}$. Then, (2.1) becomes a quadratic equation in y of the form:

$$\left(4(1+d)(3+d) + \frac{1}{r_{p,m,d}^{2m}}\right) y^2 - (8+4d)y + 4r_{p,m,d}^{2m} = 0,$$

which has two solutions:

$$\begin{aligned} y &= \frac{4+2d \pm 2\sqrt{(1+d)(3+d)}\sqrt{1-4r_{p,m,d}^{2m}}}{4(1+d)(3+d) + \frac{1}{r_{p,m,d}^{2m}}} \\ &\le \frac{4+2d + 2\sqrt{(1+d)(3+d)}\sqrt{1-4r_{p,m,d}^{2m}}}{4(1+d)(3+d) + \frac{1}{r_{p,m,d}^{2m}}} \\ &\le \frac{1}{2}\left(\sup \frac{2+d+\sqrt{(1+d)(3+d)}\sqrt{1-4r_{p,m,d}^{2m}}}{(1+d)(3+d) + \frac{1}{4r_{p,m,d}^{2m}}}\right), \end{aligned}$$

and therefore, it is a simple exercise to see that:

$$\begin{aligned} r_{p,m,d}^{p+m} &\le \frac{1}{2}\left(\sup_{t\in(0,1]} \frac{2+d+\sqrt{(1+d)(3+d)}\sqrt{1-t}}{(1+d)(3+d)+\frac{1}{t}}\right) \\ &= \frac{1}{2}\left(\frac{2+d+\sqrt{(1+d)(3+d)}\sqrt{1-t}}{(1+d)(3+d)+\frac{1}{t}}\right)\Bigg|_{t=\frac{2}{3+d}} \\ &= \frac{1}{3+d}, \end{aligned}$$

which completes the proof of the lemma. □

Lemma 2. *Suppose that m, $p \in \mathbb{N}$, $m \le p$, $d \in (0,1]$, and $r = r_{p,m,d}$ is as in Theorem 1, i.e., the root of (2.1) in $(0,1)$. Then:*

$$\frac{1}{r^{p-m}}\left(2+d-\sqrt{(1+d)(3+d)}\sqrt{1-r^{2p}}\right) = \frac{1}{2}.$$

Proof. Suppose that $m < p$ and let $y = r^{p-m}$. Then, (2.1) reduces to a quadratic equation in y:

$$y^2 - (8+4d)y + 4(1+d)(3+d)r^{2p} + 4 = 0,$$

which has two solutions:

$$\begin{aligned} y_1 &= 4+2d+2\sqrt{(1+d)(3+d)}\sqrt{1-r^{2p}} > 1 \quad \text{and} \\ y_2 &= 4+2d-2\sqrt{(1+d)(3+d)}\sqrt{1-r^{2p}}. \end{aligned}$$

The solution $y = y_1$ is impossible, because all positive roots of the initial equation must be less than 1. Therefore:

$$y = y_2 = 2\left(2 + d - \sqrt{(1+d)(3+d)}\sqrt{1 - r^{2p}}\right).$$

Now, consider the case $m = p$. In this case:

$$r_{m,m,d} = \left(\frac{3+4d}{4(1+d)(3+d)}\right)^{\frac{1}{2m}},$$

so that:

$$\begin{aligned}
&\frac{1}{r^{p-m}}\left(2 + d - \sqrt{(1+d)(3+d)}\sqrt{1 - r^{2m}}\right)\\
&= 2 + d - \sqrt{(1+d)(3+d)}\sqrt{1 - \frac{3+4d}{4(1+d)(3+d)}}\\
&= 2 + d - \left(\frac{2d+3}{2}\right) = \frac{1}{2},
\end{aligned}$$

and the proof is complete. □

Lemma 3. [25]. *Let $0 < R \leq 1$. If $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is analytic and satisfies the inequality $|g(z)| \leq 1$ in $\mathbb{D}$. Then, the following sharp inequality holds:*

$$\sum_{k=1}^{\infty} |b_k|^2 R^{pk} \leq R^p \frac{(1 - |b_0|^2)^2}{1 - |b_0|^2 R^p}. \tag{3.1}$$

Lemma 4. [41]. *If $f \in \mathcal{B}$ has the expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then:*

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq |a_0| + \frac{r}{1-r}(1 - |a_0|^2).$$

4. Bohr's Inequality for the Class of Harmonic Mappings

4.1. Proof of Theorem 1

Given that $|g'(z)| \leq d|h'(z)|$ for some $d \in (0, 1]$, where:

$$h(z) = \sum_{k=0}^{\infty} a_k z^{pk+m} \quad \text{and}\, g(z) = \sum_{k=0}^{\infty} b_k z^{pk+m}.$$

We integrate inequality $|g'(z)|^2 \leq d^2|h'(z)|^2$ over the circle $|z| = r$ and get:

$$\sum_{k=0}^{\infty} (pk+m)^2 |b_k|^2 r^{2(pk+m-1)} \leq d^2 \sum_{k=0}^{\infty} (pk+m)^2 |a_k|^2 r^{2(pk+m-1)}.$$

We integrate the last inequality with respect to r^2 and obtain:

$$\sum_{k=0}^{\infty} (pk+m)|b_k|^2 r^{2(pk+m)} \leq d^2 \sum_{k=0}^{\infty} (pk+m)|a_k|^2 r^{2(pk+m)}.$$

One more integration (after dividing by r^2) gives:

$$\sum_{k=0}^{\infty} |b_k|^2 r^{2(pk+m)} \leq d^2 \sum_{k=0}^{\infty} |a_k|^2 r^{2(pk+m)},$$

which (since $|a_0| = |b_0|$ by hypothesis) yields:

$$\sum_{k=1}^{\infty} |b_k|^2 r^{pk} \leq d^2 \sum_{k=1}^{\infty} |a_k|^2 r^{pk} \quad \text{for } r < 1. \tag{4.1}$$

For simplicity, we suppose that $\|h\|_\infty = 1$.

Following the idea from [25] (see also [26, Proof of Theorem 1]), one can obtain first that:

$$B(h,r) = r^m \sum_{k=1}^{\infty} |a_k| r^{pk} \leq \frac{r^{p+m}(1-a^2)}{\sqrt{1-a^2 r^p \rho^p}} \frac{1}{\sqrt{1-\rho^{-p} r^p}}, \tag{4.2}$$

where $a = |a_0|$, and for any $\rho > 1$, such that $\rho r \leq 1$. Indeed, we may let $h(z) = z^m t(z^p)$, where $t(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$. Also, let $r = r_{p,m,d}$ and $|a_0| = a$. Then, as in Ref. [26, Proof of Theorem 1], it follows easily that:

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_k| r^{pk} &\leq \sqrt{\sum_{k=1}^{\infty} |a_k|^2 \rho^{pk} r^{pk}} \sqrt{\sum_{k=1}^{\infty} \rho^{-pk} r^{pk}} \\
&\leq \sqrt{r^p \rho^p \frac{(1-a^2)^2}{1-a^2 r^p \rho^p}} \sqrt{\frac{\rho^{-p} r^p}{1-\rho^{-p} r^p}} \\
&= \frac{r^p(1-a^2)}{\sqrt{1-a^2 r^p \rho^p}} \frac{1}{\sqrt{1-\rho^{-p} r^p}},
\end{aligned} \tag{4.3}$$

for any $\rho > 1$, such that $\rho r \leq 1$. In the first and the second steps above, we have used the classical Cauchy–Schwarz inequality, and (3.1) with $R = \rho r$ in Lemma 3, respectively. Hence, (4.2) follows.

Second, using the classical Cauchy–Schwarz inequality and (4.1), we see that:

$$\begin{aligned}
\sum_{k=1}^{\infty} |b_k| r^{pk} &\leq \sqrt{\sum_{k=1}^{\infty} |b_k|^2 \rho^{pk} r^{pk}} \sqrt{\sum_{k=1}^{\infty} \rho^{-pk} r^{pk}} \\
&\leq d \sqrt{\sum_{k=1}^{\infty} |a_k|^2 \rho^{pk} r^{pk}} \sqrt{\frac{\rho^{-p} r^p}{1-\rho^{-p} r^p}} \quad (\text{by } (4.1)),
\end{aligned}$$

and thus, by (4.3), we have:

$$B(g,r) = r^m \sum_{k=1}^{\infty} |b_k| r^{pk} \leq \frac{d r^{p+m}(1-a^2)}{\sqrt{1-a^2 r^p \rho^p}} \frac{1}{\sqrt{1-\rho^{-p} r^p}}, \tag{4.4}$$

for any $\rho > 1$, such that $\rho r \leq 1$. Consequently, by combining the inequalities (4.2) and (4.4), we get:

$$B_H(f,r) = r^m \left(|a_0| + |b_0| + \sum_{k=1}^{\infty} |a_k| r^{pk} + \sum_{k=1}^{\infty} |b_k| r^{pk} \right) \leq 2r^m \left(a + \frac{1+d}{2} \frac{r^p(1-a^2)}{\sqrt{1-a^2 r^p \rho^p}} \frac{1}{\sqrt{1-\rho^{-p} r^p}} \right). \tag{4.5}$$

We wish to maximize the right-hand side of above. For this, we need to consider the cases $a \geq r^p$ and $a < r^p$, separately. Note that our choice of ρ is such that $\rho r \leq 1$.

Case 1. Assume that $a \geq r^p$.

In this case, we set $\rho = \frac{1}{\sqrt[p]{a}}$ and obtain from (4.5) that:

$$B_H(f,r) \leq 2r^m \psi(a) \quad \text{for}\, a \geq r^p, \tag{4.6}$$

where we let $\alpha = r^p$ and:

$$\psi(x) = x + \frac{\alpha(1+d)}{2} \cdot \frac{1-x^2}{1-\alpha x}, \quad x \in [0,1].$$

Simple computation shows that, when $\alpha \geq \frac{1}{2+d}$, $\psi(x)$ attains its maximum at $x = x_1$, where:

$$x_1 = \left(1 - \sqrt{\frac{1+d}{3+d}} \sqrt{1-\alpha^2} \right) \frac{1}{\alpha},$$

and thus, $\psi(x) \leq \psi(x_1)$. On the other hand, when $\alpha < \frac{1}{2+d}$, $\psi(x)$ is monotonically increasing for $x \in [0,1]$, so that $\psi(x) \leq \psi(1) = 1$. Consequently, for the $r = r_{p,m,d}$ defined as in Theorem 1 and $\alpha \geq \frac{1}{2+d}$, it follows from (4.6) that:

$$B_H(f,r) \leq 2r^m \psi(x_1) = \frac{2}{r^{p-m}} \left(2 + d - \sqrt{(1+d)(3+d)} \sqrt{1-r^{2p}} \right) = 1, \tag{4.7}$$

where we have used Lemma 2 for the equality sign on the right. When $\alpha < \frac{1}{2+d}$, we have $\psi(x) \leq \psi(1) = 1$ and thus:

$$B_H(f,r) \leq 2r^m \psi(1) = 2r^m.$$

Case 2. Assume that $a < r^p$.

In this case, we set $\rho = \frac{1}{r}$ and obtain from (4.5) that:

$$B_H(f,r) \leq 2r^m \left(a + \frac{1+d}{2} \cdot r^p \frac{\sqrt{1-a^2}}{\sqrt{1-r^{2p}}} \right) \leq (3+d) r^{p+m} \leq 1. \tag{4.8}$$

Here, the second inequality on the right follows from the argument that we omitted the critical point:

$$a = \frac{\sqrt{1-r^{2p}}}{\sqrt{1 + \left(\frac{(1+d)^2}{4} - 1 \right) r^{2p}}},$$

because it is less than r^p only in the case $r^{2p} > \frac{2}{3+d} \geq \frac{1}{2}$, which contradicts with Lemma 1. The third inequality on the right in (4.8) follows from Lemma 1.

Therefore, in both cases, for all $a \in [0, 1)$, we have:

(i) when $\alpha = r^p \geq \frac{1}{2+d}$:

$$B_H(f, r) \leq 1 \quad \text{for } r \leq r_{p,m,d},$$

where $r_{p,m,d}$ is defined as in Theorem 1.

(ii) when $\alpha = r^p < \frac{1}{2+d}$, since $\max\{2r^m, (3+d)r^{p+m}\} = 2r^m$, we have:

$$B_H(f, r) \leq 2r^m.$$

In summary, if $r^m = \frac{1}{2}$ and $r^p < \frac{1}{2+d}$, that is, if $\frac{p}{m} > \log_2(2+d)$, we have by the second case above:

$$B_H(f, r) \leq 2r^m \leq 1, \quad \text{for } r \leq \sqrt[m]{\frac{1}{2}}.$$

The extremal function for the case $d = 1$ is $f(z) = h(z) + \overline{\lambda h(z)}$ with $h(z) = z^m$ and $|\lambda| = 1$.

When $1 \leq \frac{p}{m} \leq \log_2(2+d)$, we apply the first case above to obtain:

$$B(f, r) \leq \frac{1}{2} \quad \text{for } r \leq r_{p,m,d},$$

where $r_{p,m,d}$ is defined as in Theorem 1.

For the case $d = 1$, sharpness follows if we consider $f(z) = h(z) + \overline{\lambda h(z)}$ with $|\lambda| = 1$:

$$h(z) = z^m \left(\frac{z^p - a}{1 - az^p}\right), \quad a = \left(1 - \frac{\sqrt{1 - r_{p,m,1}^{2p}}}{\sqrt{2}}\right) \frac{1}{r_{p,m,1}^p},$$

and then calculate the Bohr radius for it. It coincides with r. □

4.2. Proof of Theorem 2

Without loss of generality, we may assume that:

$$\max\{\|h\|_\infty, \|g\|_\infty\} = 1.$$

Case 3. Assume that $1 \leq \frac{p}{m} \leq \log_2 3$.

It follows from Corollary 1(1) and the hypothesis that $B(h, r) \leq \frac{1}{2}$ and $B(g, r) \leq \frac{1}{2}$ for $r \leq r_{p,m}$, where $r_{p,m}$ is as in Theorem 2. Adding these two inequalities shows that:

$$B_H(f, r) = B(h, r) + B(g, r) \leq 1 \quad \text{for } r \leq r_{p,m}.$$

Case 4. Assume that $\frac{p}{m} > \log_2 3$.

Applying the method of the previous case, Corollary 1(2) gives:

$$B_H(f, r) = B(h, r) + B(g, r) \leq 1, \quad \text{for } r \leq \sqrt[m]{\frac{1}{2}}.$$

The extremal functions given in the statement are easy to verify. □

5. Bohr-Type Inequalities for Harmonic Mappings with a Multiple Zero at the Origin

5.1. Proof of Theorem 3

By assumption, $|g'(z)| \leq d|h'(z)|$ for some $d \in [0, 1)$. Then, $\omega_f = \frac{g'}{h'}$ is analytic in the punctured disk $0 < |z| < 1$ and has removable singularity at the origin with:

$$\lim_{z \to 0} \omega_f(z) = \frac{b_{pk+m}}{a_{pk+m}},$$

so that $|b_{pk+m}| \leq d|a_{pk+m}| < |a_{pk+m}|$.

Since $h \in \mathcal{B}_{pk+m}$, by applying Lemma 4 to the function $H(z) = \sum_{n=0}^{\infty} A_n z^{pn}$, $A_n = a_{p(n+k)+m}$, one has:

$$\sum_{n=0}^{\infty} |A_n| r^{pn} + \left(\frac{1}{1 + |A_0|} + \frac{r^p}{1 - r^p} \right) \sum_{n=1}^{\infty} |A_n|^2 r^{2pn} \leq |A_0| + \left(1 - |A_0|^2\right) \frac{r^p}{1 - r^p}. \tag{5.1}$$

Multiplying both sides of the inequality (5.1) by the number r^{pk+m}, and then adding the term $r^{pk+m}|A_0|^2 \left(\frac{1}{1+|A_0|} + \frac{r^p}{1-r^p} \right)$ to both sides, we have:

$$\begin{aligned} M_{pk+m}(h, r) &\leq r^{pk+m} \left[|A_0| + \frac{r^p}{1 - r^p} + \frac{|A_0|^2}{1 + |A_0|} \right] \\ &= r^{pk+m} \left(\frac{r^p}{1 - r^p} + G(|A_0|) \right), \end{aligned} \tag{5.2}$$

where $G(t) = t + t^2(1 + t)^{-1}$. Since $G'(t) > 0$ on $[0, 1]$, it follows that $G(t) \leq G(1) = 3/2$, and thus, (5.2) implies:

$$M_{pk+m}(h, r) \leq r^{pk+m} \left(\frac{r^p}{1 - r^p} + \frac{3}{2} \right) = r^{pk+m} \left(\frac{3 - r^p}{2(1 - r^p)} \right). \tag{5.3}$$

Next, since $|g'(z)| \leq d|h'(z)|$ for $z \in \mathbb{D}$, we have (cf. [11]):

$$\sum_{n=k}^{\infty} |b_{pn+m}|\, r^{pn+m} \leq d \sum_{n=k}^{\infty} |a_{pn+m}|\, r^{pn+m} \quad \text{for } r \leq 1/\sqrt[p]{3}, \tag{5.4}$$

and, as in the proof of Theorem 1:

$$\sum_{n=k}^{\infty} |b_{pn+m}|^2\, r^{p(2n-k)+m} \leq d^2 \sum_{n=k}^{\infty} |a_{pn+m}|^2\, r^{p(2n-k)+m}. \tag{5.5}$$

Thus, we conclude from (5.2), (5.4), and (5.5) that:

$$\begin{aligned} N_{pk+m}(g, r) \leq d \sum_{n=k}^{\infty} |a_{pn+m}|\, r^{pn+m} &+ d^2 \left(\frac{1}{1 + |a_{pk+m}|} + \frac{r^p}{1 - r^p} \right) \\ &\times \sum_{n=k}^{\infty} |a_{pn+m}|^2\, r^{p(2n-k)+m}, \end{aligned}$$

which by combining with (5.2) gives:

$$M_{pk+m}(h, r) + N_{pk+m}(g, r)$$

$$\leq (1+d)\sum_{n=k}^{\infty}|a_{pn+m}|\,r^{pn+m}+(1+d^2)\left(\frac{1}{1+|a_{pk+m}|}+\frac{r^p}{1-r^p}\right)$$
$$\times\sum_{n=k}^{\infty}|a_{pn+m}|^2\,r^{p(2n-k)+m}$$
$$=(d-d^2)\sum_{n=k}^{\infty}|a_{pn+m}|\,r^{pn+m}+(1+d^2)M_{pk+m}(h,r)$$
$$\leq (d-d^2)M_{pk+m}(h,r)+(1+d^2)M_{pk+m}(h,r)=(d+1)M_{pk+m}(h,r),$$

which, by (5.3), is less than or equal to 1 if:

$$r^{pk+m}\left(\frac{3-r^p}{2(1-r^p)}\right)\leq\frac{1}{1+d},\quad \text{i.e., } t_k(r)\geq 0,$$

where $t_k(r)$ is given by (2.7); that is:

$$t_k(r)=\frac{2}{d+1}(1-r^p)-r^{pk+m}\,(3-r^p)\,.$$

This proves the first part of the assertion.

Now, we prove the uniqueness of the solution in $(0,1)$ of $t_k(r)=0$, we compute that $t_k(0)=2/(d+1)>0$, $t_k(1)=-2<0$, and:

$$t_k'(r)=-\frac{2p}{d+1}r^{p-1}-3(pk+m)r^{pk+m-1}+(p(k+1)+m)r^{p(k+1)+m-1}$$
$$=-p\left(\frac{2}{d+1}r^{p-1}-r^{p(k+1)+m-1}\right)-(pk+m)r^{pk+m-1}\,(3-r^p)$$
$$\leq -p\left(r^{p-1}-r^{p(k+1)+m-1}\right)-(pk+m)r^{pk+m-1}\,(3-r^p)<0,$$

showing that $t_k(r)$ is a decreasing function of r in $(0,1)$, and thus, $t_k(r)=0$ has a unique root in $(0,1)$. □

5.2. Proof of Theorem 4

By assumption $h\in\mathcal{B}_{pk+m}$. Therefore, as in the proof of Theorem 3, we can apply Lemma 4 to the function $H(z)=\sum_{n=0}^{\infty}A_nz^{pn}$, $A_n=a_{p(n+k)+m}$. Thus, (5.1) holds. Multiplying the inequality (5.1) by r^{pk+m} gives:

$$M^1_{pk+m}(h,r)\leq r^{pk+m}\left[|A_0|+\left(1-|A_0|^2\right)\frac{r^p}{1-r^p}\right].$$

Now, we can maximize the right-hand side with respect to $|A_0|$ by fixing r. A simple calculation shows that we arrive at the maximum value $r^{pk+m}M(r)$ which is achieved at $|A_0|=1$, if $r\in\left[0,\frac{1}{\sqrt[p]{3}}\right]$, and at $|A_0|=\frac{1-r^p}{2r^p}$ in the remaining cases. Thus, we have the maximum value:

$$\frac{r^{p(k-1)+m}\left(5r^{2p}-2r^p+1\right)}{4\,(1-r^p)},$$

and therefore:

$$M^1_{pk+m}(h,r)\leq r^{pk+m}\left[|A_0|+\left(1-|A_0|^2\right)\frac{r^p}{1-r^p}\right]$$

$$\leq \frac{r^{p(k-1)+m}\left(5r^{2p}-2r^p+1\right)}{4\left(1-r^p\right)}.$$

Again, as $g \in \mathcal{B}_{pk+m}$, we have similarly the inequality:

$$\begin{aligned} M^1_{pk+m}(g,r) &\leq r^{pk+m}\left[|B_0| + \left(1-|B_0|^2\right)\frac{r^p}{1-r^p}\right] \\ &\leq \frac{r^{p(k-1)+m}\left(5r^{2p}-2r^p+1\right)}{4\left(1-r^p\right)}, \end{aligned}$$

where $B_0 = b_{pk+m}$. Adding the two resulting inequalities yields that:

$$M^1_{pk+m}(h,r) + M^1_{pk+m}(g,r) \leq \frac{r^{p(k-1)+m}\left(5r^{2p}-2r^p+1\right)}{2\left(1-r^p\right)}.$$

Hence, the desired inequality (2.8), i.e., $M^1_{pk+m}(h,r) + M^1_{pk+m}(g,r) \leq 1$, holds whenever $L_k(r) \geq 0$, where:

$$L_k(r) = 2\left(1-r^p\right) - r^{p(k-1)+m}\left(5r^{2p}-2r^p+1\right).$$

This proves the first part of the assertion.

Next, we prove the uniqueness of the solution in $(0,1)$ of $L_k(r) = 0$. In fact, note that $L_k(0) = 2 > 0$, $L_k(1) = -4 < 0$, and:

$$L_k'(r) = -pr^{p-1}\left(2 - r^{p(k-2)+m}\right) - r^{p(k-1)+m-1}Q(r^p),$$

where $Q(x) = 5(p(k+1)+m)x^2 - 2(pk+m)x + pk + m$. It follows that $Q(x) > 0$, because the discriminant of the function Q is less than 0. This gives that $L_k'(r) < 0$, and hence, $L_k(r) = 0$ has a unique root τ_k in $(0,1)$.

Finally, we verify the sharpness of the upper bound τ_k for the Bohr radius. We consider the function $f(z) = h(z) + \overline{\lambda h(z)}$, $|\lambda| = 1$, where:

$$h(z) = z^{pk+m}\left(\frac{a - z^p}{1 - az^p}\right), \quad a \in (0,1).$$

For this function, we obtain that:

$$M^1_{pk+m}(h,r) + M^1_{pk+m}(g,r) = 2r^{pk+m}\left[a + \frac{(1-a^2)r^p}{1-r^p}\right],$$

which equals 1 for $r = \tau_k$ and $a = \frac{1-\tau_k^p}{2\tau_k^p}$. This completes the proof of Theorem 4 □

5.3. Proof of Theorem 5

Let $h \in \mathcal{B}_{pk+m}$. Then, (5.2) holds, that is:

$$M_{pk+m}(h,r) \leq r^{pk+m}\left[|A_0| + \frac{r^p}{1-r^p} + \frac{|A_0|^2}{1+|A_0|}\right] \leq r^{pk+m}\left[\frac{3}{2} + \frac{r^p}{1-r^p}\right], \tag{5.6}$$

where $A_0 = a_{pk+m}$. The second inequality holds, because the function T defined by:

$$T(x) = x + \frac{x^2}{1+x}$$

is a monotonically increasing function of $x \in [0,1]$, so that: $T(x) \le T(1) = 3/2$.

Similarly, with $B_0 = b_{pk+m}$, we have:

$$M_{pk+m}(g,r) \le r^{pk+m}\left[|B_0| + \frac{r^p}{1-r^p} + \frac{|B_0|^2}{1+|B_0|}\right] \le r^{pk+m}\left[\frac{3}{2} + \frac{r^p}{1-r^p}\right], \tag{5.7}$$

where $|B_0| \in [0,1]$.

Combining (5.6) and (5.7) leads to:

$$M_{pk+m}(h,r) + M_{pk+m}(g,r) \le r^{pk+m}\left(3 + \frac{2r^p}{1-r^p}\right), \tag{5.8}$$

if and only if $w_k(r) \ge 0$, where:

$$w_k(r) = (1-r^p) - r^{pk+m}(3-r^p).$$

This proves the first part of the assertion of the theorem.

Next, to prove the uniqueness of the solution in $(0,1)$ of $w_k(r) = 0$, it is sufficient to observe that $w_k(0) = 1 > 0$, $w_k(1) = -2 < 0$, and:

$$w_k'(r) = -pr^{p-1}\left(1 - r^{pk+m}\right) - (pk+m)r^{pk+m-1}(3-r^p) < 0.$$

Finally, it is easy to verify that the extremal function has the form $f(z) = h(z) + \overline{\lambda h(z)}$, where $h(z) = z^{pk+m}$ and $|\lambda| = 1$. This completes the proof of Theorem 5. □

5.4. Proof of Theorem 6

The proof is essentially similar to the proof of Theorem 5. At first, from (5.8) and the assumption that $|A_0| = |B_0| = a$, it is obvious that the required inequality (2.10) is true if:

$$2r^{pk+m}\left(a + \frac{r^p}{1-r^p} + \frac{a^2}{1+a}\right) = 2r^{pk+m}\left[\frac{a + 2a^2 + r^p\left(1-2a^2\right)}{(1+a)(1-r^p)}\right] \le 1, \tag{5.9}$$

which holds if and only if $V_{k,a}(r) \ge 0$, where:

$$V_{k,a}(r) := (1+a)(1-r^p) - 2r^{pk+m}\left[2a^2 + a + r^p(1-2a^2)\right].$$

This proves the first part of the assertion of the theorem.

Next, we can prove the uniqueness of the solution of $V_{k,a}(r) = 0$ in $(0,1)$. It is obvious that $V_{k,a}(0) = 1 + a > 0$ and $V_{k,a}(1) = -2(1+a) < 0$. Furthermore:

$$\begin{aligned} V_{k,a}'(r) = &-pr^{p-1}\left[1 + a + 2\left(1-2a^2\right)r^{pk+m}\right] \\ &-2r^{pk+m-1}(pk+m)\left[2a^2 + a + \left(1-2a^2\right)r^p\right], \end{aligned}$$

and it is easy to obtain that $V_{k,a}'(r) < 0$, and thus, $V_{k,a}(r) = 0$ has the unique root $\varsigma_k = \varsigma_k(a)$ in the interval $(0,1)$.

Finally, we verify the sharpness of the upper bound ς_k for the Bohr radius. Consider $f(z) = h(z) + \overline{\lambda h(z)}$, where:

$$h(z) = z^{pk+m}\left(\frac{a - z^p}{1 - az^p}\right), \ a \in [0,1],$$

$|\lambda| = 1$ and $a \in [0, 1]$ is fixed. In this case, we get for the left hand side of (2.10) (for simplicity, call it as $W(r)$) takes the form:

$$\begin{aligned} W(r) &= 2r^{pk+m}\left[a + \frac{(1-a^2)r^p}{1-r^p}\right] + 2\left(\frac{1}{1+a} + \frac{r^p}{1-r^p}\right)a^2r^{pk+m} \\ &= 2r^{pk+m}\left[\frac{a + 2a^2 + r^p\left(1-2a^2\right)}{(1+a)\left(1-r^p\right)}\right]. \end{aligned}$$

Comparison of this expression with the right-hand side of the equation in formula (5.9) delivers the asserted sharpness. The proof of Theorem 6 is complete. □

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